Rutgers University
Math 151

Section 2.6: Limits Involving Infinity - Worksheet

1. Evaluate the following limits. If a limit does not exist, explain why. If a limit is infinite, specify it and determine if it is ∞ or $-\infty$.

   (a) $\displaystyle\lim_{x\to -1^-} \frac{x^2+3x+2}{(x+1)^2}$.

   Solution. Substitution gives $\frac{0}{0}$, so we need more analysis. We have

$$\lim_{x\to -1^-} \frac{x^2+3x+2}{(x+1)^2} = \lim_{x\to -1^-} \frac{(x+1)(x+2)}{(x+1)^2} = \lim_{x\to -1^-} \frac{x+2}{x+1}.$$

   In this simplified form, substitution gives $\frac{1}{0}$ so the one-sided limit is infinite. To determine if the limit is ∞ or $-\infty$, we use a sign analysis. As $x \to -1^-$, $x+2>0$ and $x+1<0$, so $\frac{x+2}{x+1}<0$. Therefore

$$\boxed{\lim_{x\to -1^-} \frac{x^2+3x+2}{(x+1)^2} = -\infty}.$$

   (b) $\displaystyle\lim_{x\to\infty} \frac{3x\sqrt{x}+2}{\sqrt{4x^3+1}}$.

   Solution. Observe that $x\sqrt{x}=\sqrt{x^3}$ when $x>0$. So

$$\begin{aligned}\lim_{x\to\infty} \frac{3x\sqrt{x}+2}{\sqrt{4x^3+1}} &= \lim_{x\to\infty} \frac{3x\sqrt{x}+2}{\sqrt{4x^3+1}} \cdot \frac{\frac{1}{\sqrt{x^3}}}{\frac{1}{\sqrt{x^3}}} \\ &= \lim_{x\to\infty} \frac{3+\frac{2}{\sqrt{x^3}}}{\sqrt{4+\frac{1}{x^3}}} \\ &= \frac{3+0}{\sqrt{4+0}} \\ &= \boxed{\frac{3}{2}}.\end{aligned}$$

   (c) $\displaystyle\lim_{x\to 2\pi} \frac{x}{\cos(x)-1}$.

   Solution. Substitution gives $\frac{2\pi}{0}$, so both one-sided limits are infinite. We need a sign analysis to determine if the limit is ∞ or $-\infty$ on each side. Observe that $\cos(x)-1 \leqslant 0$ for all x since the range of cos is $[-1,1]$. It follows that as $x\to 2\pi^+$ and $x\to 2\pi^-$, the values of $\frac{x}{\cos(x)-1}$ are negative (positive numerator and negative denominator). Hence,

$$\boxed{\lim_{x\to 2\pi} \frac{x}{\cos(x)-1} = -\infty}.$$

(d) $\lim_{x\to 2} \dfrac{x-5}{x^2-2x}$.

Solution. Substitution gives $\frac{-3}{0}$, so both one-sided limits are infinite. We need a sign analysis to determine if the limit is ∞ or $-\infty$ on each side. When $x \to 2^-$, we have $0 < x < 2$ so $x^2 - 2x = x(x-2) < 0$ and $\frac{x-5}{x(x-2)} > 0$. Therefore,

$$\lim_{x\to 2^-} \frac{x-5}{x^2-2x} = \infty.$$

When $x \to 2^+$, we have $0x > 2$ so $x^2 - 2x = x(x-2) > 0$ and $\frac{x-5}{x(x-2)} < 0$. Therefore,

$$\lim_{x\to 2^+} \frac{x-5}{x^2-2x} = -\infty.$$

We conclude that $\boxed{\lim_{x\to 2} \dfrac{x-5}{x^2-2x} \text{ does not exist}}$.

(e) $\lim_{x\to -\infty} \dfrac{x^3+2}{\sqrt{16x^6+1}}$.

