Section 2.6

Limits Involving Infinity

Learning Goals

Learning Goal	*Homework Problems*
2.6.1 Find limits using a graph of the function.	1, 2.
2.6.2 Compute limits of functions as x approaches infinity or negative infinity using appropriate algebraic manipulation.	3-36.
2.6.3 Find infinite limits.	37-62.
2.6.4 Graph rational functions and identify any asymptotes of the function.	63-68.
2.6.5 Use limits to find domains, ranges, and asymptotes of functions.	69-74.
2.6.6 Find and graph functions with given conditions.	75-85.
2.6.7 Compute the limits of differences of functions at infinity.	86-92.
2.6.8 Find and graph oblique asymptotes of rational functions.	105-110.
2.6.9 Answer conceptual questions involving limits at infinity, infinite limits, and asymptotes.	83-85.

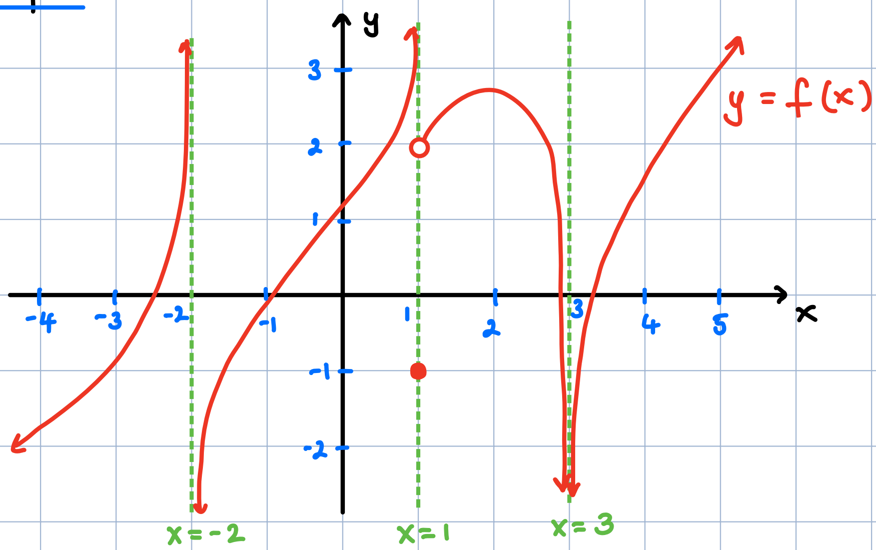
Infinite limits:

$\lim_{x \to a} f(x) = \infty$ means that $f(x)$ gets arbitrarily large / grows without bounds as x approaches a.

$\lim_{x \to a} f(x) = -\infty$ means that $f(x)$ gets arbitrarily large and negative / decreases without bounds as x approaches a.

Graphically, $x = a$ is a vertical asymptote of f if either $\lim_{x \to a^-} f(x)$ is infinite or $\lim_{x \to a^+} f(x)$ is infinite.

Example:



$x = -2$, $x = 1$ and $x = 3$ are VAs of f.

$\lim_{x \to -2^-} f(x) = \infty$, $\lim_{x \to -2^+} f(x) = -\infty$, $\lim_{x \to -2} f(x)$ DNE

$\lim_{x \to 1^-} f(x) = \infty$, $\lim_{x \to 1^+} f(x) = 2$, $\lim_{x \to 1} f(x)$ DNE

$\lim_{x \to 3^-} f(x) = -\infty$, $\lim_{x \to 3^+} f(x) = -\infty$, $\lim_{x \to 3} f(x) = -\infty$.

Remarks :

- Technically, $\lim_{x\to a} f(x) = \infty$ is a particular case of of limit that does not exist. But we always specify $\pm\infty$ when applicable. A limit exists when it is finite

- A vertical asymptote is a line, so it must always be given as an equation: " $x = 3$ " and not " 3 ".

