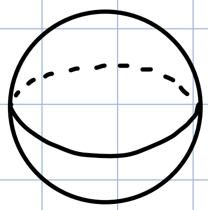
Section 3.10

Related Rates

Learning Goals

Learning Goal	*Homework Problems*
3.10.1 Solve related rates problems in which variables are related by a given equation.	1-10, 13-19, 29, 35.
3.10.2 Solve related rates problems using familiar geometric or trigonometric identities.	11, 12, 20-28, 30-34, 36-47.

Conceptual introduction: imagine you are inflating a spherical balloon. Both the volume V and radius r are changing



over time: V and r are both functions of the time t.

However, the variables V and r are not independent: they are related by the equation:

$$V = \frac{4}{3}\pi r^3 .$$

Therefore, the rates of change $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are also related.

$$V = \frac{4}{3}\pi r^3 \xrightarrow[\text{implicit diff.}]{\frac{d}{dt}} \frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$$

$$\boxed{\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}}$$ equation relating the rates of change

In a related rates problem, we are given a rate of change and we look for another one.

Examples: 1) In the inflating balloon situation above, assume that when the radius is 2 in, the volume increases at 4 in^3/sec. Find the rate at which the radius changes at that time.

- Write equations: $\begin{cases} V = \frac{4}{3}\pi r^3 \\ \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \end{cases}$

- Write information: $\begin{cases} r = 2 \\ \frac{dV}{dt} = 4 \end{cases}$

- Substitute information into equations:

$$\begin{cases} V = \frac{4}{3}\pi(2)^3 \\ 4 = 4\pi(2)^2 \frac{dr}{dt} \end{cases}$$

- Solve for goal, here $\frac{dr}{dt}$:

$$4 = 4\pi(2)^2 \frac{dr}{dt} \Rightarrow \boxed{\frac{dr}{dt} = \frac{1}{4\pi} \text{ in/sec} \simeq 0.083 \text{ in/sec}}$$

Steps to solve related rates problem:

- Step 1: draw a picture and name variables
- Step 2: write down relations between variables
- Step 3: use implicit differentiation to differentiate relations with respect to time t and get relations between rates of change.
- Step 4: substitute given information into all relations
- Step 5: solve for the goal

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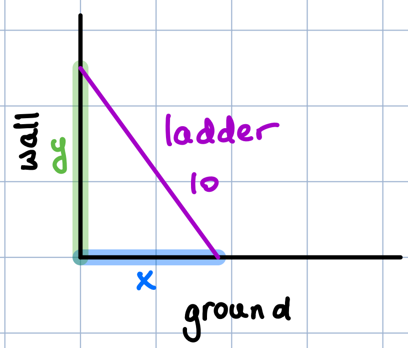
- Do NOT substitute in the value of any non-constant variable before differentiating

- Always include units in final answer

2) A ladder of length 10 ft is leaning against a wall. The bottom of the ladder slides away from the wall at 2 ft/sec. How fast is the top of the ladder sliding down the wall when the top is 8 ft from the ground?

• Step 1: draw a picture and name variables



• Step 2: write down relations between variables.

$$x^2 + y^2 = 10^2 = 100$$

• Step 3: differentiate relations with respect to time t

$$\frac{d}{dt}\left(\begin{array}{l} x^2 + y^2 = 100 \\ 2x\frac{dx}{dt} + 2y\frac{dy}{dx} = 0 \end{array} \right.$$

• Step 4: substitute given information into all relations

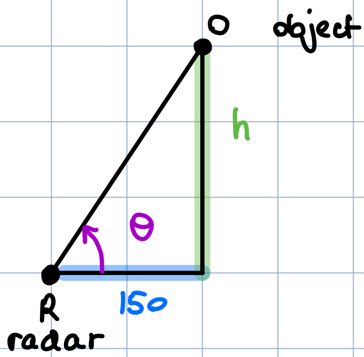
$$\left\{ \begin{array}{l} x^2 + y^2 = 100 \\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \end{array} \right. \xrightarrow[\frac{dx}{dt} = 2]{y = 8} \left\{ \begin{array}{l} x^2 + 8^2 = 100 \\ 2x(2) + 2(8)\frac{dy}{dt} = 0 \end{array} \right.$$

• Step 5: solve for the goal, here $\frac{dy}{dt}$.

$$\left\{ \begin{array}{l} x^2 + 64 = 100 \rightarrow x^2 = 36 \rightarrow x = 6 \quad (x > 0) \\ 4x + 16\frac{dy}{dt} = 0 \rightarrow 4 \cdot 6 + 16\frac{dy}{dt} = 0 \rightarrow \frac{dy}{dt} = -\frac{24}{16} \end{array} \right.$$

$$\boxed{\frac{dy}{dt} = -\frac{3}{2} \text{ ft/sec}}$$

3) An object is falling straight down, tracked by a radar positioned 150 ft away from the impact point. When the object is 300 ft in the air, it is falling at 10 m/sec. How fast is the viewing angle of the radar changing at that time?



