Section 3.7

Implicit Differentiation

Learning Goals

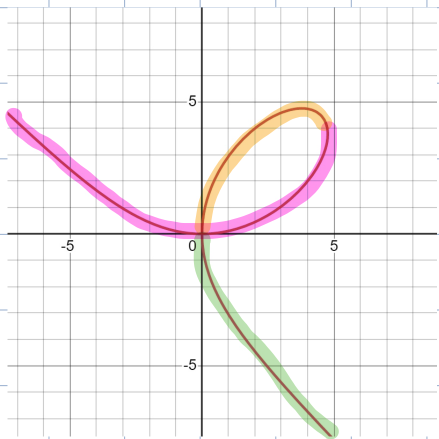
Learning Goal	*Homework Problems*
3.7.1 Compute derivatives for implicitly defined functions by applying the Chain Rule.	1-58.
3.7.2 Compute the slope of the line tangent or normal to the graph of an implicitly defined curve at a given point. Find points at which the curve has a given slope.	31--49, 51-54, 59-66.
3.7.3 Find higher order derivatives using implicit differentiation.	21-30, 55-58.
3.7.4 Answer conceptual questions using implicit differentiation.	50, 52, 55-57.

Conceptual introduction:

Explicit functions: $y = f(x)$ relation is solved for y as a function of x.

Implicit functions: general relations between x and y defining functions locally.

Example: $x^3 + y^3 = 9xy$ (Follium of Descartes)



Each highlighted part is the graph of a function that we call implicit since we cannot solve algebraically for y in $x^3 + y^3 = 9xy$.

How to find $y' = \frac{dy}{dx}$ for an implicit function ?

i) Differentiate both sides with respect to x, treating y as a function of x (chain rule).

ii) Collect the terms with $y' = \frac{dy}{dx}$ on one side.

iii) Solve for $y' = \frac{dy}{dx}$.

Examples: 1) Find $\frac{dy}{dx}$ for the following.

a) $y^2 + 5x = e^y$

b) $x^2 y = e^{2y} - 7x$

c) $\tan(x+y) = x + \sin(y)$

d) $\sin(x)\sin(y) = \sin(xy)$

e) $e^{x^2+y^2} + 3y = 0$

f) $\cot(6e^{\sin(y)}) = x^2 - y^2$

a) $y^2 + 5x = e^y$

$\frac{d}{dx}$

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(5x) = \frac{d}{dx}(e^y)$$
$$2yy' + 5 = e^y y'$$
$$2yy' - e^y y' = -5$$
$$y'(2y - e^y) = -5$$
$$\boxed{y' = -\frac{5}{2y - e^y}}$$

b) $x^2 y = e^{2y} - 7x$

$\frac{d}{dx}$

$$\frac{d}{dx}(x^2)y + x^2\frac{dy}{dx} = \frac{d}{dx}(e^{2y}) - \frac{d}{dx}(7x)$$

$$2xy + x^2 y' = 2e^{2y}y' - 7$$
$$x^2 y' - 2e^{2y}y' = -7 - 2xy$$
$$y'(x^2 - 2e^{2y}) = -7 - 2xy \Rightarrow \boxed{y' = \frac{-7 - 2xy}{x^2 - 2e^{2y}}}$$

b) $\tan(x+y) = x + \sin(y)$

$\frac{d}{dx}$

$$\sec(x+y)^2 \frac{d}{dx}(x+y) = \frac{dx}{dx} + \cos(y)\frac{dy}{dx}$$
$$\sec(x+y)^2(1+y') = 1 + \cos(y)y'$$
$$\sec(x+y)^2 + \sec(x+y)^2 y' = 1 + \cos(y)y'$$
$$\sec(x+y)^2 y' - \cos(y)y' = 1 - \sec(x+y)^2$$
$$y'(\sec(x+y)^2 - \cos(y)) = 1 - \sec(x+y)^2$$
$$\boxed{y' = \frac{1 - \sec(x+y)^2}{\sec(x+y)^2 - \cos(y)}}$$

