Rutgers University
Math 151

Sections 3.3, 3.5: Differentiation Rules - Worksheet Solutions

1. Calculate the derivatives of the following functions.

(a) $f(x) = 5x^4 - 8\sqrt[5]{x} - e^4$.

Solution.

$$f'(x) = 5\frac{d}{dx}(x^4) - 8\frac{d}{dx}(x^{1/5}) - \frac{d}{dx}(e^4) = \boxed{20x^3 - \frac{8}{5}x^{-4/5} - 0}.$$

(b) $f(x) = 7x\cos(x)e^x$.

Solution.

$$f'(x) = \frac{d}{dx}(7x)\cos(x)e^x + 7x\frac{d}{dx}(\cos(x))\,e^x + 7x\cos(x)\frac{d}{dx}(e^x) = \boxed{7\cos(x)e^x - 7x\sin(x)e^x + 7\cos(x)e^x}.$$

(c) $f(x) = ex^e + 4\frac{\sqrt{x}}{\sin(x)}$.

Solution.

$$f'(x) = e\frac{d}{dx}(x^e) + 4\frac{d}{dx}\left(\frac{\sqrt{x}}{\sin(x)}\right) = \boxed{e^2x^{e-1} + 4\frac{\frac{\sin(x)}{2\sqrt{x}} - \sqrt{x}\cos(x)}{\sin(x)^2}}.$$

(d) $f(x) = \frac{3}{5+x^4}$.

Solution.

$$f'(x) = \frac{\frac{d}{dx}(3)\left(5+x^4\right) - 3\frac{d}{dx}\left(5+x^4\right)}{(5+x^4)^2} = \boxed{\frac{-12x^3}{(5+x^4)^2}}.$$

(e) $f(x) = 3\sin(1)7^x - x^{4/3}$.

Solution.

$$f'(x) = 3\sin(1)\frac{d}{dx}(7^x) - \frac{d}{dx}\left(x^{4/3}\right) = \boxed{3\sin(1)\ln(7)7^x - \frac{4}{3}x^{1/3}}.$$

(f) $f(x) = \dfrac{x^2}{xe^x - 1}$.

Solution.

$$f'(x) = \frac{\frac{d}{dx}\left(x^2\right)(xe^x - 1) - x^2 \frac{d}{dx}(xe^x - 1)}{(xe^x - 1)^2} = \boxed{\frac{2x(xe^x - 1) - x^2(e^x + xe^x)}{(xe^x - 1)^2}}.$$

(g) $f(x) = 2^x x^2$.

Solution.

$$f'(x) = \frac{d}{dx}(2^x)\,x^2 + 2^x \frac{d}{dx}\left(x^2\right) = \ln(2)2^x x^2 + 2^x(2x) = \boxed{\ln(2)2^x x^2 + 2^{x+1}x}.$$

(h) $f(x) = \dfrac{\cos(x)}{\sin(x) + 1}$.

Solution.

$$f'(x) = \frac{\frac{d}{dx}(\cos(x))(\sin(x) + 1) - \cos(x)\frac{d}{dx}(\sin(x) + 1)}{(\sin(x) + 1)^2} = \boxed{\frac{-\sin(x)(\sin(x) + 1) - \cos(x)^2}{(\sin(x) + 1)^2}}.$$

(i) $f(x) = \dfrac{x\cos(x)\sin(x)}{5^x}$.

Solution.

$$f'(x) = \frac{\frac{d}{dx}(x\cos(x)\sin(x))\,5^x - x\cos(x)\sin(x)\frac{d}{dx}(5^x)}{(5^x)^2}$$
$$= \boxed{\frac{(\cos(x)\sin(x) - x\sin(x)^2 + x\cos(x)^2)5^x - \ln(5)x\cos(x)\sin(x)5^x}{5^{2x}}}$$

2. (a) Find the points on the graph of $f(x) = 2\sec(x) + \tan(x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, where the tangent line is horizontal.

Solution. The tangent line is horizontal when $f'(x) = 0$. Here, we have $f'(x) = 2\sec(x)\tan(x) + \sec(x)^2 = \sec(x)(2\tan(x) + \sec(x))$. So we get the equation

$$\sec(x)(2\tan(x) + \sec(x)) = 0$$

which produces $\sec(x) = 0$ or $2\tan(x) + \sec(x) = 0$. The equation $\sec(x) = 0$ has no solution, while the other equation gives

$$2\tan(x) + \sec(x) = 0$$

$$
\begin{aligned}
\frac{2\sin(x)+1}{\cos(x)} &= 0 \\
\sin(x) &= -\frac{1}{2} \\
x = -\frac{\pi}{6} &\text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2}.
\end{aligned}
$$

For this value of x, we have $y = 2\sec\left(-\frac{\pi}{6}\right) + \tan\left(-\frac{\pi}{6}\right) = \frac{4}{\sqrt{3}} - \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$, so we have obtained the point $\boxed{\left(-\frac{\pi}{6}, \sqrt{3}\right)}$.

