Rutgers University
Math 151

Section 2.5: Continuity - Worksheet Solutions

1. For each function, find the values of the constants a, b that make it continuous.

(a) $f(x) = \begin{cases} 3x - b & \text{if } x \leqslant 1 \\ ax + 4 & \text{if } 1 < x \leqslant 3 \\ bx - 2a & \text{if } x > 3 \end{cases}.$

Solution. Each piece of f being continuous, it suffices to test for continuity at the transition points $x = 1$ and $x = 3$. At $x = 1$, we have

$$\begin{aligned} &\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 3x - b = 3 - b, \\ &\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} ax + 4 = a + 4, \\ &f(1) = 3(1) - b = 3 - b. \end{aligned}$$

So the continuity test gives the condition $3 - b = a + 4$, or $a + b = -1$. At $x = 3$, we have

$$\begin{aligned} &\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} ax + 4 = 3a + 4, \\ &\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} bx - 2a = 3b - 2a, \\ &f(3) = 3a + 4. \end{aligned}$$

So the continuity test gives the condition $3a + 4 = 3b - 2a$, or $5a - 3b = -4$. Therefore, for f to be continuous, the constants a, b must satisfy the equations

$$\begin{cases} a + b = -1, \\ 5a - 3b = -4. \end{cases}$$

To finish, we need to solve this system of two linear equations. Adding 3 times the first equation to the second one gives $8a = -7$, so $\boxed{a = -\frac{7}{8}}$. Then we get $\boxed{b = \frac{15}{8}}$.

(b) $f(x) = \begin{cases} bx + 4 & \text{if } x < 1 \\ a & \text{if } x = 1 \\ \dfrac{x^{-1} - 1}{x^2 - 1} & \text{if } x > 1 \end{cases}.$

Solution. Each piece of f is continuous. This is obvious for the piece for $x < 1$, as it is a linear function. For the piece for $x > 1$, observe that the roots of the denominator are $x = -1, 1$. Therefore, the denominator does not cancel for $x > 1$ and the piece is a well-defined rational function (therefore continuous). So it suffices to test for continuity at $x = 1$. We have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} bx + 4 = b + 4,$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^{-1}-1}{x^2-1} = \lim_{x\to 1^+} \frac{\frac{1-x}{x}}{(x-1)(x+1)} = \lim_{x\to 1^+} -\frac{1}{x(x+1)} = -\frac{1}{2},$$
$$f(1) = a.$$

So the continuity test gives the condition $b+4 = -\frac{1}{2} = a$. This gives the values $\boxed{a = -\frac{1}{2}}$ and $\boxed{b = -\frac{9}{2}}$.

(c) **[Advanced]** $f(x) = \begin{cases} \dfrac{\sin(ax)}{3x} & \text{if } x < 0 \\ b & \text{if } x = 0 \\ \dfrac{x^2+5x}{\sqrt{x+4}-2} & \text{if } x > 0 \end{cases}.$

Solution. The pieces for $x < 0$ and $x > 0$ are both continuous (well-defined common functions). So it suffices to test for continuity at $x = 0$. We have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin(ax)}{3x} \cdot \frac{ax}{ax} = \left(\lim_{x\to 0^-} \frac{\sin(ax)}{ax}\right)\left(\lim_{x\to 0^-} \frac{ax}{3x}\right) = \frac{a}{3},$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2+5x}{\sqrt{x+4}-2} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \lim_{x\to 0^+} \frac{x(x+5)\left(\sqrt{x+4}+2\right)}{x+4-4} = \lim_{x\to 0^+} (x+5)\left(\sqrt{x+4}+2\right) = 20$$
$$f(1) = b.$$

So we get the conditions $\frac{a}{3} = 20 = b$. This gives the solutions $\boxed{a = 60}$ and $\boxed{b = 20}$.

