Section 3.6

Chain Rule

Learning Goals

Learning Goal	*Homework Problems*
3.6.1 Identify "inside" and "outside" functions in a composition and apply the chain rule, along with any other appropriate rules of differentiation.	1-105, 107-113.
3.6.2 Answer conceptual questions involving the chain rule.	81-92, 97, 98, 106-108, 111-113.

Conceptual introduction: which of these derivatives can we compute using product or quotient rule?

A $\frac{d}{dx}(x^2\cos(x))$	B $\frac{d}{dx}(\cos(x)^2)$
C $\frac{d}{dx}(\cos(x^2))$	D $\frac{d}{dx}\left(\frac{\cos(x)}{x^2}\right)$

A : $\frac{d}{dx}(\overset{u}{x^2}\overset{v}{\cos(x)}) = \overset{u'}{2x}\,\overset{v}{\cos(x)} + \overset{u}{x^2}\overset{v'}{(-\sin(x))}$ (product)

B : $\frac{d}{dx}(\cos(x)^2) = \frac{d}{dx}(\overset{u}{\cos(x)}\,\overset{v}{\cos(x)}) = \overset{u'}{(-\sin(x))}\,\overset{v}{\cos(x)} + \overset{u}{\cos(x)}\overset{v'}{(-\sin(x))}$

D : $\frac{d}{dx}\left(\frac{\overset{u}{\cos(x)}}{x^2}\right) = \frac{\overset{u'}{(-\sin(x))}\,\overset{v}{x^2} - \overset{u}{\cos(x)}\overset{v'}{(2x)}}{\underset{v^2}{(x^2)^2}}$ (quotient)

However, we cannot compute C. using product rule because C. is a composition of two functions:

$$\cos(x^2) = (f\circ g)(x) \quad \text{with} \quad \begin{cases} f(x) = \cos(x) & \text{(outside)} \\ g(x) = x^2 & \text{(inside)} \end{cases}.$$

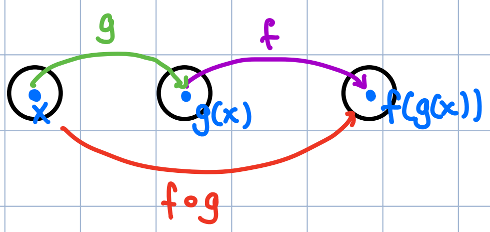
How do we compute the derivative of a composition?

Intuition: if $f(x) = 2x$ (constant rate of change $= 2$)
$g(x) = 3x$ (constant rate of change $= 3$)

then $f(g(x)) = f(3x) = 2\cdot 3x = 6x$ (rate of change $= 2\cdot 3$)

$\Rightarrow$ the rate of change of $f\circ g$ is the product of the rates of change of f, g.

Chain Rule :



$$\boxed{(f\circ g)'(x) = \underbrace{f'(g(x))}_{\text{derivative of outside evaluated at inside}}\ \underbrace{g'(x)}_{\text{derivative of inside}}}$$

Leibniz notation: if $y = f(u)$, $u = g(x)$: $\boxed{\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}}$

Examples :

1) $\dfrac{d}{dx}\left(\cos(x^2)\right)$

outside $f(x) = \cos(x)$, $f'(x) = -\sin(x)$

inside $g(x) = x^2$, $g'(x) = 2x$

$= f'(g(x))\,g'(x) = \boxed{-\sin(x^2)\,(2x)}$

2) Calculate the following derivatives.

a) $\dfrac{d}{dx}\left(\sqrt{x^5+e^x}\right)$

b) $\dfrac{d}{dx}\left(e^{3\tan(x)}\right)$

c) $\dfrac{d}{dx}\left(\left(\dfrac{1-x}{1+x}\right)^{29}\right)$

d) $\dfrac{d}{dx}\left(\sec(3x)\,e^{\sqrt{x}}\right)$

e) $\dfrac{d}{dx}\left(\cos(\sin(x^4))\right)$

f) $\dfrac{d}{dx}\left(\tan\left(\sqrt{\cos(5x+1)}\right)\right)$

g) $\dfrac{d}{dx}\left(\cot(1+\sin(2x+\sqrt[3]{x}))\right)$

a) $\dfrac{d}{dx}\left(\sqrt{x^5+e^x}\right)$

outside $f(x) = \sqrt{x}$, $f'(x) = \dfrac{1}{2\sqrt{x}}$

inside $g(x) = x^5+e^x$, $g'(x) = 5x^4+e^x$

$= \dfrac{1}{2\sqrt{x^5+e^x}}\cdot(5x^4+e^x) = \boxed{\dfrac{5x^4+e^x}{2\sqrt{x^5+e^x}}}$.