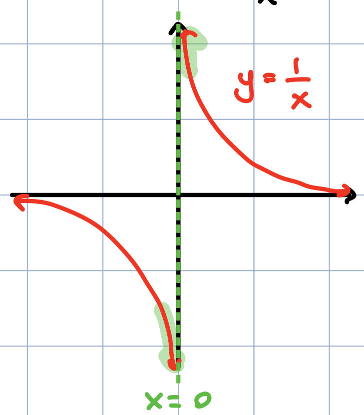
How to find infinite limits / VAs :

- If substitution at $x = a$ gives " $\frac{\text{non-zero \#}}{0}$ ", the limits as $x \to a^{\pm}$ are infinite (analyze signs on each side to determine ∞ or $-\infty$) and $x = a$ is a VA.

- If substitution at $x = a$ gives " $\frac{0}{0}$ " : indeterminate ! Do more work (algebra) to find the limit as $x \to a$.

Examples : 1) Vertical asymptotes of common functions.

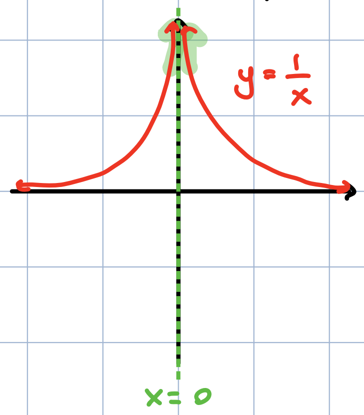
- $f(x) = \frac{1}{x}$



$\lim_{x\to 0^+} \frac{1}{x} = \infty$, $\lim_{x\to 0^-} \frac{1}{x} = -\infty$, $\lim_{x\to 0} \frac{1}{x}$ DNE

$x = 0$ is a VA of $y = \frac{1}{x}$

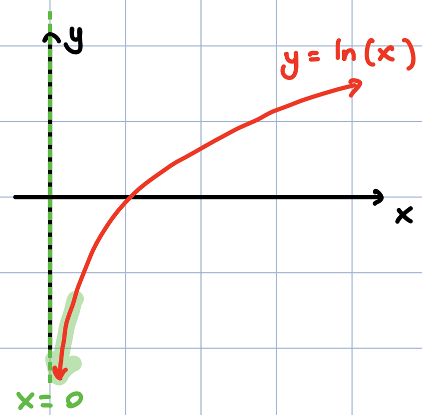
- $f(x) = \frac{1}{x^2}$



$\lim_{x \to 0^+} \frac{1}{x^2} = \infty$, $\lim_{x \to 0^-} \frac{1}{x^2} = \infty$, $\lim_{x \to 0} \frac{1}{x^2} = \infty$

$x = 0$ is a VA of $y = \frac{1}{x^2}$

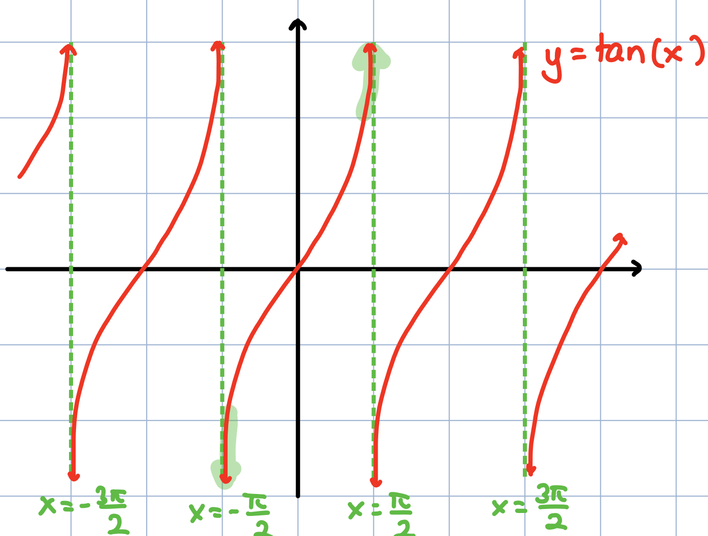
- $f(x) = \ln(x)$



$\lim_{x \to 0^+} \ln(x) = -\infty$

$x = 0$ is a VA of $y = \ln(x)$

- $f(x) = \tan(x)$



f has infinitely many VAs:

$x = \frac{\pi}{2} + n\pi$ for any integer n.

2) Find the VAs of $f(x) = \dfrac{3x^2 - 6x}{x^4 - 4x^2}$.