(b) Find the points on the graph of $f(x) = \dfrac{1}{1-2x}$ where the tangent line passes through the origin.

Solution. The tangent line to the graph of f at $x = a$ passes through $\left(a, \frac{1}{1-2a}\right)$ and has slope

$$f'(a) = -\frac{-2}{(1-2a)^2} = \frac{2}{(1-2a)^2}.$$

So the equation of the tangent is $y - \frac{1}{1-2a} = \frac{2}{(1-2a)^2}(x-a)$. The tangent line passes through the origin if this equation is satisfied for $(x, y) = (0, 0)$, which gives the condition

$$
\begin{aligned}
0 - \frac{1}{1-2a} &= \frac{2}{(1-2a)^2}(0-a) \\
\frac{1}{1-2a} &= \frac{2a}{(1-2a)^2} \\
1 - 2a &= 2a \\
4a &= 1 \\
a &= \frac{1}{4}.
\end{aligned}
$$

For this value of a, we have $y = \frac{1}{1-2(1/4)} = 2$, so we have obtained the point $\boxed{\left(\frac{1}{4}, 2\right)}$.

(c) [**Advanced**] Find the values of the constant a for which the tangent lines to the graph of $f(x) = x^3 + 3x^2 + 5x$ at $x = a$ and $x = a + 1$ are parallel.

Solution. We have $f'(x) = 3x^2 + 6x$. The tangent lines to f at $x = a$ and $x = a + 1$ are parallel when $f'(a) = f'(a+1)$, which gives

$$
\begin{aligned}
3a^2 + 6a &= 3(a+1)^2 + 6(a+1) \\
3a^2 + 6a &= 3a^2 + 6a + 3 + 6a + 6 \\
6a &= -9 \\
&\boxed{a = -\frac{3}{2}}.
\end{aligned}
$$

3. Find the second derivative of the functions below.

(a) $f(x) = x^3 e^x$.

Solution.

$$f'(x) = 3x^2 e^x + x^3 e^x = (3x^2 + x^3)e^x,$$
$$f''(x) = (6x + 3x^2)e^x + (3x^2 + x^3)e^x = \boxed{(6x + 6x^2 + x^3)e^x}.$$

(b) $f(x) = \dfrac{3x+5}{2x+7}$.

Solution.

$$f'(x) = \frac{3(2x+7) - 2(3x+5)}{(2x+7)^2} = -\frac{1}{(2x+5)^2},$$
$$f''(x) = -\frac{(0)(2x+5)^2 - 1(2(2x+5) + (2x+5)(2))}{(2x+5)^4} = \boxed{\frac{4}{(2x+5)^3}}.$$

(c) $f(x) = \dfrac{7\cos(x)}{x}$.

Solution.

$$f'(x) = 7\frac{-\sin(x)x - \cos(x)}{x^2},$$
$$f''(x) = 7\frac{(-\cos(x)x - \sin(x) + \sin(x))x^2 - (-\sin(x)x - \cos(x))(2x)}{x^4} = \boxed{-7\frac{\cos(x)x^2 - 2\sin(x)x + 2\cos(x)}{x^3}}.$$

4. Suppose that f is a differentiable function such that $y = -2x + 1$ is tangent to the graph of f at $x = 3$. Evaluate the following

(a) $f(3)$.

Solution. $\boxed{f(1) = -2(3) + 1 = -5}$.

(b) $f'(3)$.

Solution. $\boxed{f'(3) = -2}$.

(c) $\dfrac{d}{dx}\left(2f(x) - x^3\right)_{|x=3}$.

Solution.

$$\frac{d}{dx}\left(2f(x) - x^3\right)_{|x=3} = \left(2f'(x) - 3x^2\right)_{|x=3} = 2f'(3) - 3 \cdot 3^2 = \boxed{-37}.$$

(d) $\frac{d}{dx}\left(\frac{f(x)}{x}\right)_{|x=3}$.

Solution.

$$\frac{d}{dx}\left(\frac{f(x)}{x}\right)_{|x=3} = \frac{f'(x)x - f(x)}{x^2}_{|x=3} = \frac{f'(3)3 - f(3)}{3^2} = -\frac{1}{9}$$

(e) **[Advanced]** $\frac{d}{dx}\left(e^x f(x)^2\right)_{|x=3}$.

Solution.

$$\begin{aligned}
\frac{d}{dx}\left(e^x f(x)^2\right)_{|x=3} &= \frac{d}{dx}\left(e^x f(x) f(x)\right)_{|x=3} \\
&= \left(e^x f(x) f(x) + e^x f'(x) f(x) + e^x f(x) f'(x)\right)_{x=3} \\
&= e^3 f(3)^2 + 2e^3 f'(3) f(3) \\
&= \boxed{45e^3}.
\end{aligned}$$