Solution. Observe that $\sqrt{x^6} = |x^3| = -x^3$ when $x < 0$. So

$$\begin{aligned}
\lim_{x\to -\infty} \frac{x^3+2}{\sqrt{16x^6+1}} &= \lim_{x\to -\infty} \frac{x^3+2}{\sqrt{x^6\left(16+\frac{1}{x^6}\right)}} \\
&= \lim_{x\to -\infty} \frac{x^3+2}{-x^3\sqrt{16+\frac{1}{x^6}}} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \\
&= \lim_{x\to -\infty} \frac{1+\frac{2}{x^3}}{-\sqrt{16+\frac{1}{x^6}}} \\
&= \frac{1+0}{-\sqrt{16+0}} \\
&= \boxed{-\frac{1}{4}}.
\end{aligned}$$

(f) $\lim_{t\to\infty} \sqrt{9t^2+8t} - \sqrt{9t^2-5t}$.

Solution.

$$\begin{aligned}
\lim_{t\to\infty} \sqrt{9t^2+8t} - \sqrt{9t^2-5t} &= \lim_{t\to\infty} \left(\sqrt{9t^2+8t} - \sqrt{9t^2-5t}\right) \frac{\sqrt{9t^2+8t} + \sqrt{9t^2-5t}}{\sqrt{9t^2+8t} + \sqrt{9t^2-5t}} \\
&= \lim_{t\to\infty} \frac{\left(\sqrt{9t^2+8t}\right)^2 - \left(\sqrt{9t^2-5t}\right)^2}{\sqrt{9t^2+8t} + \sqrt{9t^2-5t}} \\
&= \lim_{t\to\infty} \frac{9t^2+8t-\left(9t^2-5t\right)}{\sqrt{9t^2+8t} + \sqrt{9t^2-5t}} \\
&= \lim_{t\to\infty} \frac{13t}{\sqrt{9t^2+8t} + \sqrt{9t^2-5t}} \cdot \frac{\frac{1}{t}}{\frac{1}{t}}
\end{aligned}$$

$$= \lim_{t\to\infty} \frac{13}{\sqrt{9+\frac{8}{t}} + \sqrt{9-\frac{5}{t}}}$$

$$= \frac{13}{\sqrt{9+0}+\sqrt{9+0}}$$

$$= \boxed{\frac{13}{6}}.$$

[Advanced]

(a) $\displaystyle\lim_{\theta\to-\infty} \frac{2\theta + 5\sin(3\theta)}{7\theta}$.

Solution. Observe that

$$\frac{2\theta+5\sin(3\theta)}{7\theta} = \frac{2}{7} + \frac{5\sin(3\theta)}{7\theta}.$$

Since $-1 \leqslant \sin(3\theta) \leqslant 1$, we have $-5 \leqslant 5\sin(3\theta) \leqslant 5$ and $-\frac{5}{7\theta} \leqslant \frac{5\sin(3\theta)}{7\theta} \leqslant \frac{5}{7\theta}$. Additionally, we have

$$\lim_{\theta\to-\infty} -\frac{5}{7\theta} = \lim_{\theta\to-\infty} \frac{5}{7\theta} = 0.$$

So by the Squeeze Theorem, $\lim_{\theta\to-\infty} \frac{5\sin(3\theta)}{7\theta} = 0$. Therefore

$$\lim_{\theta\to-\infty} \frac{2\theta+5\sin(3\theta)}{7\theta} = \lim_{\theta\to-\infty} \frac{2}{7} + \frac{5\sin(3\theta)}{7\theta} = \frac{2}{7} + 0 = \boxed{\frac{2}{7}}.$$

(b) $\displaystyle\lim_{x\to0^+} \left(\frac{1}{\sqrt[3]{x}} - \frac{1}{\sqrt{x}}\right)$.

Solution. For $x > 0$, we have

$$\frac{1}{\sqrt[3]{x}} - \frac{1}{\sqrt{x}} = \frac{1}{x^{1/3}} - \frac{1}{x^{1/2}} = \frac{x^{1/6}}{x^{1/3+1/6}} - \frac{1}{x^{1/2}} = \frac{x^{1/6}}{x^{1/2}} - \frac{1}{x^{1/2}} = \frac{x^{1/6}-1}{x^{1/2}}.$$

Substituting $x = 0$ in this expression would give $\frac{-1}{0}$, so we know that the one-sided limit is infinite. To determine if the limit is ∞ or $-\infty$, we look at the sign of the expression. For $x > 0$, $\sqrt{x} > 0$. As $x \to 0^+$, x is close to 0 so $x^{1/6} - 1 < 0$. It follows that $\dfrac{x^{1/6}-1}{x^{1/2}} < 0$ and

$$\boxed{\lim_{x\to0^+} \left(\frac{1}{\sqrt[3]{x}} - \frac{1}{\sqrt{x}}\right) = -\infty}.$$

(c) $\displaystyle\lim_{t\to\infty} \frac{t\arctan(3t)}{\sqrt{t^2+1}}$.