d) $\sin(x)\sin(y) = \sin(xy)$

$\frac{d}{dx}$

$$\frac{d}{dx}(\sin(x))\sin(y) + \sin(x)\frac{d}{dx}(\sin(y)) = \cos(xy)\left(\frac{dx}{dx}y + x\frac{dy}{dx}\right)$$
$$\cos(x)\sin(y) + \sin(x)\cos(y)y' = \cos(xy)(y + xy')$$

$$\sin(x)\cos(y)y' - \cos(xy)xy' = \cos(xy)y - \cos(x)\sin(y)$$

$$y'(\sin(x)\cos(y) - \cos(xy)x) = \cos(xy)y - \cos(x)\sin(y)$$

$$\boxed{y' = \frac{\cos(xy)y - \cos(x)\sin(y)}{\sin(x)\cos(y) - \cos(xy)x}}$$

e) $e^{x^2+y^2} + 3y = 0$

$\frac{d}{dx}$

$$e^{x^2+y^2}\frac{d}{dx}(x^2+y^2) + 3y' = 0$$

$$e^{x^2+y^2}(2x + 2yy') + 3y' = 0$$

$$2e^{x^2+y^2}x + 2e^{x^2+y^2}yy' + 3y' = 0$$

$$y'(2e^{x^2+y^2} + 3) = -2e^{x^2+y^2}x$$

$$\boxed{y' = -\frac{2e^{x^2+y^2}x}{2e^{x^2+y^2} + 3}}$$

f) $\cot(6e^{\sin(y)}) = x^2 - y^2$

$\frac{d}{dx}$

$$-\csc^2(6e^{\sin(y)})6e^{\sin(y)}\cos(y)y' = 2x - 2yy'$$

$$2yy' - 6\csc^2(6e^{\sin(y)})e^{\sin(y)}\cos(y)y' = 2x$$

$$y'(2 - 6\csc^2(6e^{\sin(y)})e^{\sin(y)}\cos(y)) = 2x$$

$$\boxed{y' = \frac{2x}{2 - 6\csc^2(6e^{\sin(y)})e^{\sin(y)}\cos(y)}}$$

2) Find an equation of the tangent and normal lines to $2x^2 + 3y^2 = 21$ at the point $(3,1)$. Also find $\frac{d^2y}{dx^2}$.

First, we find $y' = \frac{dy}{dx}$.

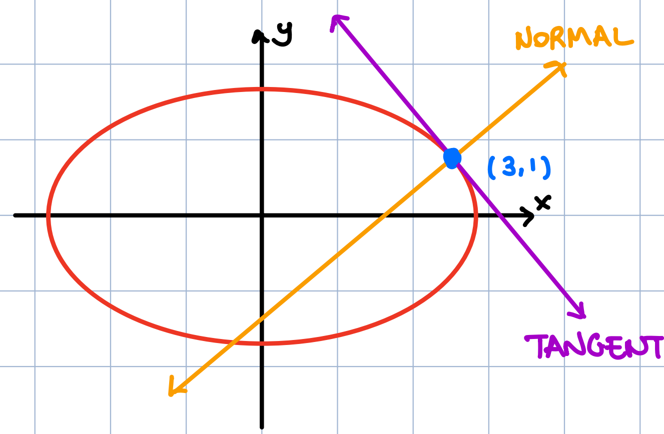
$2x^2 + 3y^2 = 21$

$\frac{d}{dx}$

$$4x + 6yy' = 0 \Rightarrow 6yy' = -4x \Rightarrow y' = -\frac{4x}{6y} = -\frac{2x}{3y}.$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = -\frac{2}{3} \cdot \frac{y - xy'}{y^2}$$

$$= \boxed{-\frac{2}{3} \cdot \frac{y - x\left(-\frac{2x}{3y}\right)}{y^2}}$$



Slope of tangent at $(3,1)$:

$$\left.\frac{dy}{dx}\right|_{\substack{x=3 \\ y=1}} = -\frac{2 \cdot 3}{3 \cdot 1} = -2$$

Equation of tangent at $(3,1)$: $\boxed{y-1 = -2(x-3)}$.

