Sections 3.1, 3.2

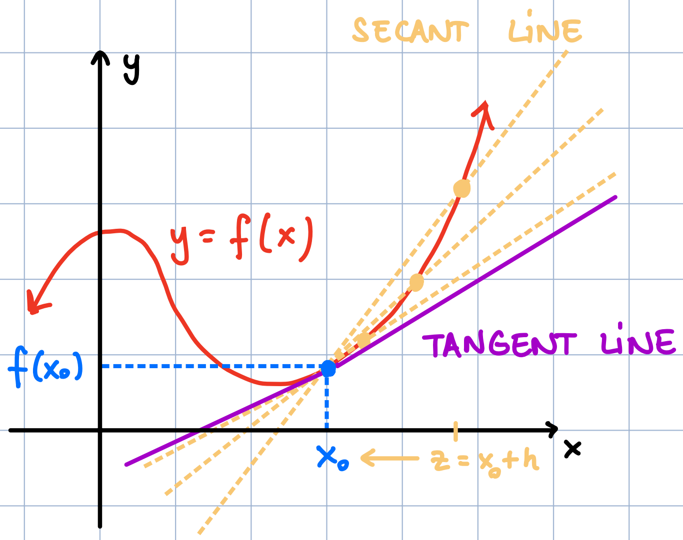
Derivatives and Tangent Lines

Learning Goals

Learning Goal	*Homework Problems*
3.1.1 Estimate a derivative by visual inspection of a graph.	1-4, 23-28, 39-48.
3.1.2 Compute a derivative as a limit of a difference quotient. Recognize a function as non-differentiable at a point when this limit does not exist.	5-36, 39-48.
3.1.3 Interpret the derivative as the slope of a graph or of a tangent line. Find the equation of a line tangent to a graph using the derivative at a point.	5-22, 27, 28, 33-36.
3.1.4 Interpret the derivative as an instantaneous rate of change.	23, 24, 29-32.

Learning Goal	*Homework Problems*
3.2.1 Understand Leibniz notation for derivatives.	7-12, 19-22.
3.2.2 Compute the derivative of a function using limits.	1-26, 37-42, 51-58, 60.
3.2.3 Understand how the derivative of a function f relates to the graph of f. Recognize a function as non-differentiable at a point based on the behavior of its graph.	27-32, 34-36, 45-54, 63.
3.2.4 Compute the one-sided derivative of a function at a point using one-sided limits. Use one-sided limits and continuity properties to determine whether a function is differentiable at a certain point.	37-42, 45-50, 63.
3.2.5 Answer conceptual questions involving differentiation.	55-60, 63.

Motivation: calculate the instantaneous rate of change of a function at a point.



Slope of secant line:

$$m_{sec} = \frac{f(x_0+h) - f(x_0)}{h}$$

↳ average rate of change on $[x_0, x_0+h]$.

Slope of tangent line at $x = x_0$:

$$\boxed{m_{tan} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

This is the instantaneous rate of change of f at x_0.

Definition: the derivative of f at x_0 is the number

$$\boxed{f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{z \to x_0} \frac{f(z) - f(x_0)}{z - z_0}}$$

We say that f is differentiable at x_0 if $f'(x_0)$ exists.

The tangent line of f at x_0 is the line passing through $(x_0, f(x_0))$ and having slope $f'(x_0)$. It has equation

$$\boxed{\begin{aligned} y - f(x_0) &= f'(x_0)(x - x_0) \\ \text{or} \quad y &= f'(x_0)(x - x_0) + f(x_0) \end{aligned}}$$

The function $y = f'(x)$ is called the derivative of f.

Interpretation:

> $f'(x_0)$ is the instantaneous rate of change / slope of the graph of f at x_0

Other notation for derivatives: if $y = f(x)$:

$$f' = \underbrace{\frac{df}{dx} = \frac{dy}{dx}}_{\text{a function}}$$

$$\underbrace{f'(x_0) = \left.\frac{df}{dx}\right|_{x=x_0} = \left.\frac{dy}{dx}\right|_{x=x_0}}_{\text{the value of } f' \text{ at } x_0 \text{: a number representing the slope of } y=f(x) \text{ at } x=x_0}$$

Examples: 1) For $f(x) = x^2 + 3x - 1$, find $f'(1)$ and an equation of the tangent line to the graph of f at $x = 1$.