Solution.

$$\lim_{t\to\infty} \frac{t\arctan(3t)}{\sqrt{t^2+1}} = \lim_{t\to\infty} \frac{t\arctan(3t)}{\sqrt{t^2+1}} \cdot \frac{\frac{1}{t}}{\frac{1}{t}}$$

$$= \lim_{t\to\infty} \frac{\arctan(3t)}{\sqrt{1+\frac{1}{t^2}}}$$
$$= \frac{\frac{\pi}{2}}{\sqrt{1+0}}$$
$$= \boxed{\frac{\pi}{2}}.$$

2. Find the vertical and horizontal asymptotes of the following functions, if any. Also, determine the limit to the left and right of any vertical asymptote.

   (a) $f(x) = \dfrac{x^2-3x-4}{\sqrt{x}-2}$.

   Solution. To find potential vertical asymptotes, we set the denominator equal to 0. This gives $\sqrt{x}-2=0$, that is $x=4$. Substituting 4 in $f(x)$ gives $\frac{0}{0}$, so we need to do more analysis to determine if $x=4$ is indeed a vertical asymptote. The limit at 4 is

   $$\lim_{x\to4} \frac{x^2-3x-4}{\sqrt{x}-2}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to4}\frac{(x-4)(x+1)(\sqrt{x}+2)}{x-4} = \lim_{x\to4}(x+1)(\sqrt{x}+2) = (4+1)(\sqrt{4}+2)20.$$

   Since the limit as $x\to4$ is finite, $x=4$ is not a vertical asymptote. So $\boxed{f \text{ has no vertical asymptote}}$.

   To find the horizontal asymptotes, we compute the limits at ∞ and $-\infty$. Note that f is undefined for $x<0$, so only the limit at ∞ makes sense. We have

   $$\begin{aligned}\lim_{x\to\infty}\frac{x^2-3x-4}{\sqrt{x}-2} &= \lim_{x\to\infty}\frac{x^2-3x-4}{\sqrt{x}-2}\cdot\frac{\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x}}}\\ &= \lim_{x\to\infty}\frac{x^{3/2}-3x^{1/2}-\frac{4}{\sqrt{x}}}{1-\frac{2}{\sqrt{x}}}\\ &= \frac{+\infty}{1}\\ &= \infty.\end{aligned}$$

   Therefore, $\boxed{f \text{ has no horizontal asymptote}}$.

   (b) $f(x) = \dfrac{x^2-1}{|x+1|^3}$.

   Solution. To find potential vertical asymptotes, we set the denominator equal to 0. This gives $|x+1|^3=0$, that is $x=-1$. Substituting -1 in $f(x)$ gives $\frac{0}{0}$, so we need to do more analysis to determine if $x=-1$ is indeed a vertical asymptote. The left and right limit at -1 are

   $$\lim_{x\to-1^+}\frac{x^2-1}{|x+1|^3} = \lim_{x\to-1^+}\frac{(x-1)(x+1)}{(x+1)^3} = \lim_{x\to-1^+}\frac{(x-1)}{(x+1)^2} = -\infty,$$
   $$\lim_{x\to-1^-}\frac{x^2-1}{|x+1|^3} = \lim_{x\to-1^-}\frac{(x-1)(x+1)}{-(x+1)^3} = \lim_{x\to-1^-} -\frac{(x-1)}{(x+1)^2} = \infty.$$

   So $\boxed{x=-1 \text{ is the one vertical asymptote of } f}$.

