Section 2.5

Continuity

Learning Goals

Learning Goal	*Homework Problems*
2.5.1 Determine where functions are continuous using the function's graph and if they can be made continuous by changing their values at certain points.	1-10.
2.5.2 Determine where functions are continuous (algebraically) by applying the continuity test.	11-40.
2.5.3 Find values that make a function continuous.	41-54.
2.5.4 Apply the intermediate value theorem to find solutions of equations.	55-60, 73-80.
2.5.5 Analyze concepts involving continuous functions.	61-72.

Continuity: f is continuous if its graph has no holes or breaks.

Precise definition: $\boxed{f \text{ is continuous at } c \text{ if } \lim_{x \to c} f(x) = f(c)}$

This means that:

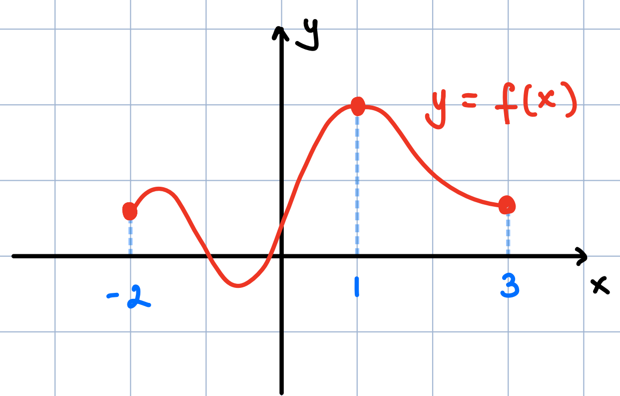
- $f(c)$ exists (c is in the domain of f)
- $\lim_{x \to c} f(x)$ exists $\left(\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) \right)$
- $f(c) = \lim_{x \to c} f(x)$.

↳ These three conditions are called the continuity test.

We say that f is discontinuous (or has a discontinuity) at c if f is not continuous at c.
We say that f is continuous on an interval I if f is continuous at every point c in I.

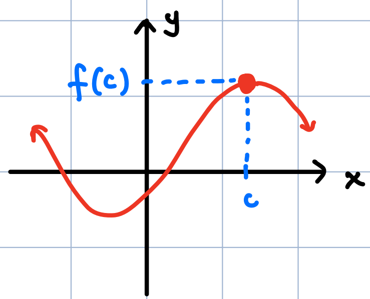
One-sided continuity:

- f is left-continuous at c if $\lim_{x \to c^-} f(x) = f(c)$.
- f is right-continuous at c if $\lim_{x \to c^+} f(x) = f(c)$.

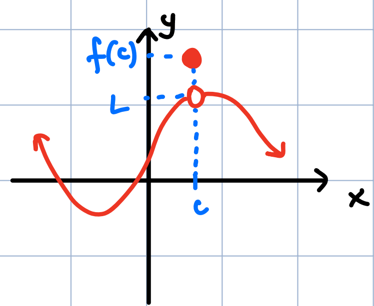


f is left continuous at 3
right continuous at -2
continuous at 1
continuous on [-2,3]

Examples : 1)

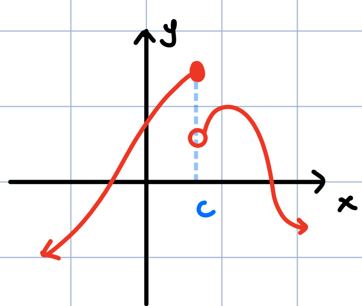


<u>continuous</u> at c since $\lim_{x \to c} f(x) = f(c)$



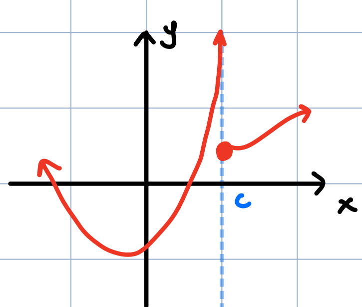
<u>removable discontinuity</u> at $x = c$

$\lim_{x \to c}$ exists but is not equal to $f(c)$.



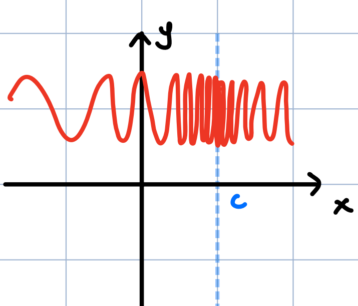
<u>jump discontinuity</u> at $x = c$.

$\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$



<u>infinite discontinuity</u> at $x = c$.