Relations:

$$\tan(\theta) = \frac{h}{150}$$

$$\frac{d}{dt}$$

$$\sec(\theta)^2 \frac{d\theta}{dt} = \frac{1}{150} \cdot \frac{dh}{dt}$$

$$\begin{cases} \tan(\theta) = \dfrac{h}{150} \\ \sec(\theta)^2 \dfrac{d\theta}{dt} = \dfrac{1}{150} \dfrac{dh}{dt} \end{cases} \xrightarrow[\frac{dh}{dt} = -10 \text{ falling}]{h = 300} \begin{cases} \tan(\theta) = \dfrac{300}{150} = 2 \\ \sec(\theta)^2 \dfrac{d\theta}{dt} = -\dfrac{10}{150} = -\dfrac{1}{15}. \end{cases}$$

Now we solve for $\frac{d\theta}{dt}$.

$$\begin{cases} \tan(\theta) = 2 \rightarrow \sec(\theta)^2 = 1 + \tan(\theta)^2 = 1 + 2^2 = 5 \\ \sec(\theta)^2 \dfrac{d\theta}{dt} = -\dfrac{1}{15} \rightarrow 5 \dfrac{d\theta}{dt} = -\dfrac{1}{15} \end{cases}$$

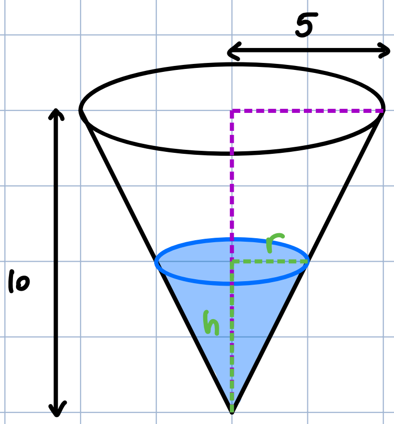
$$\boxed{\frac{d\theta}{dt} = -\frac{1}{75} \text{ rad/sec}}.$$

Alternative way to solve: $\theta = \arctan\left(\frac{h}{150}\right)$

$$\Rightarrow \frac{d\theta}{dt} = \frac{1}{\sqrt{1 + \left(\frac{h}{150}\right)^2}} \cdot \frac{1}{150} \cdot \frac{dh}{dt}$$

$$= \frac{1}{\sqrt{1 + \left(\frac{300}{150}\right)^2}} \cdot \frac{1}{150} \cdot (-10) = -\frac{1}{75} \text{ rad/sec.}$$

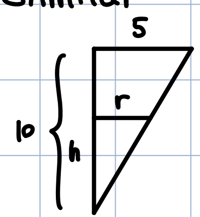
4) Water runs out of a conical tank of height 10 ft and radius 5 ft at a rate of 9 ft^3/min. How fast is the water level decreasing when there are 18π ft^3 of water left in the tank?



We need a relation between the water level h and the volume of water V.

$$V = \frac{1}{3}\pi r^2 h$$

Similar triangles: $\frac{r}{h} = \frac{5}{10} \Rightarrow r = \frac{h}{2}$.



So $$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

$\frac{d}{dt}$

$$\frac{dV}{dt} = \frac{\pi}{12}(3h^2)\frac{dh}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}.$$

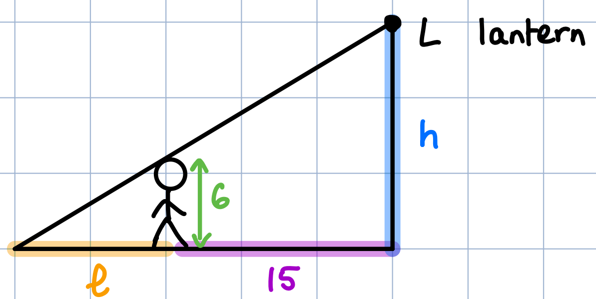
$$\begin{cases} V = \frac{\pi}{12}h^3 \\ \frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt} \end{cases} \xrightarrow[\frac{dV}{dt} = -9]{V = 18\pi} \begin{cases} 18\pi = \frac{\pi}{12}h^3 \\ -9 = \frac{\pi}{4}h^2\frac{dh}{dt} \end{cases}$$

We now solve for the goal, here $\frac{dh}{dt}$.

