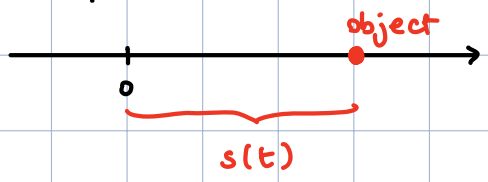
Section 3.4

Rates of Change

Learning Goals

Learning Goal	*Homework Problems*
3.4.1 Interpret the derivative as an instantaneous rate of change. Solve applications involving rate of change.	23-32.
3.4.2 Solve linear motion problems using derivatives. Find an object's displacement, velocity, speed, and acceleration.	1-18, 19a, 19b, 20-22, 33-36.
3.4.3 Analyze the motion of an object given a graph of its position, velocity, or acceleration.	15-18, 20-22, 33-36.

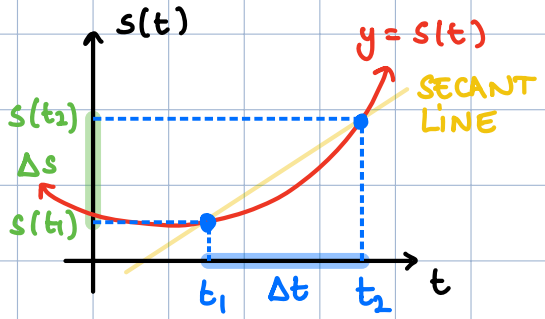
Conceptual introduction :



Suppose that we track the position $s(t)$ of an object moving on an axis.

The displacement on $[t_1, t_2]$ is $\boxed{\Delta s = s(t_2) - s(t_1)}$

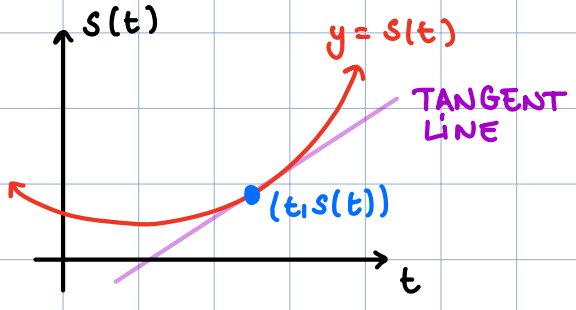
The average velocity on $[t_1, t_2]$ is



$$\boxed{v_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}}$$

↳ slope of secant line on the graph of the position

The instantaneous velocity is



$$\boxed{v(t) = s'(t) = \frac{ds}{dt} = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}}$$

↳ slope of tangent line on the graph of the position

On any interval where $v(t) > 0$, the object moves forward.

On any interval where $v(t) < 0$, the object moves backward.

↳ the sign of the velocity indicates the direction of the movement.

The speed is $|v(t)|$: it indicates how fast the object is moving, without specification on the direction.

The rate at which the velocity changes is called the acceleration:

$$\boxed{a(t) = v'(t) = s''(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}}$$

It measures how quickly the object picks up or loses velocity.

Examples: 1) Suppose the position of a body moving on an axis is given by $s = \frac{48}{t} + 3t + 5$ for $t \geq 1$.

a) Find the displacement and average velocity on $[1,3]$.

b) Find the velocity and acceleration.

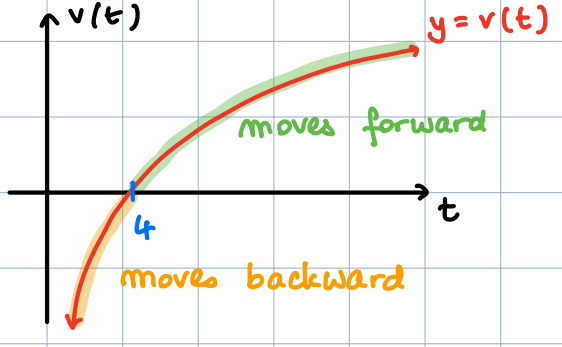
c) Does the body ever change direction?

a) $\Delta s = s(3) - s(1) = \left(\frac{48}{3} - 3\cdot 3 + 5\right) - \left(\frac{48}{1} - 3 + 5\right) = \boxed{-41}$

$v_{av} = \frac{\Delta s}{\Delta t} = \boxed{-\frac{41}{2}}$.

b) $v(t) = \frac{ds}{dt} = -\frac{48}{t^2} + 3$

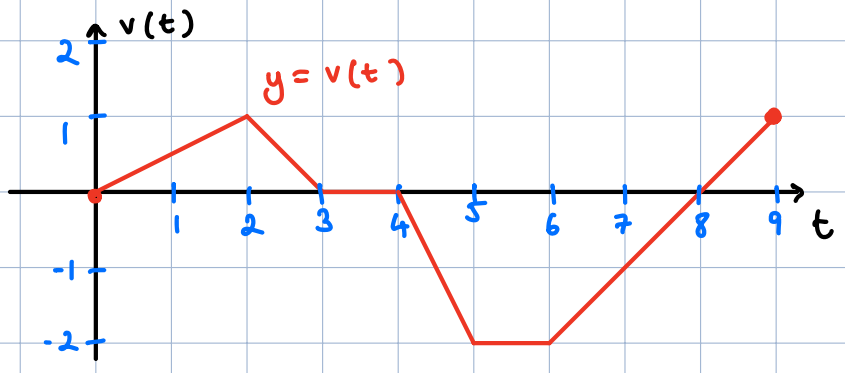
c) The body changes direction when $v(t)$ changes sign.



