Rutgers University
Math 151

Section 3.11: Linear Approximations - Worksheet Solutions

1. Use a well-chosen linear approximation to estimate the following quantities.

   (a) $\sqrt[3]{62}$

   Solution. For $f(x) = \sqrt[3]{x}$ at $x = 64$, we have $f(64) = 4$ and $f'(64) = \frac{1}{3}64^{-2/3} = \frac{1}{48}$, so

   $$L(x) = \frac{1}{48}(x - 64) + 4.$$

   Therefore, $\sqrt[3]{62} \simeq L(62) = \frac{1}{48}(62 - 64) + 4 = \boxed{\frac{95}{24}}$.

   (b) $e^{-0.8}$

   Solution. For $f(x) = e^x$ at $x = 0$, we have $f(0) = 1$ and $f'(0) = e^0 = 1$, so $L(x) = x + 1$. Therefore, $e^{-0.8} \simeq L(-0.8) = -0.8 + 1 = \boxed{0.2}$.

   (c) $\sqrt{49.6}$

   Solution. For $f(x) = \sqrt{x}$ at $x = 49$, we have $f(49) = 7$ and $f'(64) = \frac{1}{2}49^{-1/2} = \frac{1}{14}$, so

   $$L(x) = \frac{1}{14}(x - 49) + 7.$$

   Therefore, $\sqrt{49.6} \simeq L(49.6) = \frac{1}{14}(49.6 - 49) + 7 = \boxed{\frac{493}{70}}$.

   (d) $\ln(1 + 5\sin(0.06))$

   Solution. For $f(x) = \ln(1+5\sin(x))$ at $x = 0$, we have $f(0) = 0$ and $f'(0) = \left(\frac{5\cos(x)}{1 + 5\sin(x)}\right)_{|x=0} = 5$, so

   $$L(x) = 5x.$$

   Therefore, $\ln(1 + 5\sin(0.06)) \simeq L(0.06) = 5(0.06) = \boxed{0.3}$.

   (e) $\cot\left(\frac{\pi}{6} + 0.02\right) - \sqrt{3}$

   Solution. For $f(x) = \cot(x)$ at $x = \frac{\pi}{6}$, we have $f\left(\frac{\pi}{6}\right) = \sqrt{3}$ and $f'\left(\frac{\pi}{6}\right) = -\csc^2\left(\frac{\pi}{6}\right) = -4$. So

   $$\cot\left(\frac{\pi}{6} + 0.02\right) - \sqrt{3} = \Delta f \simeq f'\left(\frac{\pi}{6}\right)\Delta x = -4(0.02) = \boxed{-0.08}.$$

(f) $\sqrt[4]{17} - \sqrt[4]{16}$

Solution. For $f(x) = \sqrt[4]{x}$ at $x = 16$, we have $f(16) = 2$ and $f'(4) = \frac{1}{4}16^{-3/4} = \frac{1}{32}$. So

$$\sqrt[4]{17} - \sqrt[4]{16} = \Delta f \simeq f'(16)\,\Delta x = \frac{1}{32} \cdot 1 = \boxed{\frac{1}{32}}.$$

2. Suppose that f is a function such that $f(3) = -7$ and $f'(3) = 2$. Use a linear approximation to estimate the following quantities.

(a) $f(3.07)$

Solution. The linearization of f at $x = 3$ is $L(x) = f'(3)(x-3) + f(3) = 2(x-3) - 7$. So

$$f(3.07) \simeq L(3.07) = 2(3.07 - 3) - 7 = \boxed{-6.86}.$$

(b) **[Advanced]** $f(1 + \cos(0.1) + e^{0.2})$

Solution. Put $g(x) = f(1 + \cos(x) + e^{2x})$. We have $g(0) = f(1 + \cos(0) + e^0) = f(3) = -7$ and

$$g'(x) = f'(1 + \cos(x) + e^{2x})\left(-\sin(x) + 2e^{2x}\right),$$

so $g'(0) = f'(3)(2) = 4$. Hence the linearization of g at $x = 0$ is $L(x) = 4x - 7$. So

$$f(1 + \cos(0.1) + e^{0.2}) = g(0.1) \simeq L(0.1) = 4(0.1) - 7 = \boxed{-6.6}.$$

3. Find the differential dy of the following functions.

(a) $y = \arcsin(3x^2)$

Solution. $dy = \dfrac{1}{\sqrt{1-(3x^2)^2}}(6x)dx = \boxed{\dfrac{6x}{\sqrt{1-9x^4}}dx}.$

(b) $y = 4\sqrt[3]{x} - \dfrac{5}{x^2} + e^3$

Solution. $\boxed{dy = \left(\dfrac{4}{3}x^{-2/3} + 10x^{-3}\right)dx}.$

(c) $y = \csc(5\theta)$

Solution. $\boxed{dy = -5\csc(5\theta)\cot(5\theta)d\theta}.$

(d) $y = 5^{3-t^2}$

Solution. $\boxed{dy = \ln(5)5^{3-t^2}(-2t)dt}.$

(e) $y = x^{\cos(2x)}$

Solution. We have $y = e^{\cos(2x)\ln(x)}$ so

$$\boxed{dy = e^{\cos(2x)\ln(x)}\left(-2\sin(x)\ln(x) + \frac{\cos(2x)}{x}\right)dx}.$$

(f) $y = \sin(3e^{-7z})$

Solution. $\boxed{dy = -21\cos(3e^{-7z})e^{-7z}dz}$.

4. The volume of a sphere is computed by measuring its diameter.

(a) Suppose that the diameter of the sphere is measured at 5 cm with a precision of 0.2 cm. What is the percentage error propagated in the computation of the volume?

Solution. Call x the diameter of the sphere. We have $V = \frac{4}{3}\pi\left(\frac{x}{2}\right)^3 = \frac{\pi}{6}x^3$. From this we deduce

$$\begin{aligned} dV &= \frac{\pi}{2}x^2 dx, \\ \Rightarrow \frac{dV}{V} &= \frac{\frac{\pi}{2}x^2 dx}{\frac{\pi}{6}x^3} = 3\frac{dx}{x}. \end{aligned}$$

The relative error propagated is

$$\frac{\Delta V}{V} \simeq \frac{dV}{V} = 3\frac{dx}{x} = 3\frac{0.2}{5} = 0.12,$$

which means that the pencentage error is $\boxed{12\%}$.

(b) **[Advanced]** Suppose that we want a measurement of the volume with an error of at most 1.5%. What is the maximum percentage error that can be made measuring the diameter?

Solution. We want to have $\frac{\Delta V}{V} = 0.015$, so

$$\frac{dx}{x} \simeq \frac{1}{3}\frac{\Delta V}{V} = 0.005.$$

Therefore, the maximum percentage error that can be made measuring the diameter is $\boxed{0.5\%}$.