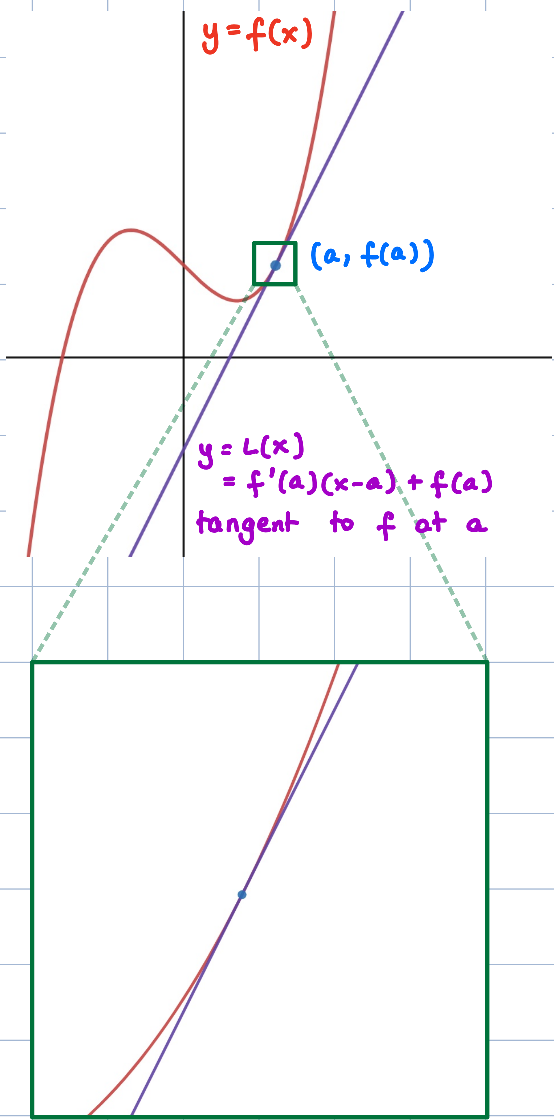
Section 3.11

Linearization and Differentials

Learning Goals

Learning Goal	*Homework Problems*
3.11.1 Estimate the value of a function using an appropriate linearization.	1-18, 64-74.
3.11.2 Compute differentials. Use them to estimate the error that propagates through a computation.	19-64, 69-74.

Conceptual introduction :



Let f be a differentiable function at $x = a$. Near $x = a$, the graph of f is very close to the tangent line at $x = a$.

$$\boxed{L(x) = f'(a)(x-a) + f(a)}$$

<u>linearization</u> of f at $x = a$

We can use $L(x)$ to approximate $f(x)$ when x is close to a.

<u>Standard linear approximation centered at a</u> :

$$\boxed{f(x) \approx L(x) \text{ if } x \text{ close to } a}$$

Equivalently, we can state the standard linear approximation at a as :

$$f(x) \approx f'(a)(x-a) + f(a)$$

or $\quad f(x) - f(a) \approx f'(a)(x-a)$

or $\quad \boxed{\Delta f \approx f'(a)\,\Delta x}$ if Δx is small.

<u>Remark</u> : in fact, $L(x)$ is the <u>best</u> linear approximation of f near $x = a$.

Examples: 1) Find the linearization of $f(x) = \sqrt{x}$ at $x=1$ and use it to estimate $\sqrt{1.06}$.

$L(x) = f'(1)(x-1) + f(1)$ with $f(1) = \sqrt{1} = 1$

$f'(1) = \frac{d}{dx}\sqrt{x}\Big|_{x=1} = \frac{1}{2\sqrt{x}}\Big|_{x=1} = \frac{1}{2}.$

$$\boxed{L(x) = \frac{1}{2}(x-1) + 1}$$

Linear approximation: $\sqrt{1.06} = f(1.06)$

$\approx L(1.06) = \frac{1}{2}(1.06-1) + 1$

$$\boxed{\sqrt{1.06} \approx 1.03}.$$

(Actual value: $\sqrt{1.06} = 1.02956301\ldots$)

2) Use linear approximation to estimate $\sqrt[3]{9}$

Here $f(x) = \sqrt[3]{x} = x^{1/3}$

$f'(x) = \frac{1}{3}x^{-2/3}.$

To find where to center our approximation, we look for a value of a close to 9 for which we know $f(a)$ and $f'(a)$ exactly.

↳ Here $a = 8$ will do.

$f(8) = 8^{1/3} = 2$

$f'(8) = \frac{1}{3}8^{-2/3} = \frac{1}{12}$

$\Big] \rightarrow L(x) = f'(8)(x-8) + f(8)$

$= \frac{1}{12}(x-8) + 2$

So $\sqrt[3]{9} = f(9)$

$\approx L(9) = \frac{1}{12}(9-8) + 2$

$$\boxed{\sqrt[3]{9} \approx \frac{25}{12}}.$$

3) Use linear approximation to estimate $\ln(0.997)$.

Here $f(x) = \ln(x)$, $f'(x) = \frac{1}{x}$

We can take $a=1$ as our center since it is close to 0.997 and we know $f(1) = \ln(1) = 0$, $f'(1) = \frac{1}{1} = 1$.

$L(x) = f'(1)(x-1) + f(1) = x-1$.

So $\ln(0.997) = f(0.997) \approx L(0.997) = 0.997-1$

$$\boxed{\ln(0.997) \approx -0.003}$$

4) Use linear approximation to estimate $\tan\left(\frac{\pi}{4} - 0.08\right)$

Here $f(x) = \tan(x)$, $f'(x) = \sec(x)^2$.

We can take $a = \frac{\pi}{4}$ since it is close to $\frac{\pi}{4} - 0.08$ and we know the exact values of $f\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$ and $f'\left(\frac{\pi}{4}\right) = \sec\left(\frac{\pi}{4}\right)^2 = (\sqrt{2})^2 = 2$.