Potential VAs : set denominator to 0.

$$x^4 - 4x^2 = 0 \Rightarrow x^2(x^2-4) = 0 \Rightarrow x^2(x-2)(x+2) = 0$$

$$\Rightarrow x = 0, 2, -2.$$

Test each value :

- At $x = -2$ substitution gives "$\dfrac{3\cdot 4+12}{0}$" : $\boxed{x = -2 \text{ is a VA}}$

- At $x = 0$ substitution gives $\dfrac{0}{0}$: more work needed.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{3\cancel{x}\cancel{(x-2)}}{x^{\cancel{2}}\cancel{(x-2)}(x+2)}$$ factor and simplify

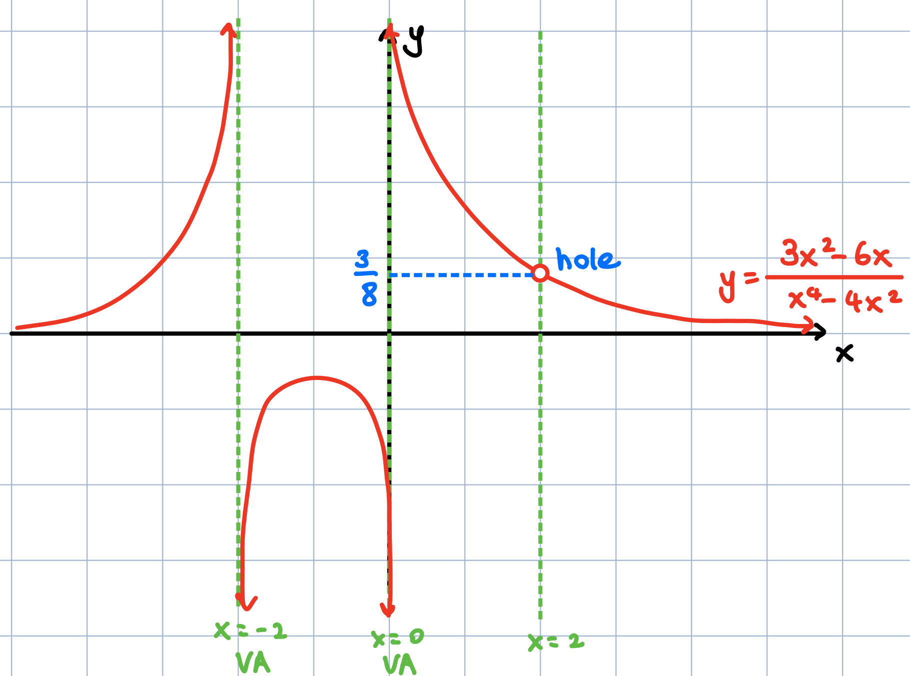
$$= \lim_{x\to 0} \frac{3}{x(x+2)} = \text{"}\frac{3}{0}\text{"}$$ so $\boxed{x = 0 \text{ is a VA.}}$

- At $x = 2$ substitution gives $\dfrac{0}{0}$: more work needed.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{3\cancel{x}\cancel{(x-2)}}{x^{\cancel{2}}\cancel{(x-2)}(x+2)}$$ factor and simplify

$$= \lim_{x\to 2} \frac{3}{x(x+2)} = \frac{3}{8}$$ so $x = 2$ is not a VA.

($x = 2$ is a hole.)



3) Find the VAs of $f(x) = \dfrac{\sin(4x)}{x^2 - 3x}$.

Potential VAs : $x^2 - 3x = 0 \Rightarrow x(x-3) = 0 \Rightarrow x=0,\ x=3.$

- At $x = 0$: substitution gives $\frac{0}{0}$: more work needed.

$$\lim_{x\to 0} \frac{\sin(4x)}{x(x-3)} \cdot \frac{4x}{4x} = \lim_{x\to 0} \frac{\sin(4x)}{4x} \cdot \frac{4\cancel{x}}{\cancel{x}(x-3)}$$

$$= \Big(\underbrace{\lim_{x\to 0} \frac{\sin(4x)}{4x}}_{=1}\Big)\Big(\underbrace{\lim_{x\to 0} \frac{4}{x-3}}_{=\frac{4}{0-3}}\Big) = 1 \cdot \frac{4}{0-3} = -\frac{4}{3} : x = 0 \text{ is not a VA.}$$

special trig limit

- At $x = 3$ substitution gives $\dfrac{\sin(12)}{0}$: $\boxed{x = 3 \text{ is a VA.}}$

4) Calculate $\displaystyle\lim_{x\to -3^-} \frac{5x}{6+2x}$ and $\displaystyle\lim_{x\to -3^+} \frac{5x}{6+2x}$.