$$\begin{cases} 18\pi = \frac{\pi}{12}h^3 \to h^3 = 18\cdot 12 = 216 \to h = 6 \\ -9 = \frac{\pi}{4}h^2\frac{dh}{dt} \to -9 = \frac{\pi}{4}(6)^2\frac{dh}{dt} \to \frac{dh}{dt} = -\frac{9\cdot 4}{\pi\cdot 6^2} \end{cases}$$

$$\boxed{\begin{aligned} \frac{dh}{dt} &= -\frac{1}{\pi} \text{ ft/min} \\ &\simeq 0.32 \text{ ft/min} \end{aligned}}$$

5) A person 6 ft tall is standing 15 ft from a point P directly beneath a falling lantern. When the lantern is 10 ft above the ground, it is falling at 5 ft / sec. At what rate is the person's shadow changing at that time?



Similar triangles:

$$\frac{h}{6} = \frac{\ell + 15}{\ell} = 1 + \frac{15}{\ell}$$

$$\frac{d}{dt}$$

$$\frac{1}{6} \cdot \frac{dh}{dt} = -\frac{15}{\ell^2} \cdot \frac{d\ell}{dt}$$

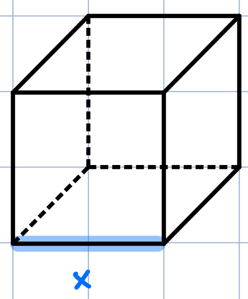
$$\begin{cases} \frac{h}{6} = 1 + \frac{15}{\ell} \\ \frac{1}{6}\frac{dh}{dt} = -\frac{15}{\ell^2}\frac{d\ell}{dt} \end{cases} \xrightarrow[\frac{dh}{dt} = -5]{h = 10} \begin{cases} \frac{10}{6} = 1 + \frac{15}{\ell} \\ -\frac{5}{6} = -\frac{15}{\ell^2}\frac{d\ell}{dt} \end{cases}$$

We now solve for the goal, here $\frac{d\ell}{dt}$.

$$\begin{cases} \frac{5}{3} = 1 + \frac{15}{\ell} \rightarrow \frac{15}{\ell} = \frac{2}{3} \rightarrow \ell = \frac{45}{2} \\ -\frac{5}{6} = -\frac{15}{\ell^2}\frac{d\ell}{dt} \rightarrow \frac{d\ell}{dt} = \frac{5\ell^2}{6 \cdot 15} = \frac{45^2 \cdot 5}{2^2 \cdot 6 \cdot 15} \end{cases}$$

$$\boxed{\frac{d\ell}{dt} = \frac{225}{8} \text{ ft/ sec}}$$

6) The total surface area of a cube is changing at a rate of 12 in^2/sec when the volume is 1,000 in^3. At what rate is the volume changing at that time?



Surface area : $S = 6x^2$

Volume : $V = x^3$

$$\frac{d}{dt}\left(S = 6x^2 \right) \quad \frac{dS}{dt} = 12x \frac{dx}{dt}$$

$$\frac{d}{dt}\left(V = x^3 \right) \quad \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$$\begin{cases} S = 6x^2 \\ V = x^3 \\ \frac{dS}{dt} = 12x \frac{dx}{dt} \\ \frac{dV}{dt} = 3x^2 \frac{dx}{dt} \end{cases} \xrightarrow[\frac{dS}{dt} = 12]{V = 1000} \begin{cases} S = 6x^2 \\ 1000 = x^3 \\ 12 = 12x \frac{dx}{dt} \\ \frac{dV}{dt} = 3x^2 \frac{dx}{dt} \end{cases}$$

We now solve for $\frac{dV}{dt}$.

$$\begin{cases} S = 6x^2 \\ 1000 = x^3 \longrightarrow x = 10 \\ 12 = 12x \frac{dx}{dt} \longrightarrow \frac{dx}{dt} = \frac{1}{10}. \\ \frac{dV}{dt} = 3x^2 \frac{dx}{dt} \longrightarrow \frac{dV}{dt} = 3 \cdot 10^2 \cdot \frac{1}{10} \end{cases}$$

$$\boxed{\frac{dV}{dt} = 30 \ in^3/sec}.$$

Remark: we can also solve this problem by writing a relation between V and S eliminating the variable x.

$$\left. \begin{array}{l} V = x^3 \\ S = 6x^2 \end{array} \right\} \quad S = 6V^{2/3}$$

$$\frac{dS}{dt} = 6 \frac{2}{3} V^{-1/3} \frac{dV}{dt} = \frac{4}{V^{1/3}} \frac{dV}{dt} \xrightarrow[\frac{dS}{dt} = 12]{V = 1000} 12 = \frac{4}{1000^{1/3}} \cdot \frac{dV}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{120}{4} = 30 \ in^3/sec.$$