To find the horizontal asymptotes of f, we compute the limits at ∞ and $-\infty$. We have

$$\lim_{x\to\infty} \frac{x^2-1}{|x+1|^3} = \lim_{x\to\infty} \frac{x^2-1}{(x+1)^3}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{1}{x^3}}{\left(1+\frac{1}{x}\right)^3} = \frac{0-0}{(1+0)^3} = 0,$$

$$\lim_{x\to-\infty} \frac{x^2-1}{|x+1|^3} = \lim_{x\to-\infty} \frac{x^2-1}{-(x+1)^3}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to-\infty} -\frac{\frac{1}{x}-\frac{1}{x^3}}{\left(1+\frac{1}{x}\right)^3} = -\frac{0-0}{(1+0)^3} = 0.$$

So $\boxed{y=0 \text{ is the one horizontal asymptote of } f}$.

(c) $f(x) = \dfrac{7+2e^x}{5e^x-4}$.

Solution. To find potential vertical asymptotes, we set the denominator equal to 0. This gives $5e^x-4=0$, that is $e^x=\frac{4}{5}$, so $x=\ln\left(\frac{4}{5}\right)$. Substituting this value in $f(x)$ gives $\frac{7+2\cdot\frac{4}{5}}{0}$. This has the form $\frac{\text{non-zero number}}{0}$, so $\boxed{x=\ln\left(\frac{4}{5}\right) \text{ is the one vertical asymptote of } f}$.

To find the horizontal asymptotes of f, we compute the limits at ∞ and $-\infty$. Recall that $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$. Therefore

$$\lim_{x\to-\infty} \frac{7+2e^x}{5e^x-4} = \frac{7+2\cdot 0}{5\cdot 0-4} = -\frac{7}{4},$$

$$\lim_{x\to\infty} \frac{7+2e^x}{5e^x-4}\cdot\frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim_{x\to\infty} \frac{\frac{7}{e^x}+2}{5-\frac{4}{e^x}} = \frac{0+2}{5-0} = \frac{2}{5}.$$

So $\boxed{y=-\frac{7}{4} \text{ and } y=\frac{2}{5} \text{ are the two horizontal asymptotes of } f}$.

(d) $f(x) = \dfrac{\sqrt{x^2+25}+3x}{2x+5}$.

Solution. To find potential vertical asymptotes, we set the denominator equal to 0. This gives $2x+5=0$, or $x=-\frac{2}{5}$. Substituting this value in $f(x)$ gives the form $\frac{\text{non-zero number}}{0}$. It follows that $\boxed{x=-\frac{2}{5} \text{ is the one vertical asymptote of } f}$.

To find the horizontal asymptotes of f, we calculate the limits at ∞ and $-\infty$. We have

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{x^2+25}+3x}{2x+5} &= \lim_{x\to-\infty} \frac{\sqrt{x^2\left(1+\frac{25}{x^2}\right)}+3x}{2x+5} \\
&= \lim_{x\to-\infty} \frac{|x|\sqrt{1+\frac{25}{x^2}}+3x}{2x+5} \\
&= \lim_{x\to-\infty} \frac{-x\sqrt{1+\frac{25}{x^2}}+3x}{2x+5}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} \quad (x<0) \\
&= \lim_{x\to-\infty} \frac{-\sqrt{1+\frac{25}{x^2}}+3}{2+\frac{5}{x}}
\end{aligned}$$

$$= \frac{-\sqrt{1+0}+3}{2+0}$$
$$= 1,$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{x^2+25}+3x}{2x+5} &= \lim_{x\to\infty} \frac{\sqrt{x^2\left(1+\frac{25}{x^2}\right)}+3x}{2x+5} \\
&= \lim_{x\to\infty} \frac{|x|\sqrt{1+\frac{25}{x^2}}+3x}{2x+5} \\
&= \lim_{x\to\infty} \frac{x\sqrt{1+\frac{25}{x^2}}+3x}{2x+5} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \quad (x>0) \\
&= \lim_{x\to\infty} \frac{\sqrt{1+\frac{25}{x^2}}+3}{2+\frac{5}{x}} \\
&= \frac{\sqrt{1+0}+3}{2+0} \\
&= 2.
\end{aligned}$$

So $\boxed{y = 1 \text{ and } y = 2 \text{ are the two horizontal asymptotes of } f}$.

(e) $f(x) = \dfrac{\sin(7x)}{x^2+3x}$.

