Rutgers University
Math 151

Section 3.7: Implicit Differentiation - Worksheet Solutions

1. Calculate $\dfrac{dy}{dx}$ for the following curves.

   (a) $e^{5xy} + 11\tan(x) = y^2$

   Solution. Differentiating both sides with respect to x gives

$$\begin{aligned}
5e^{5xy}(y + xy') + 11\sec(x)^2 &= 2yy' \\
5e^{5xy}xy' - 2yy' &= -11\sec(x)^2 - 5e^{5xy}y \\
(5e^{5xy}x - 2y)y' &= -11\sec(x)^2 - 5e^{5xy}y \\
y' &= \boxed{\frac{-11\sec(x)^2 - 5e^{5xy}y}{5e^{5xy}x - 2y}}
\end{aligned}$$

   (b) $x^3 - x\sin(y) = 3xy$

   Solution. Differentiating both sides with respect to x gives

$$\begin{aligned}
3x^2 - \sin(y) - x\cos(y)y' &= 3y + 3xy' \\
3xy' + x\cos(y)y' &= 3x^2 - \sin(y) - 3y \\
(3x + x\cos(y))y' &= 3x^2 - \sin(y) - 3y \\
y' &= \boxed{\frac{3x^2 - \sin(y) - 3y}{3x + x\cos(y)}}
\end{aligned}$$

   (c) $\sqrt{x^2 + y^2} = 3^y$

   Solution. Differentiating both sides with respect to x gives

$$\begin{aligned}
\frac{2x + 2yy'}{2\sqrt{x^2 + y^2}} &= \ln(3)3^y y' \\
x + yy' &= \ln(3)3^y\sqrt{x^2 + y^2}\,y' \\
\ln(3)3^y\sqrt{x^2 + y^2}\,y' - yy' &= x \\
\left(\ln(3)3^y\sqrt{x^2 + y^2} - y\right)y' &= x \\
y' &= \boxed{\frac{x}{\ln(3)3^y\sqrt{x^2 + y^2} - y}}
\end{aligned}$$

(d) $x^4 + 6xy^2 + 5y^3 = 0$

Solution. Differentiating both sides with respect to x gives

$$\begin{aligned} 4x^3 + 6y^2 + 12xyy' + 15y^2y' &= 0 \\ \left(12xy + 15y^2\right) y' &= -4x^3 - 6y^2 \\ y' &= \boxed{\frac{-4x^3 - 6y^2}{12xy + 15y^2}} \end{aligned}$$

2. Consider the curve of equation $x^2 + 6xy - y^2 = 40$. Find the points on the curve, if any, where the tangent line is (a) horizontal, (b) vertical, (c) [**Advanced**] perpendicular to $y = 2x + 9$.

Solution. First, let us differentiate the relation with respect to x:

$$\begin{aligned} 2x + 6y + 6xy' - 2yy' &= 0 \\ x + 3y + 3xy' - yy' &= 0. \end{aligned}$$

(a) The tangent line is horizontal when $y' = 0$. Using this in the previous equation, we get $x + 3y = 0$, or $x = -3y$. Plugging this in the equation of the curve gives $(-3y)^2 + 6(-3y)y - y^2 = 40$, or $-10y^2 = 40$. This equation has no solution, so there are no points on the curve where the tangent line is horizontal.

(b) Solving for y' in the previous equation gives $y' = -\dfrac{x + 3y}{3x - y}$, so the tangent line is vertical when $y = 3x$. Plugging this in the equation of the curve gives $x^2 + 6x(3x) - (3x)^2 = 40$, or $10x^2 = 40$. We get $x^2 = 4$, that is $x = 2$ (which gives $y = 6$) and $x = -2$ (which gives $y = -6$). So the points where the tangent line is vertical are $\boxed{(2, 6), (-2, -6)}$.

(c) The tangent line is perpendicular to $y = 2x+9$ when $y' = -\frac{1}{2}$. Plugging this in $2x+6y+6xy'-2yy' = 0$ gives $2x + 6y - 3x + y = 0$, or $x = 7y$. Substituting $x = 7y$ in the equation of the curve gives

$$\begin{aligned} (7y)^2 + 6(7y)y - y^2 &= 40 \\ 90y^2 &= 40 \\ y^2 &= \frac{4}{9} \\ y &= \pm\frac{2}{3}. \end{aligned}$$

For $y = \frac{2}{3}$, we get $x = \frac{14}{3}$ and for $y = -\frac{2}{3}$, we get $x = -\frac{14}{3}$. Therefore, the points on the curve where the tangent line is perpendicular to $y = 2x + 9$ are $\boxed{\left(\frac{14}{3}, \frac{2}{3}\right), \left(-\frac{14}{3}, -\frac{2}{3}\right)}$.