Slope of normal at $(3,1)$ $= -\dfrac{1}{\text{slope of tangent}} = -\dfrac{1}{-2} = \dfrac{1}{2}$

Equation of normal at $(3,1)$: $\boxed{y-1 = \frac{1}{2}(x-3)}$.

3) Find the points on the ellipse $x^2 + xy + 4y^2 = 15$ where the tangent line is a) horizontal and b) vertical.

First, find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left(x^2 + xy + 4y^2 = 15\right)$$

$$2x + y + xy' + 8yy' = 0$$

$$y'(x+8y) = -y - 2x$$

$$y' = -\frac{y+2x}{x+8y}$$

a) To find the points (x,y) where the tangent line is horizontal, we need to solve:

$$\begin{cases} \dfrac{dy}{dx} = 0 \rightarrow & \text{num} = 0 \rightarrow y+2x = 0 \rightarrow y = -2x \\ & \text{denom} \neq 0 \rightarrow x+8y \neq 0 \\ x^2 + xy + 4y^2 = 15 & \end{cases}$$

plug in original equation and solve

$$x^2 + x(-2x) + 4(-2x)^2 = 15$$

$$x^2 - 2x^2 + 16x^2 = 15$$

$$15x^2 = 15 \Rightarrow x^2 = 1 \Rightarrow \begin{array}{c} x = 1 \\ y = -2x = -2 \end{array} \quad \text{or} \quad \begin{array}{c} x = -1 \\ y = -2x = 2 \end{array}$$

We get the points $\boxed{(1,-2) \text{ and } (-1,2)}$

b) To find the points (x,y) where the tangent line is vertical, we need to solve:

$$\begin{cases} \dfrac{dy}{dx} = \text{"}\infty\text{"} \rightarrow \begin{array}{l} \text{num} \neq 0 \longrightarrow y + 2x \neq 0 \\ \text{denom} = 0 \longrightarrow x + 8y = 0 \rightarrow x = -8y \end{array} \\ x^2 + xy + 4y^2 = 15 \end{cases}$$