$L(x) = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right)$

$= 2\left(x - \frac{\pi}{4}\right) + 1$

So $\tan\left(\frac{\pi}{4} - 0.08\right) = f\left(\frac{\pi}{4} - 0.08\right) \approx L\left(\frac{\pi}{4} - 0.08\right) = 2(-0.08) + 1$

$$\boxed{\tan\left(\frac{\pi}{4} - 0.08\right) \approx 0.84}.$$

5) Suppose the tangent line to f at $x=1$ is $y = 2x+3$.

Estimate : a) $f(1.04)$ b) $g(0.96)$ where $g(x) = 2f(x)e^{1-x}$.

a) The linearization of f at $x=1$ is $L(x) = 2x+3$

So $f(1.04) \approx L(1.04) = 2(1.04) + 3 = \boxed{5.08}$

b) $g(1) = 2f(1)e^{1-1} = 2\cdot 5\cdot e^0 = 10.$

$$g'(1) = \frac{d}{dx}\left(2f(x)e^{1-x}\right)\Big|_{x=1} = \left(2f'(x)e^{1-x} - 2f(x)e^{1-x}\right)\Big|_{x=1}$$

$$= 2f'(1)e^{1-1} - 2f(1)e^{1-1} = 4-10 = -6.$$

So $L(x) = -6(x-1)+10.$

$g(0.96) \simeq L(0.96) = -6(0.96-1)+10$

$\boxed{g(0.96) \approx 10.24}$.

Differentials and errors:

The differential of f is $\boxed{df = f'(x)\,dx}$ where dx is a new independent variable, and df depends on both x and dx.

Examples: if

$y = x^2$, then $dy = 2x\,dx$

$y = \cos(e^{3x})$, then $dy = -3\sin(e^{3x})e^{3x}\,dx$

$s = 2t + t^3$, then $ds = (2+3t^2)\,dt$

$u = w\sin(6w)$, then $du = (\sin(6w) + 6w\cos(6w))\,dw$

etc.

Linear approximation: if $dx = \Delta x$, $\boxed{\Delta f \simeq df = f'(x)\,\Delta x}$.

Intuitively, df tells us how sensitive the output of f is to a change in the input x - the greater $f'(x)$ the more sensitive. We can use this idea to estimate errors propagating through a computation.

Errors: if x is measured with error Δx, then the propagated error is $\boxed{\underbrace{\Delta f}_{\text{exact error}} \approx \underbrace{f'(x)\Delta x}_{\text{approximation of the error}}}$

Relative error: $\boxed{\dfrac{\Delta f}{f(x)} \approx \dfrac{f'(x)\Delta x}{f(x)}}$

Percentage error: $\boxed{\dfrac{\Delta f}{f(x)} \cdot 100 \approx \dfrac{f'(x)\Delta x}{f(x)} \cdot 100}$

Examples: 1) Suppose the side of a cube is measured at 2 cm with an error of ± 0.25 cm. What is the propagated error in the computation of the volume?

$$V = x^3 \Rightarrow dV = 3x^2\,dx$$

$$\underbrace{\Delta V}_{\text{error in volume}} \approx 3x^2 \underbrace{\Delta x}_{\text{error in measurement of side}} = 3 \cdot 2^2 \cdot 0.25 = \boxed{3 \text{ cm}^3}$$

Relative error: $\dfrac{\Delta V}{V} = \dfrac{3}{2^3} = \dfrac{3}{8} = 0.375$

Percentage error: $\dfrac{\Delta V}{V} \cdot 100 = \boxed{37.5\ \%}$.

2) In the example above, suppose we want an error of at most 10 % in the measurement of the volume. What is the maximum percentage error that can be made measuring the side?

This time, we know $\dfrac{\Delta V}{V} = 0.1$ and we want $\dfrac{\Delta x}{x}$.

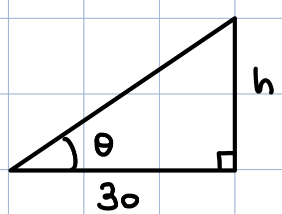
We have $\Delta V \approx 3x^2 \Delta x$

$$\Rightarrow \quad \Delta x \approx \frac{\Delta V}{3x^2}$$

$$\frac{\Delta x}{x} \approx \frac{\Delta V}{3x^3} = \frac{1}{3} \cdot \frac{\Delta V}{V} = \frac{0.1}{3} = 0.0333\ldots$$

The maximum percentage error that can be made in the measurement of the side is $\boxed{33.3\%}$.

3) Suppose you stand 30 ft from a building and calculate the height of the building by measuring the elevation angle to the top. You measure 60°. How accurate must the measurement be to have an error of at most 1% on the height?



$h = 30\tan(\theta) \Rightarrow dh = 30\sec(\theta)^2 d\theta$

So $\Delta h \simeq 30\sec(\theta)^2 \Delta\theta$.

We know $\frac{\Delta h}{h} = 0.01$ and we want $\frac{\Delta\theta}{\theta}$.

$$\Delta h = 0.01 \cdot h = 0.01 \cdot 30\tan(\theta) = 0.01 \cdot 30\tan\left(\frac{\pi}{3}\right) = 0.3\sqrt{3}$$

So $$\Delta\theta = \frac{\Delta h}{30\sec(\theta)^2} = \frac{0.3\sqrt{3}}{30 \cdot 4} = 0.0025\sqrt{3} \text{ rad.}$$

$$\frac{\Delta\theta}{\theta} = \frac{0.0025\sqrt{3}}{\frac{\pi}{3}} = 0.00413.$$

The maximal error that can be made on the measurement of θ is $\boxed{0.41\%}$.