Rutgers University
Math 151

Section 3.1-2: Derivatives and Tangent Lines - Worksheet Solutions

1. For the functions below, find the value of the derivative and an equation of the tangent line at the point indicated. (You must use the limit definition of the derivative in this problem - you cannot use derivative rules.)

   (a) $f(x) = \dfrac{3x}{1-2x}$ at $x = 1$.

   Solution. We have $f(1) = -3$ and

$$\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\
&= \lim_{h\to 0} \frac{\frac{3(1+h)}{1-2(1+h)} - (-3)}{h} \\
&= \lim_{h\to 0} \frac{\frac{3+3h}{-1-2h} + 3}{h} \\
&= \lim_{h\to 0} \frac{3+3h+3(-1-2h)}{h(-1-2h)} \\
&= \lim_{h\to 0} \frac{3+3h-3-6h}{h(-1-2h)} \\
&= \lim_{h\to 0} \frac{-3h}{h(-1-2h)} \\
&= \lim_{h\to 0} \frac{-3}{-1-2h} \\
&= \frac{-3}{-1-2\cdot 0} \\
&= \boxed{3}.
\end{aligned}$$

   So the tangent line passes through $(1, -3)$ and has slope 3. Therefore, it has equation $\boxed{y = 3(x-1) - 3}$.

   (b) $f(x) = \sqrt{5x-1}$ at $x = 2$.

   Solution. We have $f(2) = 3$ and

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{5(2+h)-1} - 3}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{9+5h} - 3}{h} \cdot \frac{\sqrt{9+5h}+3}{\sqrt{9+5h}+3} \\
&= \lim_{h\to 0} \frac{(\sqrt{9+5h})^2 - 3^2}{h(\sqrt{9+5h}+3)}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{h\to 0} \frac{9+5h-9}{h(\sqrt{9+5h}+3)} \\
&= \lim_{h\to 0} \frac{5h}{h(\sqrt{9+5h}+3)} \\
&= \lim_{h\to 0} \frac{5}{\sqrt{9+5h}+3} \\
&= \frac{5}{\sqrt{9+5\cdot 0}+3} \\
&= \boxed{\frac{5}{6}}.
\end{aligned}$$

So the tangent line passes through $(2,3)$ and has slope $\frac{5}{6}$. Therefore, it has equation $\boxed{y = \frac{5}{6}(x-2)+3}$.

(c) $f(x) = 18x^{-2}$ at $x = -3$.

Solution. We have $f(-3) = 2$ and

$$\begin{aligned}
f'(-3) &= \lim_{h\to 0} \frac{f(-3+h)-f(-3)}{h} \\
&= \lim_{h\to 0} \frac{\frac{18}{(-3+h)^2}-2}{h} \\
&= \lim_{h\to 0} \frac{18-2(-3+h)^2}{h(-3+h)^2} \\
&= \lim_{h\to 0} \frac{18-2(9-6h+h^2)}{h(-3+h)^2} \\
&= \lim_{h\to 0} \frac{18-18+12h-2h^2}{h(-3+h)^2} \\
&= \lim_{h\to 0} \frac{12h-2h^2}{h(-3+h)^2} \\
&= \lim_{h\to 0} \frac{12-2h}{(-3+h)^2} \\
&= \frac{12-2\cdot 0}{(-3+0)^2} \\
&= \boxed{\frac{4}{3}}.
\end{aligned}$$

So the tangent line passes through $(-3,2)$ and has slope $\frac{4}{3}$. Therefore, it has equation $\boxed{y = \frac{4}{3}(x+3)+2}$.

(d) $f(x) = 2x^3 + 5x + 3$ at $x = -1$.

Solution. We have $f(-1) = -4$ and

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{2(-1+h)^3 + 5(-1+h) + 3 - (-4)}{h} \\
&= \lim_{h\to 0} \frac{2(-1+3h-3h^2+h^3) - 5 + 5h + 3 + 4}{h} \\
&= \lim_{h\to 0} \frac{-2+6h-6h^2+2h^3 - 5 + 5h + 3 + 4}{h} \\
&= \lim_{h\to 0} \frac{11h - 6h^2 + 2h^3}{h} \\
&= \lim_{h\to 0} 11 - 6h + 2h^2 \\
&= \boxed{11}.
\end{aligned}
$$

So the tangent line passes through $(-1, -4)$ and has slope 11. Therefore, it has equation $\boxed{y = 11(x+1) - 4}$.

[Advanced]

(e) $f(x) = 3\tan(4x)$ at $x = 0$.

Solution. We have $f(0) = 0$ and

$$
\begin{aligned}
f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\
&= \lim_{h\to 0} \frac{3\tan(4h) - 0}{h} \\
&= \lim_{h\to 0} \frac{3\sin(4h)}{h\cos(4h)} \cdot \frac{4h}{4h} \\
&= \lim_{h\to 0} \frac{3\sin(4h)}{4h} \cdot \frac{4h}{h\cos(4h)} \\
&= 3\left(\lim_{h\to 0} \frac{\sin(4h)}{4h}\right)\left(\lim_{h\to 0} \frac{4}{\cos(4h)}\right) \\
&= 3 \cdot 1 \cdot 4 \\
&= \boxed{12}.
\end{aligned}
$$

So the tangent line passes through $(0, 0)$ and has slope 12. Therefore, it has equation $\boxed{y = 12x}$.

(f) $f(x) = x^{2/3}$ at $x = 8$.

Solution. We have $f(8) = 4$. To compute $f'(8)$, we will make use of the identity $(a-b)(a^2+ab+b^2) = a^3 - b^3$. We get

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(8+h) - f(8)}{h} \\
&= \lim_{h\to 0} \frac{(8+h)^{2/3} - 4}{h} \cdot \frac{(8+h)^{4/3} + 4(8+h)^{2/3} + 16}{(8+h)^{4/3} + 4(8+h)^{2/3} + 16} \\
&= \lim_{h\to 0} \frac{\left((8+h)^{2/3}\right)^3 - 4^3}{h\left((8+h)^{4/3} + 4(8+h)^{2/3} + 16\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{(8+h)^2 - 64}{h\left((8+h)^{4/3} + 4(8+h)^{2/3} + 16\right)} \\
&= \lim_{h\to 0} \frac{64 + 16h + h^2 - 64}{h\left((8+h)^{4/3} + 4(8+h)^{2/3} + 16\right)} \\
&= \lim_{h\to 0} \frac{16h + h^2}{h\left((8+h)^{4/3} + 4(8+h)^{2/3} + 16\right)} \\
&= \lim_{h\to 0} \frac{16 + h}{(8+h)^{4/3} + 4(8+h)^{2/3} + 16} \\
&= \frac{16}{8^{4/3} + 4(8)^2 + 16} \\
&= \frac{1}{3}.
\end{aligned}
$$

So the tangent line passes through $(8,4)$ and has slope $\frac{1}{3}$. Therefore, it has equation $\boxed{y = \frac{1}{3}(x-8) + 4}$.