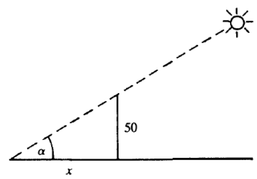
Rutgers University
Math 151

Section 3.10: Related Rates - Worksheet Solutions

1. How fast is the shadow cast on level ground by a pole 50 feet tall lengthening when the angle a of elevation of the sun is 45° and is decreasing by $\frac{1}{4}$ radian per hour? (See figure below.)



Solution. If we call x the length of the shadow and α the elevation angle of the sun, we have

$$\cot(\alpha) = \frac{x}{50}.$$

Differentiating this relation with respect to the time t, we get

$$-\csc(\alpha)^2 \frac{d\alpha}{dt} = \frac{1}{50}\frac{dx}{dt}.$$

We will now substitute the given information, $\alpha = \frac{\pi}{4}$ and $\frac{d\alpha}{dt} = -\frac{1}{4}$, in these equations. This gives us

$$\begin{cases} \cot\left(\frac{\pi}{4}\right) = \frac{x}{50}, \\ -\csc\left(\frac{\pi}{4}\right)^2\left(-\frac{1}{4}\right) = \frac{1}{50}\frac{dx}{dt}. \end{cases}$$

We want to solve for $\frac{dx}{dt}$. For this, the second equation is enough, and we get

$$\frac{dx}{dt} = -50\csc\left(\frac{\pi}{4}\right)^2\left(-\frac{1}{4}\right) = \frac{50(\sqrt{2})^2}{4} = \boxed{25 \text{ ft/hr}}.$$

2. A sphere of radius 5 in fills with water at a rate of 4 in^3/min. When the water level inside the sphere is 6 in, how fast is it increasing? *(Hint: the volume of a spherical cap of height h in a sphere of radius r is* $V = \frac{\pi}{3}(3rh^2 - h^3)$.*)*

Solution. If we call h the height of water inside the sphere and V the volume of water, we have the relation $V = \frac{\pi}{3}(15h^2 - h^3)$ (the formula given with the radius of the sphere being $r = 5$, a constant). Differentiating the relation with respect to the time t, we obtain

$$\frac{dV}{dt} = \frac{\pi}{3}\left(30h\frac{dh}{dt} - 3h^2\frac{dh}{dt}\right) = \pi\left(10h - h^2\right)\frac{dh}{dt}.$$

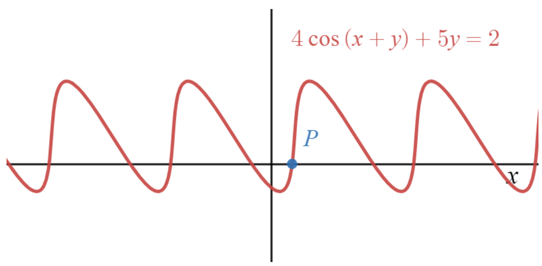
If we now substitute the information, $\frac{dV}{dt} = 4$ and $h = 6$, we get

$$\begin{cases} V = \frac{\pi}{3}(15(6)^2 - 6^3), \\ 4 = \pi\left(10(6) - (6)^2\right)\frac{dh}{dt}. \end{cases}$$

We want to solve for $\frac{dh}{dt}$. For this, we need only the second equation and we get

$$\frac{dh}{dt} = \frac{4}{\pi\left(10(6) - (6)^2\right)} = \frac{4}{24\pi} = \boxed{\frac{1}{6\pi} \text{ in/min}}.$$

3. A particle travels toward the right on the graph of the implicit function $4\cos(x + y) + 5y = 2$, see the figure below.



When the particle first crosses the positive x-axis (at the point P on the figure), its x-coordinate increases at 6 units/sec. At what rate is the y-coordinate of the particle changing at that time?

Solution. We have the relation $4\cos(x + y) + 5y = 2$. Differentiating this with respect to the time t, we get

$$-4\sin(x + y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right) + 5\frac{dy}{dt} = 0.$$

We now need to plug in the information given. When the particle passes through the point P, we have $y = 0$ and we also know that $\frac{dx}{dt} = 6$. This gives us

$$\begin{cases} 4\cos(x + 0) + 5(0) = 2, \\ -4\sin(x + 0)\left(6 + \frac{dy}{dt}\right) + 5\frac{dy}{dt} = 0. \end{cases}$$

We need to solve these relations for $\frac{dy}{dt}$. We will start by solving the first equation for x, and we'll then use this to solve for $\frac{dy}{dt}$ in the second equation. The first equation gives $4\cos(x) = 2$, that is $\cos(x) = \frac{1}{2}$. The first positive solution to this equation is $x = \frac{\pi}{3}$. Plugging this in the second equation gives