$\lim_{x \to c^-} f(x)$ or $\lim_{x \to c^+} f(x)$ is infinite



<u>essential discontinuity</u>

(e.g. infinite oscillation)

2) Common functions are continuous on their domain.
For example, $f(x) = \frac{\sqrt{3-x}}{x+7}$ is continuous on $D_f = (-\infty,-7)\cup(-7,3]$

$g(x) = \ln(x^2-4)$ is continuous on $D_g = (-\infty,-2)\cup(2,\infty)$

$h(x) = \arcsin(3x)$ is continuous on $D_h = \left[-\frac{1}{3},\frac{1}{3}\right]$.

3) Where is the function $f(x) = \begin{cases} -x^2-5x-6 & x \leq -1 \\ 5-x & -1 < x \leq 1 \\ \frac{x^2+2x-3}{x-1} & x > 1 \end{cases}$ continuous ?

We know f is continuous on $(-\infty,-1)$, $(-1,1)$ and $(1,\infty)$ (each piece is a common function). We only need to test the transition points $x = -1, 1$.

At $x = -1$:

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (-x^2-5x-6) = -2$$
$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (5-x) = 6$$

$\neq$

So $\lim_{x\to -1} f(x)$ DNE and f has a jump discontinuity at $x = -1$.

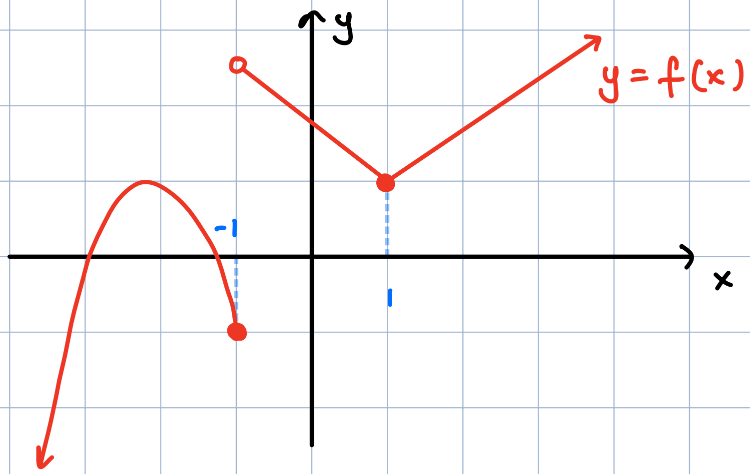
At $x = 1$:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (5-x) = 4$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^2+2x-3}{x-1} = \lim_{x\to 1^+} \frac{\cancel{(x-1)}(x+3)}{\cancel{x-1}} = 4$$

$f(1) = 4$.

So $\lim_{x\to 1} f(x) = f(1) = 4 \therefore f$ is continuous at $x=1$.

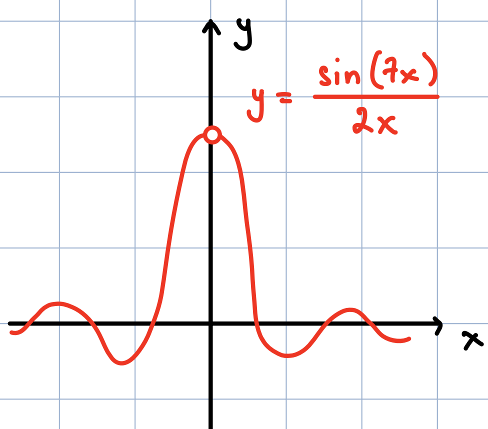
Conclusion: f is continuous on $\boxed{(-\infty, -1) \cup (-1, \infty)}$.



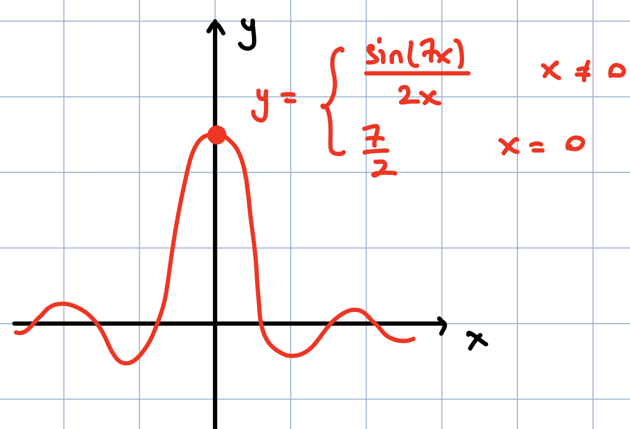
4) The function $f(x) = \dfrac{\sin(7x)}{2x}$ is continuous on $(-\infty, 0) \cup (0, \infty)$ and undefined at $x = 0$. Can we define a value of $f(0)$ that would make f continuous at $x = 0$?