$v(t) = 0 \Rightarrow \frac{48}{t^2} = 3 \Rightarrow t^2 = 16 \Rightarrow t = 4$

This happens at $t = 4$.

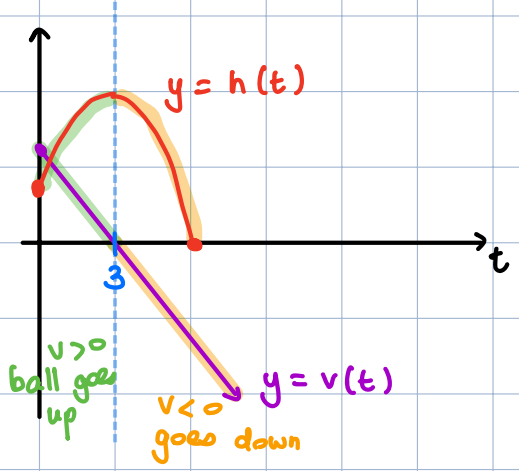
2) Suppose that $s(t)$ = position of an object moving
Suppose $v(t)$ is graphed below.



Find the intervals where the object is moving forward, backward, standing still. What is the maximum speed of the object ?

The object moves forward when $v(t) > 0$: $(1,3)$, $(8,9)$.
The object moves backward when $v(t) < 0$: $(4,8)$
The object stands still when $v(t) = 0$: $(3,4)$, $t = 8$
Maximum speed : $|v(t)|$ maximum on $[5,6]$ (equal to 2).

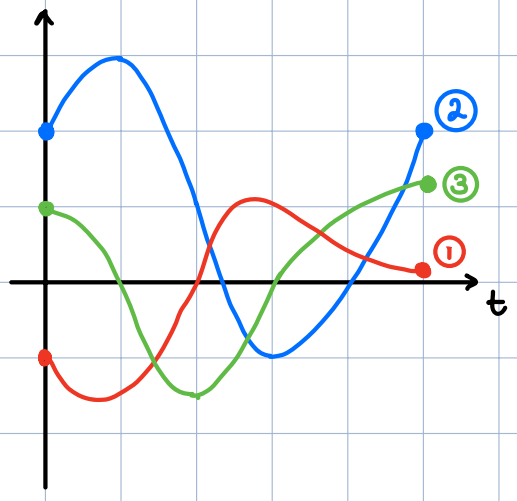
3) Suppose you throw a ball in the air. Its altitude is given by $h(t) = -5t^2 + 30t + 6$
When does the ball start is descent ? What is the maximal altitude reached by the ball ?



$h(t) = -5t^2 + 30t + 6$
$v(t) = -10t + 30 = 0$ when $t = 3$
$\Rightarrow$ Ball starts descending at $\boxed{t = 3}$.
Max. altitude = $h(3) = \boxed{51}$.

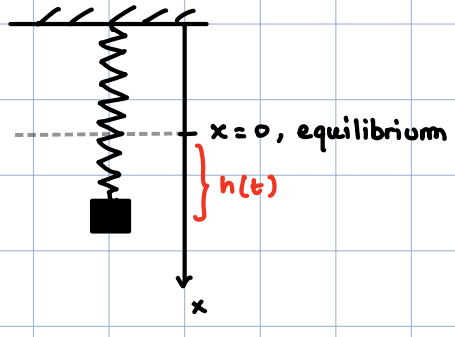
Remark : in this example, we have $a(t) = v'(t) = -10$.
So the acceleration is constant : key feature of free fall.

4) This figure shows the graph of position, velocity and acceleration. Which is which ?



Position s : ② , Velocity $v = \frac{ds}{dt}$: ③ , Acceleration $a = \frac{dv}{dt}$: ①

5) A weight is attached to a spring and oscillates around its equilibrium position following the law $h(t) = 5\cos(t)$.



a) Find velocity and acceleration
b) Find displacement on $[0, \frac{\pi}{4}]$.
c) At what times in the interval $[0, 2\pi]$ is the weight standing still ?

a) $v(t) = h'(t) = -5\sin(t)$
$a(t) = v'(t) = -5\cos(t)$

b) $\Delta h = h\left(\frac{\pi}{4}\right) - h(0) = 5\cos\left(\frac{\pi}{4}\right) - 5\cos(0) = \boxed{5\left(\frac{\sqrt{2}}{2} - 1\right)}$

c) The weight is standing still when $v(t) = 0 \Rightarrow -5\sin(t) = 0$

$\Rightarrow$ $\boxed{t = 0, \pi, 2\pi}$.