Substitution of $x = -3$ in $\dfrac{5x}{6+2x}$ gives "$\dfrac{-15}{0}$", so we know that each one-sided limit is infinite.

- $\displaystyle\lim_{x\to -3^-} \frac{5x}{6+2x} = \text{"}\frac{-15}{0^-}\text{"} = \boxed{+\infty}$

if $x < -3$, $6+2x < 0$

- $\displaystyle\lim_{x\to -3^+} \frac{5x}{6+2x} = \text{"}\frac{-15}{0^+}\text{"} = \boxed{-\infty}$

if $x > -3$, $6+2x > 0$

Limits at infinity:

- $\lim\limits_{x \to \infty} f(x) = L$ means that $f(x)$ approaches L as x gets arbitrarily large / grows without bounds

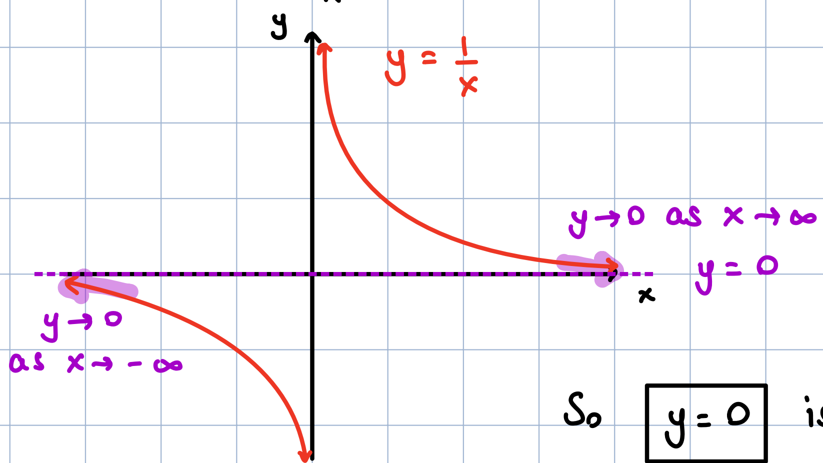
- $\lim\limits_{x \to -\infty} f(x) = L$ means that $f(x)$ approaches L as x gets arbitrarily large and negative / decreases without bounds.

Graphically, if $\lim\limits_{x \to \infty} f(x) = L$ or $\lim\limits_{x \to -\infty} f(x) = L$, the line $y = L$ is a <u>horizontal asymptote</u> of f.

Examples:

1) A function can have 0, 1 or 2 horizontal asymptotes.

- $f(x) = \frac{1}{x}$

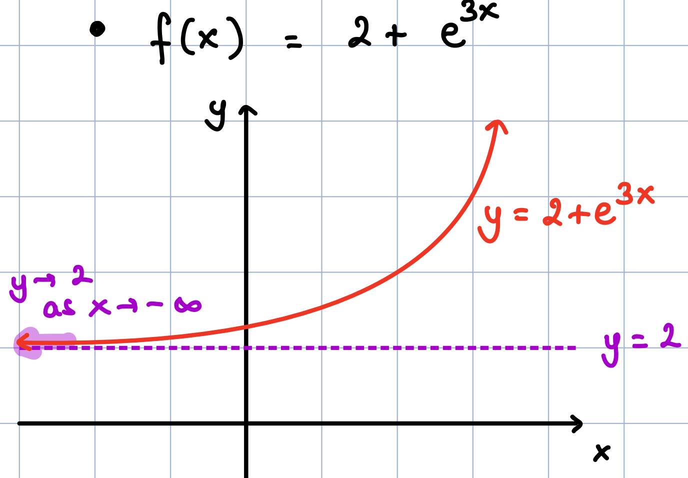


$$\boxed{\lim_{x \to \infty} \frac{1}{x} = 0 = \lim_{x \to -\infty} \frac{1}{x}}$$

(Any limit of the form "$\frac{\text{some number}}{\text{infinity}}$" is 0)

So $\boxed{y = 0}$ is the only horizontal asymptote.