$$\lim_{x \to 0} \frac{\sin(7x)}{2x} = \lim_{x \to 0} \underbrace{\frac{\sin(7x)}{2x}}_{\to 1} \cdot \frac{7\cancel{x}}{2\cancel{x}} = \frac{7}{2}.$$

So defining $\boxed{f(0) = \frac{7}{2}}$ gives continuity at $x = 0$.



The original function has a removable discontinuity at $x = 0$



Continuous extension at $x = 0$

5) Find the values of a, b such that the function

$$f(x) = \begin{cases} ax - 4 & x < 1 \\ 2a + b & x = 1 \\ \dfrac{x-1}{\sqrt{x}-1} & x > 1 \end{cases}$$ is continuous on $\mathbb{R}$.

At $x = 1$ we want $\lim\limits_{x\to 1^-} f(x) = f(1) = \lim\limits_{x \to 1^+} f(x)$

- $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} ax - 4 = a - 4$
- $f(1) = 2a + b$
- $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \dfrac{x-1}{\sqrt{x}-1} \cdot \dfrac{\sqrt{x}+1}{\sqrt{x}+1} = \lim\limits_{x\to 1^+} \dfrac{\cancel{(x-1)}(\sqrt{x}+1)}{\cancel{x-1}} = \sqrt{1}+1 = 2.$

So we get the conditions $a - 4 = 2a + b = 2$, which we must solve for a, b.

$a - 4 = 2 \Rightarrow \boxed{a = 6}$

$2a + b = 2 \Rightarrow b = 2 - 2a = 2 - 12$

$\boxed{b = -10}$.

6) Find the values of a, b such that the function

$$f(x) = \begin{cases} 5 & x < -1 \\ ax + b & -1 \leq x \leq 2 \\ \dfrac{x-2}{\frac{7}{x+5} - 1} & x > 2 \end{cases}$$ is continuous on $\mathbb{R}$.

Each piece is continuous. At $x = -1$, we want $\lim\limits_{x\to -1^-} f(x) = f(-1) = \lim\limits_{x\to -1^+} f(x)$.

- $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1} 5 = 5$
- $f(-1) = a(-1) + b = -a + b$

- $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} ax+b = -a+b$

So we get the condition $\underline{-a+b = 5}$

At $x=2$, we want $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

- $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} ax+b$
- $f(2) = 2a+b$
- $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \dfrac{x-2}{\frac{7}{x+5}-1} = \lim_{x\to 2^+} \dfrac{x-2}{\frac{7-(x+5)}{x+5}} = \lim_{x\to 2^+} \dfrac{(x-2)(x+5)}{2-x}$

$$= \lim_{x\to 2^+} \frac{\cancel{(x-2)}(x+5)}{-\cancel{(x-2)}} = \lim_{x\to 2^+} -(x+5) = -7.$$

So we get the condition $\underline{2a+b = -7.}$

We must now solve the conditions $\begin{cases} 2a+b = -7 \\ -a+b = 5 \end{cases}$ for a, b.

Subtract the two: $2a+b-(-a+b) = -7-5$

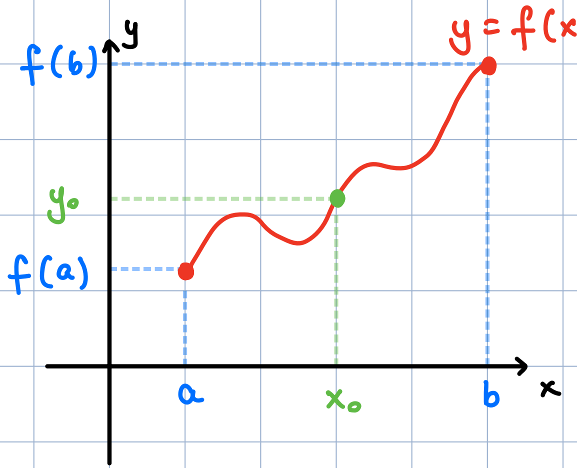
$$3a = -12$$

$$\boxed{a = -4}$$

Then $b = a+5$

$\boxed{b = 1}$.