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2 + 3(1+h) - 1 - 3}{h}$$

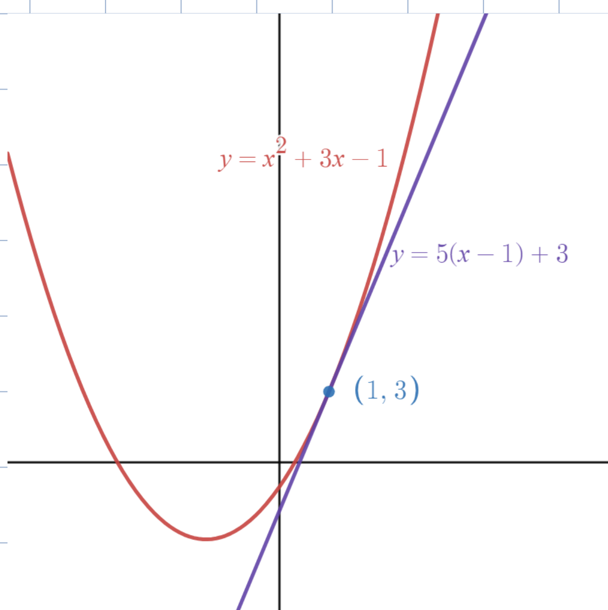
$$= \lim_{h\to 0} \frac{\cancel{1} + 2h + h^2 + \cancel{3} + 3h \cancel{-1} - \cancel{3}}{h} = \lim_{h\to 0} \frac{5h + h^2}{h} = \lim_{h\to 0} 5 + h = \boxed{5}.$$

Tangent line at $x = 1$:

- has slope $f'(1) = 5$
- passes through $(1, f(1)) = (1, 3)$

$y - 3 = 5(x-1)$

or $\boxed{y = 5(x-1) + 3}$.



2) For $f(x) = \frac{6}{x-1}$, find $f'(4)$ and equations of the tangent and normal lines to the graph of f at $x = 4$.

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{\frac{6}{4+h-1} - \frac{6}{4-1}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{6}{3+h} - 2}{h} = \lim_{h \to 0} \frac{6 - 2(3+h)}{h(3+h)} = \lim_{h \to 0} \frac{6 - 6 - 2h}{h(3+h)}$$

$$= \lim_{h \to 0} \frac{-2\cancel{h}}{\cancel{h}(3+h)} = \lim_{h \to 0} -\frac{2}{3+h} = -\frac{2}{3}$$

Tangent line at $x = 4$:

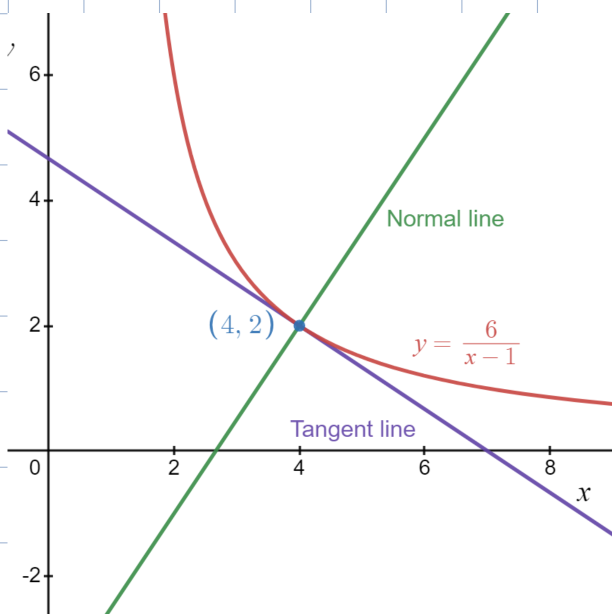
- has slope $f'(4) = -\frac{2}{3}$
- passes through $(4, f(4)) = (4, 2)$

$y - 2 = -\frac{2}{3}(x-4)$ or $\boxed{y = -\frac{2}{3}(x-4) + 2}$

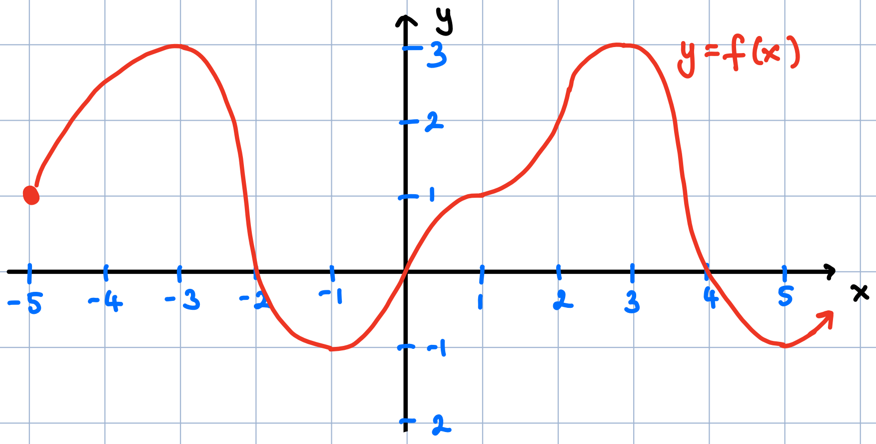
Normal line at $x = 4$ is perpendicular to the tangent line, so:

- has slope $-\frac{1}{f'(4)} = \frac{3}{2}$
- passes through $(4, f(4)) = (4, 2)$

$\boxed{y = \frac{3}{2}(x-4) + 2}$.