- $f(x) = 2 + e^{3x}$

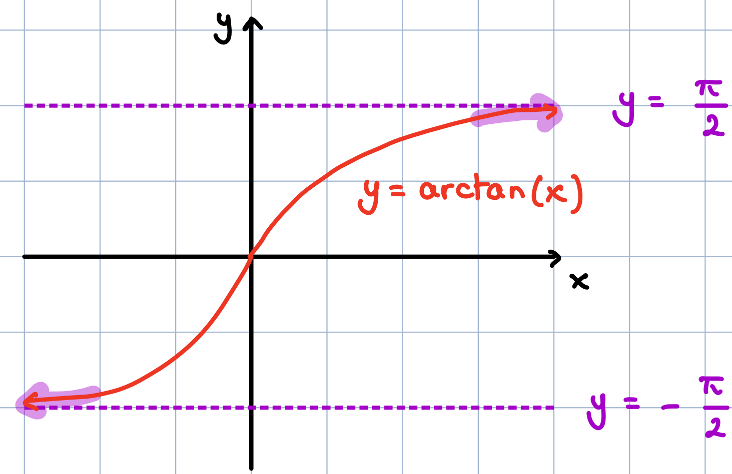


$$\lim_{x \to -\infty} (2 + e^{3x}) = 2 + 0 = 2$$

$$\lim_{x \to \infty} (2 + e^{3x}) = \infty$$

So $\boxed{y = 2}$ is the only horizontal asymptote.

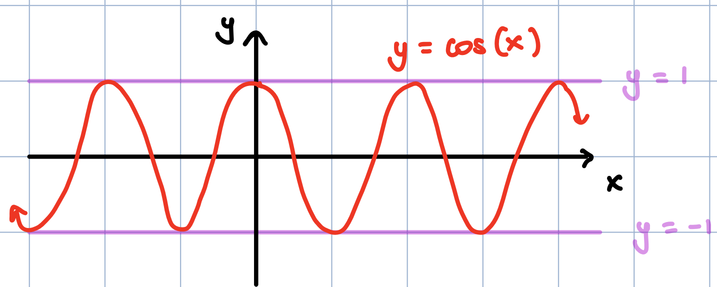
- $f(x) = \arctan(x)$



$$\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$$
$$\lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}$$

So $\boxed{y = \frac{\pi}{2}}$, $\boxed{y = -\frac{\pi}{2}}$ are the horizontal asymptotes.

- $f(x) = \cos(x)$



$\lim_{x\to\infty} \cos(x)$ DNE

$\lim_{x\to-\infty} \cos(x)$ DNE

So no horizontal asymptote.

2) Find the horizontal asymptotes of $f(x) = \dfrac{2x^4+3x^2+1}{5x^4+2x-4}$

$$\lim_{x\to\infty} \frac{2x^4+3x^2+1}{5x^4+2x-4}$$

$\frac{\infty}{\infty}$: indeterminate form, rewrite using algebra

$$= \lim_{x\to\infty} \frac{2x^4+3x^2+1}{5x^4+2x-4} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}}$$

divide by the largest power appearing in denominator

$$= \lim_{x\to\infty} \frac{2+\underbrace{\frac{3}{x^2}}_{\to 0}+\underbrace{\frac{1}{x^4}}_{\to 0}}{5+\underbrace{\frac{2}{x^3}}_{\to 0}-\underbrace{\frac{4}{x^4}}_{\to 0}} = \frac{2+0+0}{5+0-0} = \frac{2}{5}.$$

Similarly, $$\lim_{x\to\infty} \frac{2x^4+3x^2+1}{5x^4+2x-4} = \lim_{x\to-\infty} \frac{2+\underbrace{\frac{3}{x^2}}_{\to 0}+\underbrace{\frac{1}{x^4}}_{\to 0}}{5+\underbrace{\frac{2}{x^3}}_{\to 0}-\underbrace{\frac{4}{x^4}}_{\to 0}} = \frac{2}{5}.$$

So f has one horizontal asymptote: $\boxed{y = \frac{2}{5}}$.