b) b) $\frac{d}{dx}\left(e^{3\tan(x)}\right)$

outside $f(x) = e^x, \quad f'(x) = e^x$

inside $g(x) = 3\tan(x), \quad g'(x) = 3\sec(x)^2$

$$= e^{3\tan(x)}\left(3\sec(x)^2\right) = \boxed{3e^{3\tan(x)}\sec(x)^2}.$$

c) $\frac{d}{dx}\left(\left(\frac{1-x}{1+x}\right)^{29}\right)$

outside $f(x) = x^{29}, \quad f'(x) = 29x^{28}$

inside $g(x) = \frac{1-x}{1+x}, \quad g'(x) = \frac{-1\cdot(1+x) - (1-x)\cdot 1}{(1+x)^2} = \frac{-2}{(1+x)^2}$

$$= \boxed{29\left(\frac{1-x}{1+x}\right)^{28}\frac{-2}{(1+x)^2}}$$

d) $\frac{d}{dx}\left(\sec(3x)\,e^{\sqrt{x}}\right)$ first product rule

$$= \frac{d}{dx}\left(\sec(3x)\right)e^{\sqrt{x}} + \sec(3x)\frac{d}{dx}\left(e^{\sqrt{x}}\right)$$

chain rule on each part

$$= \boxed{\sec(3x)\tan(3x)(3)e^{\sqrt{x}} + \sec(3x)\,e^{\sqrt{x}}\frac{1}{2\sqrt{x}}}.$$

e) $\frac{d}{dx}\left(\cos(\sin(x^4))\right)$ composite of three functions: use chain rule twice.

$$= \boxed{-\sin(\sin(x^4))\cos(x^4)(4x^3)}$$

f) $\frac{d}{dx}\left(\tan\left(\sqrt{\cos(5x+1)}\right)\right)$ this time we have 4 layers

$$= \boxed{\sec\left(\sqrt{\cos(5x+1)}\right)^2 \cdot \frac{1}{2\sqrt{\cos(5x+1)}}\left(-\sin(5x+1)\right)(5)}$$

g) $\frac{d}{dx}\left(\cot\left(1+\sin\left(2x+\sqrt[3]{x}\right)\right)\right)$

$$= \boxed{-\csc^2\left(1+\sin\left(2x+\sqrt[3]{x}\right)\right)\cdot\cos\left(2x+\sqrt[3]{x}\right)\cdot\left(2+\frac{1}{3}x^{-2/3}\right)}$$

3) Use the table of values to calculate:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	2	0	-2	4
0	-1	1	3	2
1	0	-1	7	-2

a) $(f\circ g)'(0)$ b) $(g\circ f)'(0)$

c) $(f\circ f)'(0)$ d) $\frac{d}{dx}(f(2^x))\big|_{x=0}$

e) $\frac{d}{dx}(g(x)\sin(5f(x)))\big|_{x=1}$

a) $(f\circ g)'(0) = f'(g(0))\,g'(0) = f'(1)\,g'(0) = 7\cdot 2 = \boxed{14}$

b) $(g\circ f)'(0) = g'(f(0))\,f'(0) = g'(-1)\,f'(0) = 4\cdot 3 = \boxed{12}$

c) $(f\circ f)'(0) = f'(f(0))\cdot f'(0) = f'(-1)\,f'(0) = (-2)\cdot 3 = \boxed{-6}$

d) $\frac{d}{dx}(f(2^x))\big|_{x=0}$ inside $g(x) = 2^x$, $g'(x) = \ln(2)2^x$

$$= f'(2^x)\ln(2)2^x\big|_{x=0} = f'(2^0)\ln(2)2^0 = f'(1)\ln(2) = \boxed{7\ln(2)}$$

e) $\frac{d}{dx}(g(x)\sin(5f(x)))\big|_{x=1}$

$$= g'(x)\sin(5f(x)) + g(x)\frac{d}{dx}(\sin(5f(x)))\big|_{x=1}$$ (product)

$$= g'(x)\sin(5f(x)) + g(x)\cos(5f(x))\,5f'(x)\big|_{x=1}$$

$$= g'(1)\sin(5f(1)) + g(1)\cos(5f(1))\,5f'(1)$$

$$= -2\sin(0) + (-1)\cos(0)\,5\cdot 7 = \boxed{-35}$$

4) Let $f(x) = \frac{e^{x^2}}{3x^2+1}$. Find all horizontal tangent lines on the graph of f.