2) The graph of f is given below.

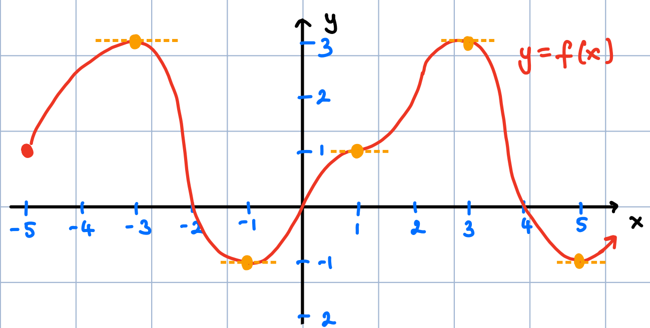


a) Solve $f'(x) = 0$

b) Solve $f'(x) > 0$ and $f'(x) < 0$.

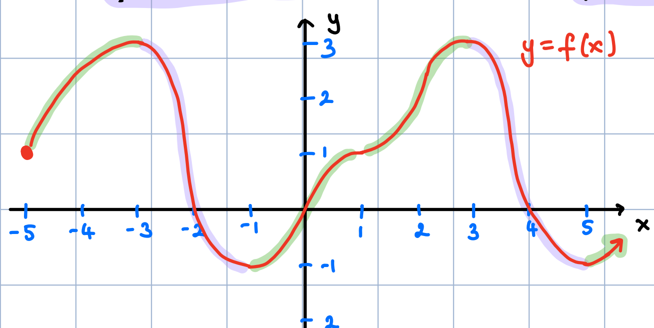
c) Which is largest/least?
$f'(-4)$, $f'(2)$, $f'(-2.5)$

a) $f'(x) = 0$ where the tangent line is horizontal, i.e. the graph of f is flat.



This happens at $\boxed{x = -3, -1, 1, 3, 5}$.

b) $f'(x) > 0$ means f has positive slope and goes upward.
$f'(x) < 0$ means f has negative slope and goes downward.



$f'(x) > 0$: $(-5,-3), (-1,1), (1,3), (5,\infty)$

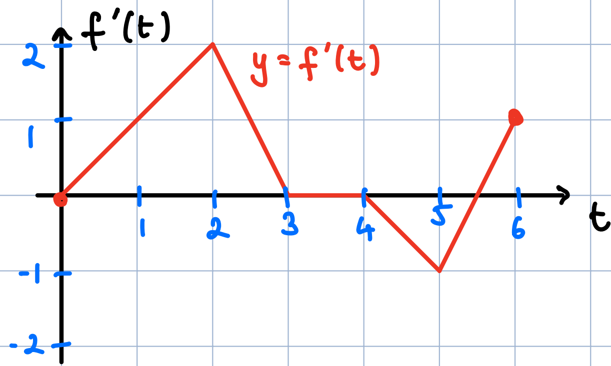
$f'(x) < 0$: $(-3,-1), (3,5)$.

c) The graph of f is steeper at $x = 2$ than it is at $x = -4$ so $f'(-4) < f'(2)$.
Graph has negative slope at -2.5 so $\boxed{f'(-2.5) < 0 < f'(-4) < f'(2)}$.

3) Suppose that $f(t)$ = position of an object moving along an axis in ft, t in sec.

$f'(t)$ = instantaneous velocity of the object in ft / sec.

Suppose f' is graphed below.



The object <u>moves forward</u> when $f'(t) > 0$: $(0,3), (5.5,6)$

<u>Moves backward</u> when $f'(t) < 0$: $(4, 5.5)$.

The object is <u>stopped</u> when $f'(t) = 0$: $(3,4)$ and $t = 5.5$.

The object <u>accelerates</u> when f' is increasing: $(0,2), (5,6)$.

The object <u>decelerates</u> when f' is decreasing: $(2,3), (4,5)$.

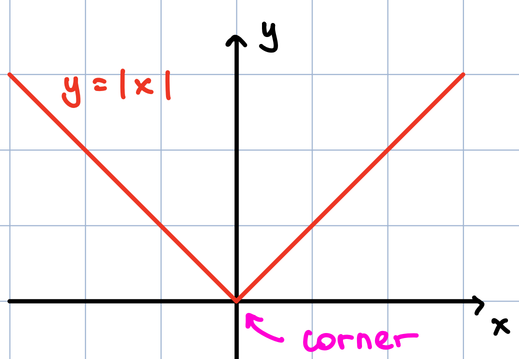
4) For $f(x) = |x|$, calculate $f'(-2)$ and $f'(0)$.