3) Find the horizontal asymptotes of $f(x) = \dfrac{\sqrt{16x^2+1}+x}{x-2}$.

- $\displaystyle\lim_{x\to\infty} \frac{\sqrt{16x^2+1}+x}{x-2}$ $\quad \frac{\infty}{\infty}$: indeterminate form, rewrite using algebra

$$= \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{16+\frac{1}{x^2}}+x}{x-2}$$ factor out x^2 from radical

$$= \lim_{x\to\infty} \frac{|x|\sqrt{16+\frac{1}{x^2}}+x}{x-2}$$ $\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

$x > 0$
so $|x| = x$

$$= \lim_{x\to\infty} \frac{x\sqrt{16+\frac{1}{x^2}}+x}{x-2} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$ divide by highest power in denominator

$$= \lim_{x\to\infty} \frac{\sqrt{16+\overbrace{\frac{1}{x^2}}^{\to 0}}+1}{1-\underbrace{\frac{2}{x}}_{\to 0}} = \frac{\sqrt{16+0}+1}{1-0} = 5$$

- $$\lim_{x\to-\infty} \frac{\sqrt{16x^2+1}+x}{x-2} = \lim_{x\to-\infty} \frac{|x|\sqrt{16+\frac{1}{x^2}}+x}{x-2} = \lim_{x\to-\infty} \frac{-x\sqrt{16+\frac{1}{x^2}}+x}{x-2} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$x < 0$
so $|x| = -x$

$$= \lim_{x\to-\infty} \frac{-\sqrt{16+\overbrace{\frac{1}{x^2}}^{\to 0}}+1}{1-\underbrace{\frac{2}{x}}_{\to 0}} = \frac{-\sqrt{16+0}+1}{1-0} = -3.$$

So f has two horizontal asymptotes: $\boxed{y = 5}$ and $\boxed{y = -3}$.

4) Find the horizontal asymptotes of $f(x) = \dfrac{3+2e^x}{7-4e^x}$

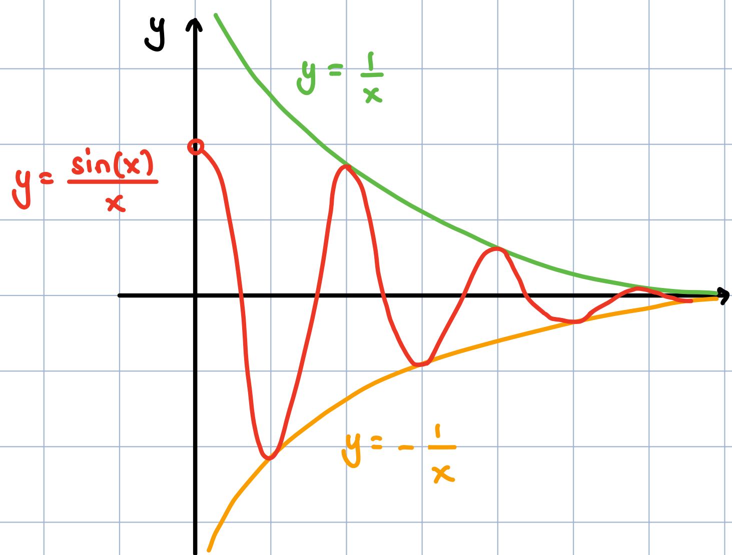
$$\lim_{x\to-\infty} \frac{3+2\overbrace{e^x}^{\to 0}}{7-4\underbrace{e^x}_{\to 0}} = \frac{3+2\cdot 0}{7-4\cdot 0} = \frac{3}{7}$$

$$\lim_{x\to\infty} \frac{3+2e^x}{7-4e^x} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim_{x\to\infty} \frac{\overbrace{\frac{3}{e^x}}^{\to 0}+2}{\underbrace{\frac{7}{e^x}}_{\to 0}-4} = \frac{2}{-4} = -\frac{1}{2}.$$

So f has two horizontal asymptotes: $\boxed{y = \frac{3}{7}}$ and $\boxed{y = -\frac{1}{2}}$.