Solution. To find potential vertical asymptotes, we set the denominator equal to 0. This gives $x(x+3) = 0$, or $x = -3, 0$. Substituting $x = -3$ in $f(x)$ gives $\frac{\sin(-21)}{0}$, which has the form $\frac{\text{non-zero number}}{0}$, so $x = -3$ is indeed a vertical asymptote of f. Subsituting $x = 0$ gives $\frac{0}{0}$, so we need more analysis to determine whether $x = 0$ is a vertical asymptote or not. We have

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(7x)}{x^2+3x} &= \lim_{x\to 0} \frac{\sin(7x)}{x(x+3)} \cdot \frac{7x}{7x} \\
&= \lim_{x\to 0} \frac{\sin(7x)}{7x} \cdot \frac{7x}{x(x+3)} \\
&= \left(\lim_{x\to 0} \frac{\sin(7x)}{7x}\right)\left(\lim_{x\to 0} \frac{7}{x+3}\right) \\
&= 1 \cdot \frac{7}{3} \\
&= \frac{7}{3},
\end{aligned}$$

so $x = 0$ is not a vertical asymptote of f. In conclusion $\boxed{x = -3 \text{ is the one vertical asymptote of } f}$.

To find the horizontal asymptotes of f, we calculate the limits at ∞ and $-\infty$. We have $-1 \leqslant \sin(7x) \leqslant 1$ so

$$-\frac{1}{x^2+3x} \leqslant \frac{\sin(7x)}{x^2+3x} \leqslant \frac{1}{x^2+3x}.$$

Additionally, $\lim\limits_{x\to\pm\infty} -\frac{1}{x^2+3x} = \lim\limits_{x\to\pm\infty} -\frac{1}{x^2+3x} = 0$. So by the Squeeze Theorem, $\lim\limits_{x\to\pm\infty} \frac{\sin(7x)}{x^2+3x} = 0$. In conclusion, $\boxed{y = 0 \text{ is the one horizontal asymptote of } f}$.

(f) $f(x) = x^2 \cos\left(\dfrac{2}{x}\right)$.

Solution. The function f is continuous on its domain, that is $(-\infty, 0) \cup (0, \infty)$. Therefore, the only potential vertical asymptote is $x = 0$. For any $x \neq 0$, we have

$$-x^2 \leqslant x^2 \cos\left(\frac{2}{x}\right) \leqslant x^2,$$

and $\lim\limits_{x\to 0} -x^2 = \lim\limits_{x\to 0} x^2 = 0$. So by the Squeeze Theorem, $\lim\limits_{x\to 0} x^2 \cos\left(\frac{2}{x}\right) = 0$. Hence, $\boxed{f \text{ has no vertical asymptote}}$.

To find the horizontal asymptotes, we must compute the limits at ∞ and $-\infty$. We have

$$\lim_{x\to -\infty} x^2 \cos\left(\frac{2}{x}\right) = \infty \cdot \cos(0) = \infty \cdot 1 = \infty,$$
$$\lim_{x\to \infty} x^2 \cos\left(\frac{2}{x}\right) = \infty \cdot \cos(0) = \infty \cdot 1 = \infty.$$

So $\boxed{f \text{ has no horizontal asymptote}}$.

[Advanced]

(g) $f(x) = \dfrac{3x\arctan(x) + 7}{x - 1}$.

Solution. To find potential vertical asymptotes, we set the denominator equal to 0. This gives $x - 1 = 0$, or $x = 1$. Substituting $x = 1$ in $f(x)$ gives $\frac{\frac{3\pi}{4}+7}{0}$. This has the form $\frac{\text{non-zero number}}{0}$, so $\boxed{x = 1 \text{ is the one vertical asymptote of } f}$.