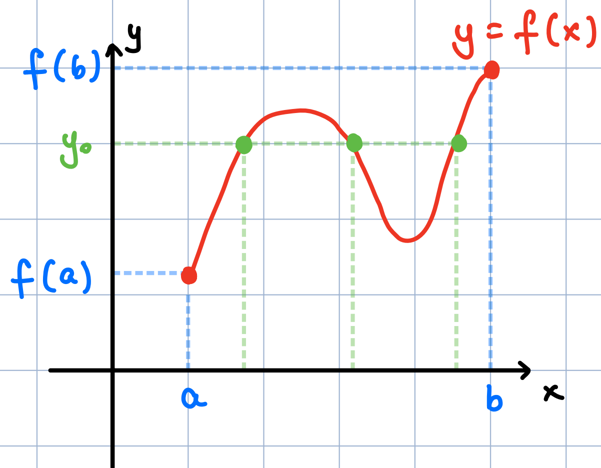
Intermediate Value Theorem: if a continuous function takes the values $f(a)$ and $f(b)$, it takes all values between $f(a)$ and $f(b)$ since the graph has no breaks/holes.

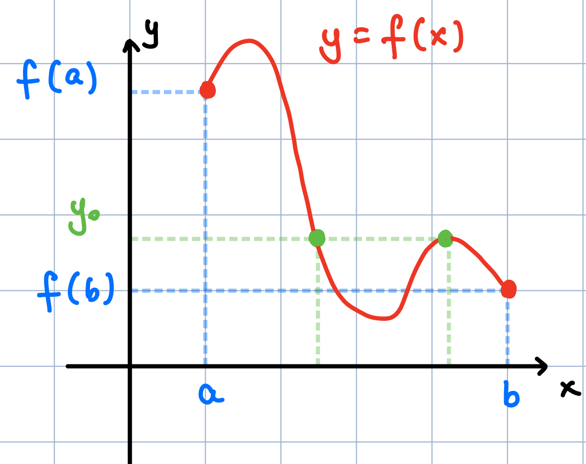


If f is continuous on $[a,b]$ and y_0 is between $f(a)$ and $f(b)$, there exists x_0 in $[a,b]$ such that $\boxed{f(x_0) = y_0}$.

Remarks:

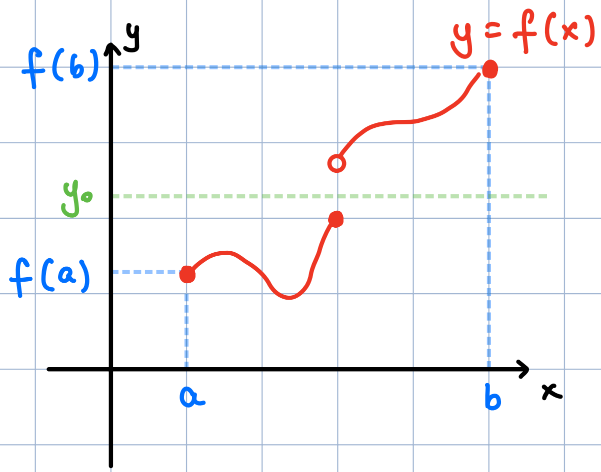
- The IVT guarantees the existence of at least one x_0, but there could be more.





- The theorem does not say how to find x_0; it just guarantees its existence.

- The theorem does not hold for discontinuous functions.



If f is discontinuous on $[a,b]$, it may skip some intermediate values between $f(a)$ and $f(b)$.

Examples: 1) Show that the equation $\cos(x) = x$ has a solution in $\left[0, \frac{\pi}{2}\right]$.

$$\cos(x) = x \iff \underbrace{\cos(x) - x}_{f(x)} = \underbrace{0}_{y_0}.$$

We know that:

- $f(x) = \cos(x) - x$ is continuous on $\left[0, \frac{\pi}{2}\right]$.
- $f(0) = \cos(0) - 0 = 1 > 0$
- $f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2} = -\frac{\pi}{2} < 0$

So $y_0 = 0$ is an intermediate value between $f(0)$ and $f\left(\frac{\pi}{2}\right)$. By the IVT, there exists x_0 in $\left[0, \frac{\pi}{2}\right]$ such that $f(x_0) = y_0 = 0$

so $\cos(x_0) - x_0 = 0$

$\cos(x_0) = x_0$: x_0 is a solution.

2) Show that the equation $xe^x = 3$ has a solution in the interval $[0, 2]$.

$$\underbrace{xe^x}_{f(x)} = \underbrace{3}_{y_0}$$

We know that:

- $f(x) = xe^x$ is continuous on $[0,2]$.
- $f(0) = 0e^0 = 0 < 3$
- $f(2) = 2e^2 > 3$ since $e^2 > 2$.