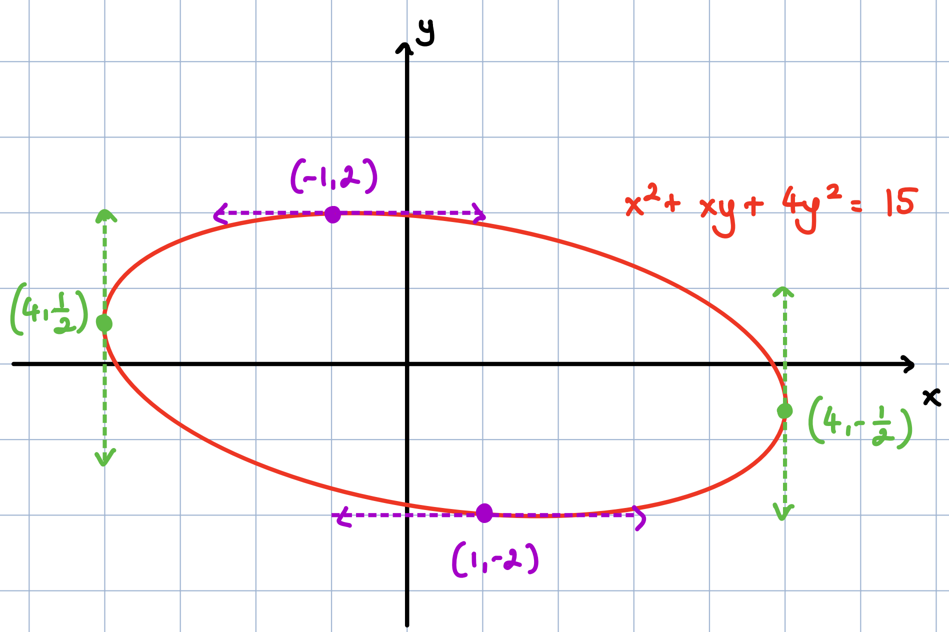
plug in original equation and solve

$$(-8y)^2 + (-8y)y + 4y^2 = 15$$

$$64y^2 - 8y^2 + 4y^2 = 15$$

$$60y^2 = 15 \Rightarrow y^2 = \frac{15}{60} = \frac{1}{4} \Rightarrow \begin{array}{c} y = \frac{1}{2} \\ x = -8y = -4 \end{array} \quad \text{or} \quad \begin{array}{c} y = -\frac{1}{2} \\ x = -8y = 4. \end{array}$$

We get the points $\boxed{\left(-4, \tfrac{1}{2}\right) \text{ and } \left(4, -\tfrac{1}{2}\right)}$



3) Find the points on $x^3 + y^3 = 9xy$ where the tangent line is
a) horizontal and b) vertical.

First find $\frac{dy}{dx}$: $\frac{d}{dx}\left(x^3 + y^3 = 9xy \right.$

$$3x^2 + 3y^2 y' = 9y + 9xy'$$
$$3y^2 y' - 9xy' = 9y - 3x^2$$
$$y'(3y^2 - 9x) = 9y - 3x^2$$
$$y' = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}.$$

a) The tangent line is horizontal when

$$\begin{cases} \frac{dy}{dx} = 0 \rightarrow \begin{array}{l} \text{num.} = 0 \rightarrow 3y - x^2 = 0 \rightarrow y = \frac{x^2}{3} \\ \text{denom.} \neq 0 \end{array} \\ x^3 + y^3 = 9xy \end{cases}$$

plug in original equation and solve

$$x^3 + \left(\frac{x^2}{3}\right)^3 = 9x\frac{x^2}{3}$$
$$x^3 + \frac{x^6}{27} = 3x^3$$
$$\frac{x^6}{27} - 2x^3 = 0$$
$$x^3\left(\frac{x^3}{27} - 2\right) = 0 \longrightarrow$$

$x = 0$

$y = \frac{x^2}{3} = 0$

$\left.\frac{dy}{dx}\right|_{\substack{x=0\\y=0}} = \text{"}\frac{0}{0}\text{"}$ undef. ×

not a solution

$$\frac{x^3}{27} - 2 = 0$$
$$x^3 = 2 \cdot 27$$
$$x = 3\sqrt[3]{2}$$
$$y = \frac{x^2}{3} = 3\sqrt[3]{4}$$

The tangent line is horizontal at the point $\boxed{(3\sqrt[3]{2}, 3\sqrt[3]{4})}$.

b) The tangent line is vertical when

$$\begin{cases} \frac{dy}{dx} = \text{"}\infty\text{"} \rightarrow \begin{array}{l} \text{num.} \neq 0 \\ \text{denom.} = 0 \rightarrow y^2 - 3x = 0 \rightarrow x = \frac{y^2}{3} \end{array} \\ x^3 + y^3 = 9xy \end{cases}$$

plug in original equation and solve

$$\left(\frac{y^2}{3}\right)^3 + y^3 = 9\frac{y^2}{3}y$$

$$\frac{y^6}{27} + y^3 = 3y^3$$

$$\frac{y^6}{27} - 2y^3 = 0$$

$$y^3\left(\frac{y^3}{27} - 2\right) = 0 \Rightarrow \begin{array}{l} y = 0 \\ x = \frac{y^2}{3} = 0 \end{array}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=0\\y=0}} = \text{“}\frac{0}{0}\text{”} \times$$