The tangent line to the graph of f is horizontal when $f'(x)=0$.
We calculate f' using quotient and chain rule.

$$f'(x) = \frac{2xe^{x^2}(3x^2+1) - e^{x^2}(6x)}{(3x^2+1)^2} = \frac{2xe^{x^2}(3x^2+1-3)}{(3x^2+1)^2} = \frac{2xe^{x^2}(3x^2-2)}{(3x^2+1)^2}.$$

$$f'(x) = 0 \Rightarrow 2xe^{x^2}(3x^2-2) = 0$$

$$\Rightarrow \quad x=0 \qquad \underset{\text{no solution}}{e^{x^2}=0} \qquad 3x^2-2=0$$

$$x^2 = \frac{2}{3}$$

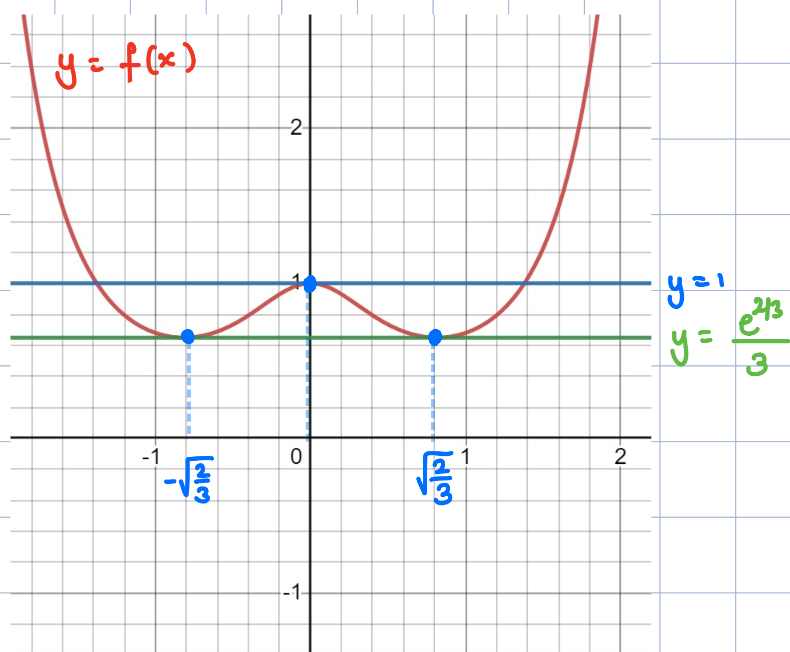
$$x = \pm\sqrt{\frac{2}{3}}$$

The tangent lines at these points are:

- $x=0$: $y = \dfrac{e^{0^2}}{3\cdot 0^2+1} = 1$
- $x=\sqrt{\frac{2}{3}}$: $y = \dfrac{e^{(\sqrt{2/3})^2}}{3(\sqrt{\frac{2}{3}})^2+1} = \dfrac{e^{2/3}}{3}$
- $x=-\sqrt{\frac{2}{3}}$: $y = \dfrac{e^{(-\sqrt{2/3})^2}}{3(-\sqrt{\frac{2}{3}})^2+1} = \dfrac{e^{2/3}}{3}$

same tangent line.

Conclusion: the horizontal tangent lines to the graph of f are $\boxed{y=1 \quad \text{and} \quad y=\frac{e^{2/3}}{3}}$



Towards implicit differentiation: if $y = f(x)$, we can use the chain rule to calculate derivatives of functions of y.

- Powers: $\frac{d}{dx}\left(f(x)^n\right) = n f(x)^{n-1} f'(x)$

  outside $= x^n$
  inside $= f(x)$

  or $\boxed{\frac{d}{dx}(y^n) = n y^{n-1} \frac{dy}{dx} = n y^{n-1} y'}$

- Exponentials: $\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} f'(x)$

  outside $= e^x$
  inside $= f(x)$

  or $\boxed{\frac{d}{dx}(e^y) = e^y \frac{dy}{dx} = e^y y'}$

⚠ $\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dy}(e^y) = e^y$ but $\frac{d}{dx}(e^y) = e^y \frac{dy}{dx}$.

Examples: calculate the following derivatives.

1) $\frac{d}{dx}(y^2 + x^3)$ 2) $\frac{d}{dx}\left(x e^y - \frac{5}{y^3}\right)$ 3) $\frac{d}{dx}(\cos(\sqrt{y}))$

1) $\frac{d}{dx}(y^2 + x^3) = \frac{d}{dx}(y^2) + \frac{d}{dx}(x^3) = \boxed{2yy' + 3x^2}$.

2) $\frac{d}{dx}\left(x e^y - \frac{5}{y^3}\right) = \frac{d}{dx}(xe^y) - \frac{d}{dx}(5y^{-3}) = \frac{dx}{dx} e^y + x \frac{de^y}{dx} - 5(-3)y^{-4} y'$

$= \boxed{e^y + xe^y y' + \frac{15y'}{y^4}}$.

3) $\frac{d}{dx}(\cos(\sqrt{y})) = -\sin(\sqrt{y}) \frac{d\sqrt{y}}{dx} = \boxed{-\sin(\sqrt{y}) \frac{1}{2\sqrt{y}} y'}$.