$-(-2+h) = 2-h$

$$f'(-2) = \lim_{h \to 0} \frac{|-2+h| - |-3|}{h} = \lim_{h \to 0} \frac{2-h-2}{h} = \lim_{h \to 0} \frac{-h}{h} = -1$$

h close to 0
so $-2+h < 0$

$$f'(0) = \lim_{h \to 0} \frac{|h|}{h} \text{ DNE since } \begin{array}{l} \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1 \\ \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = 1 \end{array}$$

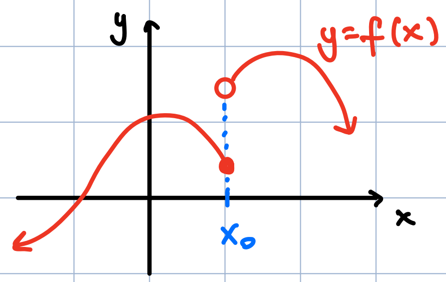


So f is not differentiable at $x = 0$.

$\Rightarrow$ there is no tangent line at $x = 0$.

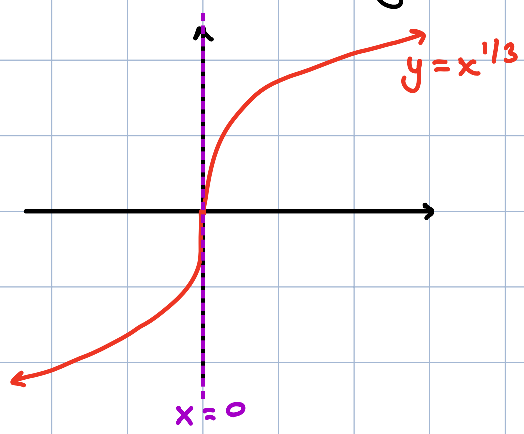
Besides corners, other behaviors can prevent the existence of a derivative at a point.

- Discontinuity :



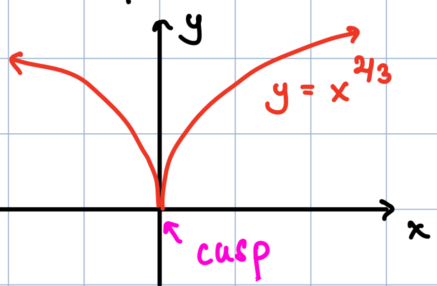
If f is not continuous at x_0, f is not differentiable at x_0.
Said differently: if $f'(x_0)$ exists, f must be continuous at x_0.

- Vertical tangent :



If the tangent at x_0 is vertical, it has no slope, so $f'(x_0)$ DNE.
Example: $y = x^{1/3}$ at $x = 0$.

- Cusp :



Slope goes to ∞ on one side and $-\infty$ on the other.
Example: $y = x^{2/3}$ at $x = 0$.

5) Is $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$ differentiable at $x = 0$?

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right).$$

Since $-1 \leq \sin\left(\frac{1}{h}\right) \leq 1$ we have $-|h| \leq h\sin\left(\frac{1}{h}\right) \leq |h|$ for any $h \neq 0$. Also, $\lim_{h \to 0} -|h| = \lim_{h \to 0} |h| = 0$. So by the

Squeeze Theorem, $\lim_{h \to 0} h\sin\left(\frac{1}{h}\right) = 0$.

So $\boxed{f \text{ is differentiable at } x=0 \text{ and } f'(0)=0}$

6) Calculate the following derivatives:

$\frac{d}{dx}(x^2)$, $\frac{d}{dx}(\sqrt{x})$, $\frac{d}{dx}\left(\frac{1}{x}\right)$

$$\frac{d}{dx}(x^2) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} 2x + h = \boxed{2x}.$$

$$\frac{d}{dx}(\sqrt{x}) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}.$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)} = \lim_{h \to 0} \frac{-h}{hx(x+h)}$$

$$= \lim_{h \to 0} -\frac{1}{x(x+h)} = -\frac{1}{x(x+0)} = \boxed{-\frac{1}{x^2}}.$$

7) Suppose that f is continuous and $\lim_{x \to 2} \frac{f(x) - 5}{x-2} = 6$. Find an equation of the tangent line to the graph of f at $x = 2$.

The only way $\lim_{x \to 2} \frac{f(x)-5}{x-2}$ is finite is if $\lim_{x \to 2} f(x) = 5$
i.e. $f(2) = 5$
(continuity).

Then $f'(2) = 6$.

So the tangent line to f at $x = 2$:

- passes through $(2,5)$
- has slope 6

equation is $y - 5 = 6(x-2)$

or $\boxed{y = 6(x-2) + 5}$.