2. Consider the function $f(x) = \begin{cases} x^2+4x+5 & \text{if } x < -2 \\ 3 & \text{if } x = -2 \\ \cos(\pi x) & \text{if } -2 < x < 3 \\ x+2 & \text{if } 3 \leqslant x \leqslant 4 \\ 6 - \ln(x-3) & \text{if } x > 4 \end{cases}.$

(a) Find the values of a for which $\lim_{x\to a} f(x)$ does not exist.

Solution. Since each piece of f is continuous (therefore has a limit at every point of its domain), it suffices to test the transition points. At $x = -2$, we have

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} x^2+4x+5 = (-2)^2 + 4(-2) + 5 = 1,$$
$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \cos(\pi x) = \cos(-2\pi) = 1.$$

Since $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x)$, $\lim_{x\to -2} f(x)$ exists.

At $x = 3$, we have

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \cos(\pi x) = \cos(3\pi) = -1,$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x+2 = 5.$$

Since $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$, $\lim_{x\to 3} f(x)$ does not exist.

At $x = 4$, we have

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} x + 2 = 6,$$
$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} 6 - \ln(x-3) = 6 - \ln(1) = 6.$$

Since $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x)$, $\lim_{x\to 4} f(x)$ exists.

In conclusion, $\lim_{x\to a} f(x)$ does not exist for $\boxed{a = 3}$.

(b) Find the values of x where f is discontinuous.

Solution. Since each piece of f is continuous, it suffices to test the transition points. We already know f is discontinuous at $x = 3$ since it does not have a limit at this point by part (a). At $x = -2$ we have $\lim_{x\to -2} f(x) = 1$ and $f(-2) = 3$, so f is discontinuous at $x = -2$. At $x = 4$, we have $\lim_{x\to 4} f(x) = 6$ and $f(4) = 6$, so f is continuous at $x = 6$.

In conclusion, f is discontinuous at $\boxed{x = -2, 3}$.

3. Show that each equation has a solution in the given interval.

(a) $x^3 = 14 + 2\sqrt{x}$ in $[0, 4]$.

Solution. We start by writing the equation as $x^3 - 2\sqrt{x} = 14$. This has the form $f(x) = y_0$ with $f(x) = x^3 - 2\sqrt{x} - 14$ and $y_0 = 14$. The function f is continuous on $[0, 4]$. We have

$$f(0) = 0^3 - 2\sqrt{0}0 < 14,$$
$$f(4) = 4^3 - 2\sqrt{4} = 60 > 14.$$

Therefore, the value $y_0 = 14$ is an intermediate value between $f(0)$ and $f(4)$. By the IVT, it follows that the equation has a solution in $[0, 4]$.

(b) $\ln(x) = 2 - x$ in $[1, e]$.

Solution. We start by writing the equation as $\ln(x) + x = 2$. This has the form $f(x) = y_0$ with $f(x) = \ln(x) + x$ and $y_0 = 2$. The function f is continuous on $[1, e]$. We have

$$f(1) = \ln(1) + 1 = 1 < 2,$$
$$f(e) = \ln(e) + e = 1 + e > 2.$$

Therefore, the value $y_0 = 2$ is an intermediate value between $f(1)$ and $f(e)$. By the IVT, it follows that the equation has a solution in $[1, e]$.

(c) **[Advanced]** $\cos(x) = \arcsin(x)$ in $[0, 1]$.

Solution. We start by writing the equation as $\cos(x) - \arcsin(x) = 0$. This has the form $f(x) = y_0$ with $f(x) = \cos(x) - \arcsin(x)$ and $y_0 = 0$. The function f is continuous on $[0, 1]$. We have

$$f(0) = \cos(0) - \arcsin(0) = 1 - 0 = 1 > 0,$$
$$f(1) = \cos(1) - \arcsin(1) = \cos(1) - \frac{\pi}{2}.$$

Observe that $\cos(1) \leqslant 1$ since the range of cos is $[-1, 1]$ and $\frac{\pi}{2} > 1$ since $\pi > 2$. Thus, $\cos(1) - \frac{\pi}{2} < 0$. It follows that the value $y_0 = 0$ is an intermediate value between $f(0)$ and $f(1)$. By the IVT, it follows that the equation has a solution in $[0, 1]$.