5) Calculate $\lim\limits_{x \to \infty} \frac{\sin(x)}{x}$.

Since $-1 \leq \sin(x) \leq 1$, we have $-\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$ if $x > 0$.



We know that:

$$\lim_{x \to \infty} -\frac{1}{x} = 0 = \lim_{x \to \infty} \frac{1}{x}.$$

So by the Squeeze Theorem,

$$\boxed{\lim_{x \to \infty} \frac{\sin(x)}{x} = 0}.$$

6) Calculate $\lim\limits_{x \to \infty} \sqrt{9x^2 + 5x} - 3x$.

$\lim\limits_{x \to \infty} \sqrt{9x^2 + 5x} - 3x$ "$\infty - \infty$" : indeterminate form : more work needed

$= \lim\limits_{x \to \infty} \left(\sqrt{9x^2 + 5x} - 3x\right) \frac{\sqrt{9x^2 + 5x} + 3x}{\sqrt{9x^2 + 5x} + 3x}$ multiply by conjugate

$$= \lim_{x \to \infty} \frac{9x^2 + 5x - (3x)^2}{\sqrt{9x^2 + 5x} + 3x} = \lim_{x \to \infty} \frac{5x}{\sqrt{9x^2 + 5x} + 3x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x \to \infty} \frac{5}{\sqrt{9 + 5/x} + 3} = \frac{5}{\sqrt{9 + 0} + 3} = \boxed{\frac{5}{6}}.$$

7) Does $f(x) = \dfrac{x^2-1}{2x+4}$ have any horizontal asymptotes?

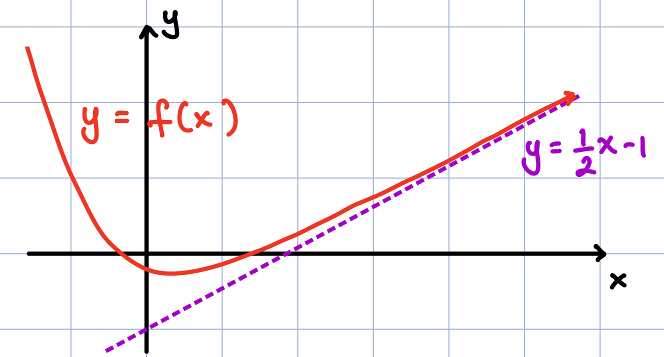
- $\lim\limits_{x\to\infty} \dfrac{x^2-1}{2x+4} \cdot \dfrac{\frac{1}{x}}{\frac{1}{x}}$ divide by highest power in denominator

$$= \lim_{x\to\infty} \frac{x - \frac{1}{x}}{2 + \frac{4}{x}} = \text{"}\frac{\infty}{2}\text{"} = \infty$$

- $\lim\limits_{x\to -\infty} \dfrac{x - \frac{1}{x}}{2 + \frac{4}{x}} = \text{"}\dfrac{-\infty}{2}\text{"} = -\infty.$

no horizontal asymptotes.

However, observe that $\dfrac{x^2-1}{2x+4} \overset{\text{long division}}{=} \dfrac{1}{2}x - 1 + \dfrac{3}{2x+4}$. $\to 0$ as $x \to \pm\infty$



So $\lim\limits_{x\to\pm\infty} f(x) - \left(\frac{1}{2}x - 1\right) = 0.$

We say that $y = \frac{1}{2}x - 1$ is an <u>oblique asymptote</u> of f

<u>In general</u>:

- We say that $y = mx + b$ is an <u>oblique asymptote</u> of f if $\lim\limits_{x\to\infty} f(x) - (mx+b) = 0$ or $\lim\limits_{x\to-\infty} f(x) - (mx+b) = 0.$

- A rational function has an oblique asymptote when the degree of the numerator is one more than the degree of the denominator.

- To find the oblique asymptote, perform the long division of numerator by denominator. The oblique asymptote is the quotient.