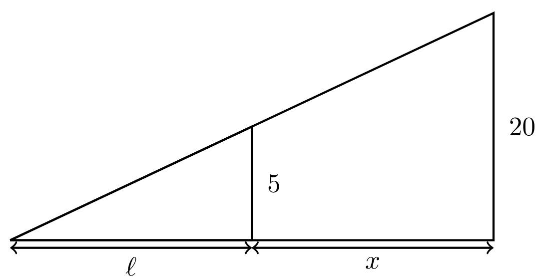
$$\begin{aligned} &-4\sin\left(\frac{\pi}{3}\right)\left(6 + \frac{dy}{dt}\right) + 5\frac{dy}{dt} = 0 \\ &-4\frac{\sqrt{3}}{2}\left(6 + \frac{dy}{dt}\right) + 5\frac{dy}{dt} = 0 \\ &-2\sqrt{3}\left(6 + \frac{dy}{dt}\right) + 5\frac{dy}{dt} = 0 \\ &-12\sqrt{3} - 2\sqrt{3}\frac{dy}{dt} + 5\frac{dy}{dt} = 0 \end{aligned}$$

$$(5-2\sqrt{3})\frac{dy}{dt} = 12\sqrt{3}$$

$$\boxed{\frac{dy}{dt} = \frac{12\sqrt{3}}{5-2\sqrt{3}} \text{ units/sec}}.$$

4. A 5-foot person is walking toward a 20-foot lamppost at the rate of 6 feet per second. How fast is the length of their shadow (cast by the lamp) changing?

   Solution. We call ℓ the length of the shadow and x the distance between the person and the lamppost, see figure below.



   Similar triangles give us the relation $\frac{\ell}{5} = \frac{\ell + x}{20}$. Differentiating this relation with respect to t gives

$$\frac{1}{5}\frac{d\ell}{dt} = \frac{1}{20}\left(\frac{d\ell}{dt} + \frac{dx}{dt}\right).$$

   We can now plug in the information, that is $\frac{dx}{dt} = -6$, and solve for $\frac{d\ell}{dt}$. We get

$$\frac{1}{5}\frac{d\ell}{dt} = \frac{1}{20}\left(\frac{d\ell}{dt} - 6\right)$$

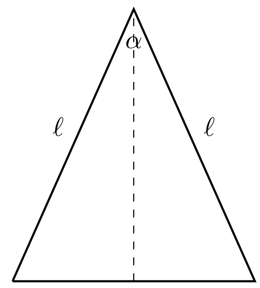
$$\frac{1}{5}\frac{d\ell}{dt} - \frac{1}{20}\frac{d\ell}{dt} = -\frac{3}{10}$$

$$\frac{3}{20}\frac{d\ell}{dt} = -\frac{3}{10}$$

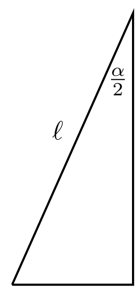
$$\boxed{\frac{d\ell}{dt} = -2 \text{ ft/sec}}.$$

5. The legs of an isosceles triangle of base 6 cm are increasing at a rate of 14 cm/hour, causing the vertex angle to decrease. When the legs are 4 cm, how fast is the vertex angle decreasing?

   Solution. Call ℓ the length of the legs of the triangle and α the vertex angle, see figure below.



Let us consider the right triangle formed by the height, one of the legs and half of the base of the isosceles triangle, see figure below.



Then we have the relation $\sin\left(\frac{\alpha}{2}\right) = \frac{3}{\ell}$. Differentiating with respect to the time t gives

$$\cos\left(\frac{\alpha}{2}\right)\frac{1}{2}\frac{d\alpha}{dt} = -\frac{3}{\ell^2}\frac{d\ell}{dt}$$

We can now plug in the information, $\ell = 6$ and $\dfrac{d\ell}{dt} = 14$ in these equations to get

$$\begin{cases} \sin\left(\frac{\alpha}{2}\right) = \frac{3}{4}, \\ \cos\left(\frac{\alpha}{2}\right)\frac{1}{2}\frac{d\alpha}{dt} = -\frac{3}{4^2}(14). \end{cases}$$

We solve for $\frac{d\alpha}{dt}$. We have

$$\cos\left(\frac{\alpha}{2}\right)^2 + \sin\left(\frac{\alpha}{2}\right)^2 = 1 \ \Rightarrow\ \cos\left(\frac{\alpha}{2}\right) = \sqrt{1 - \sin\left(\frac{\alpha}{2}\right)^2} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}.$$

Using this in the second equation, we get

$$\frac{\sqrt{7}}{8}\frac{d\alpha}{dt} = -\frac{21}{8}$$
$$\frac{d\alpha}{dt} = -\frac{21}{\sqrt{7}} = \boxed{-3\sqrt{7}\text{ rad/sec}}.$$

6. **[Advanced]** An object moves along the graph of a function $y = f(x)$. At a certain point, the slope of the graph is -4 and the y-coordinate of the object is increasing at the rate of 3 units per second. At that

point, how fast is the x-coordinate of the object changing?

Solution. Differentiating the relation $y = f(x)$ with respect to the time t gives

$$\frac{dy}{dt} = f'(x)\frac{dx}{dt}.$$

We are given the information $f'(x) = -4$ and $\frac{dy}{dt} = 3$. Plugging this in the previous equation gives

$$3 = -4\frac{dx}{dt} \quad \Rightarrow \quad \boxed{\frac{dx}{dt} = -\frac{3}{4} \text{ units/sec}}.$$