To find the horizontal asymptotes, we must compute the limits at ∞ and $-\infty$. Recall that $\lim\limits_{x\to -\infty} \arctan(x) = -\frac{\pi}{2}$ and $\lim\limits_{x\to\infty} \arctan(x) = \frac{\pi}{2}$, so we have

$$\lim_{x\to -\infty} \frac{3x\arctan(x) + 7}{x - 1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to -\infty} \frac{3\arctan(x) + \frac{7}{x}}{1 - \frac{1}{x}} = \frac{3 \cdot \left(-\frac{\pi}{2}\right) + 0}{1 - 0} = -\frac{3\pi}{2},$$
$$\lim_{x\to \infty} \frac{3x\arctan(x) + 7}{x - 1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to \infty} \frac{3\arctan(x) + \frac{7}{x}}{1 - \frac{1}{x}} = \frac{3 \cdot \frac{\pi}{2} + 0}{1 - 0} = \frac{3\pi}{2}.$$

Hence, $\boxed{y = -\dfrac{3\pi}{2} \text{ and } y = \dfrac{3\pi}{2} \text{ are the two horizontal asymptotes of } f}$.

(h) $f(x) = \dfrac{3e^{2x} - 5e^{-x}}{2e^{-x} + e^{4x}}$.

Solution. Since the denominator of $f(x)$ is positive for any value of x, the function f is continuous on $\mathbb{R}$. Hence, $\boxed{f \text{ has no vertical asymptote}}$.

To find the horizontal asymptotes, we must compute the limits at ∞ and $-\infty$. We have

$$\lim_{x\to -\infty} \frac{3e^{2x} - 5e^{-x}}{2e^{-x} + e^{4x}} \cdot \frac{\frac{1}{e^{-x}}}{\frac{1}{e^{-x}}} = \lim_{x\to -\infty} \frac{3e^{3x} - 5}{2 + e^{5x}} = \frac{3 \cdot 0 - 5}{2 + 0} = -\frac{5}{2},$$

$$\lim_{x\to\infty} \frac{3e^{2x} - 5e^{-x}}{2e^{-x} + e^{4x}} \cdot \frac{\frac{1}{e^{4x}}}{\frac{1}{e^{4x}}} = \lim_{x\to-\infty} \frac{3e^{-2x} - 5e^{-5x}}{2e^{-5x} + 1} = \frac{3 \cdot 0 - 5 \cdot 0}{2 \cdot 0 + 1} = 0.$$

Hence, $\boxed{y = -\frac{5}{2} \text{ and } y = 0 \text{ are the two horizontal asymptotes of } f}$.

(i) $f(x) = \dfrac{1 - \cos(5x)}{x^2 + x^3}$.

Solution. To find potential vertical asymptotes, we set the denominator equal to 0. This gives $x^2(x+1) = 0$, or $x = -1, 0$. Subsituting $x = -1$ in $f(x)$ gives the form $\frac{\text{non-zero number}}{0}$, so $x = -1$ is a vertical asymptote of f. Subsituting $x = 0$ gives $\frac{0}{0}$, so we need more analysis. We have

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos(5x)}{x^2+x^3} &= \lim_{x\to 0} \frac{1-\cos(5x)}{x^2(x+1)} \cdot \frac{(5x)^2}{(5x)^2} \\
&= \lim_{x\to 0} \frac{1-\cos(5x)}{(5x)^2} \cdot \frac{(5x)^2}{x^2(x+1)} \\
&= \left(\lim_{x\to 0} \frac{1-\cos(5x)}{(5x)^2}\right)\left(\lim_{x\to 0} \frac{25}{x+1}\right) \\
&= \frac{1}{2} \cdot \frac{25}{0+1} \\
&= \frac{25}{2}.
\end{aligned}$$

So $x = 0$ is not a vertical asymptote of f. It follows that $\boxed{x = -1 \text{ is the one vertical asymptote of } f}$.

To find the horizontal asymptotes, we must compute the limits at ∞ and $-\infty$. We have $-1 \leqslant \cos(5x) \leqslant 1$, so $0 \leqslant 1 - \cos(5x) \leqslant 2$ and

$$0 \leqslant \frac{1-\cos(5x)}{x^2+x^3} \leqslant \frac{2}{x^2+x^3}.$$

Also, $\lim_{x\to\pm\infty} \frac{2}{x^2+x^3} = \lim_{x\to\pm\infty} 0 = 0$. By the Squeeze Theorem, it follows that $\lim_{x\to\pm\infty} \frac{1-\cos(5x)}{x^2+x^3} = 0$. So $\boxed{y = 0 \text{ is the one horizontal asymptote of } f}$.