not a solution

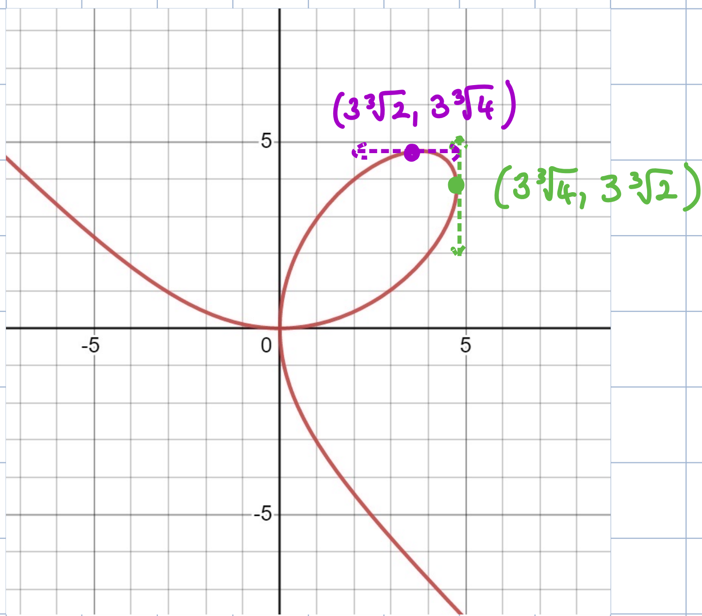
$$\frac{y^3}{27} - 2 = 0$$

$$y^3 = 2 \cdot 27$$

$$y = 3\sqrt[3]{2}$$

$$x = \frac{y^2}{3} = 3\sqrt[3]{4}$$

The tangent line is vertical at the point $\boxed{(3\sqrt[3]{4},\ 3\sqrt[3]{2})}$.



Remarks: • we could have used symmetries to answer b).
The curve is symmetric about $y = x$ since the equation is unchanged by switching x/y.
Since the tangent line at $(3\sqrt[3]{2}, 3\sqrt[3]{4})$ is horizontal, the tangent line at $(3\sqrt[3]{4}, 3\sqrt[3]{2})$ is vertical.

• There is no tangent line at $(0,0)$ because it is a " double point ".

Practice : Find $\frac{dy}{dx}$ for the following curves.

1) $\sqrt{y} = (x + \cot(y))^3$ 2) $\frac{\cos(y)}{2x+5y} = \sec(x)$

1) $\sqrt{y} = (x + \cot(y))^3$

$\frac{d}{dx}$

$$\frac{1}{2\sqrt{y}} y' = 3(x+\cot(y))^2 \frac{d}{dx}(x+\cot(y))$$

$$\frac{1}{2\sqrt{y}} y' = 3(x+\cot(y))^2 (1 - \csc(y)^2 y')$$

$$= 3(x+\cot(y))^2 - 3(x+\cot(y))^2 \csc(y)^2 y'$$

$$\frac{1}{2\sqrt{y}} y' + 3(x+\cot(y))^2 \csc(y)^2 y' = 3(x+\cot(y))^2$$

$$y'\left(\frac{1}{2\sqrt{y}} + 3(x+\cot(y))^2 \csc(y)^2\right) = 3(x+\cot(y))^2$$

$$\boxed{y' = \frac{3(x+\cot(y))^2}{\frac{1}{2\sqrt{y}} + 3(x+\cot(y))^2 \csc(y)^2}}$$

2) $\frac{\cos(y)}{2x+5y} = \sec(x)$

$\frac{d}{dx}$

$$\frac{-\sin(y)y' - \cos(y)(2+5y')}{(2x+5y)^2} = \sec(x)\tan(x)$$

$$-y'(\sin(y) + 5\cos(y)) - 2\cos(y) = \sec(x)\tan(x)$$

$$-y'(\sin(y) + 5\cos(y)) = \sec(x)\tan(x) + 2\cos(y)$$

$$\boxed{y' = -\frac{\sec(x)\tan(x) + 2\cos(y)}{\sin(y) + 5\cos(y)}}$$