So $y_0 = 3$ is an intermediate value between $f(0)$ and $f(2)$. By the IVT, there exists x_0 in $[0, 2]$ such that $f(x_0) = 3$

$x_0 e^{x_0} = 3$: x_0 is a solution.

3) Suppose that f is a continuous function such that

$$f(-1) = -3, \quad f(0) = 5, \quad f(1) = 6, \quad f(2) = 1.$$

What is the minimal number of solutions that the equation $f(x) = 2x$ must have in $[-1, 2]$?

Put $g(x) = f(x) - 2x$: g is continuous and we want to solve $g(x) = \underbrace{0}_{y_0}$.

On $[-1, 0]$: $g(-1) = f(-1) - 2(-1) = -3+2 = -1 < 0$, $g(0) = f(0) - 2\cdot 0 = 5 > 0$ — $y_0 = 0$ is an intermediate value between $g(-1)$ and $g(0)$

So $g(x) = 0$ has at least one solution in $[-1, 0]$.

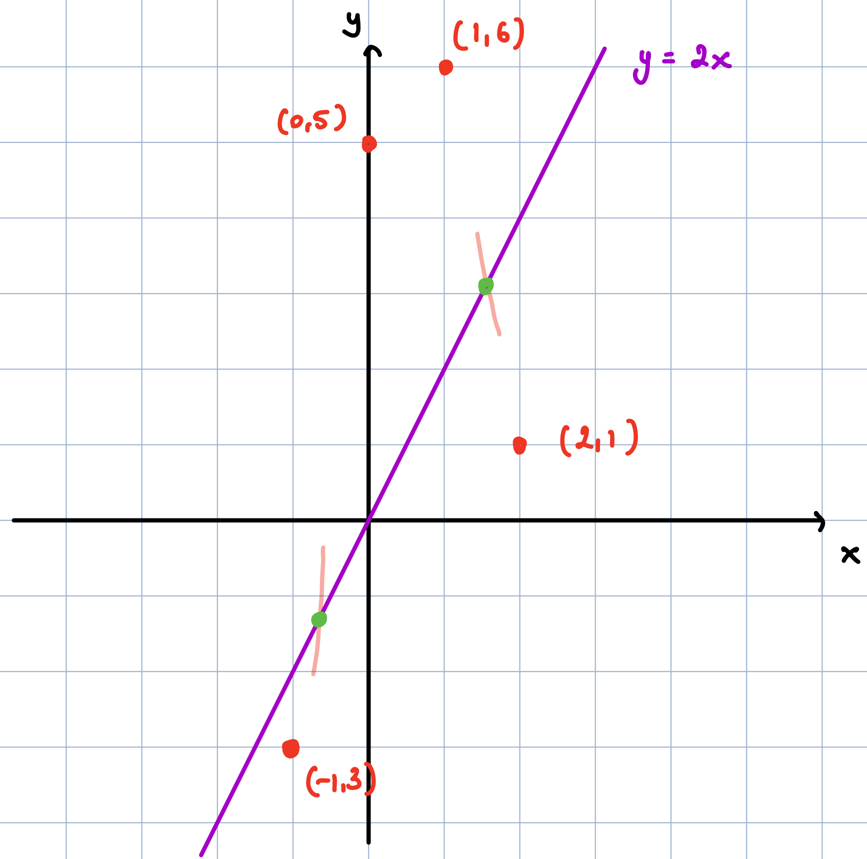
On $[0, 1]$: $g(0) = 5 > 0$, $g(1) = f(1) - 2\cdot 1 = 6 - 2 = 4 > 0$ — $y_0 = 0$ is NOT between $g(0)$ and $g(1)$

So $g(x) = 0$ is not guaranteed to have a solution in $[0,1]$.

On $[1,2]$: $g(1) = 4 > 0$
$g(2) = f(2) - 2 \cdot 2 = 1 - 4 = -3 < 0$ $\Big]$ $y_0 = 0$ is an intermediate value between $f(1)$ and $f(2)$

So $g(x) = 0$ has at least one solution in $[1,2]$.

Conclusion: $g(x) = 0$ has at least 2 solutions in $[-1,2]$.



The graph of f guaranteed to intersect $y = 2x$ at least twice on $[-1,2]$ by the IVT. There may be more than 2 solutions, but there cannot be less than 2.