Sections 3.8, 3.9

# Derivatives of Inverse Functions

## Learning Goals

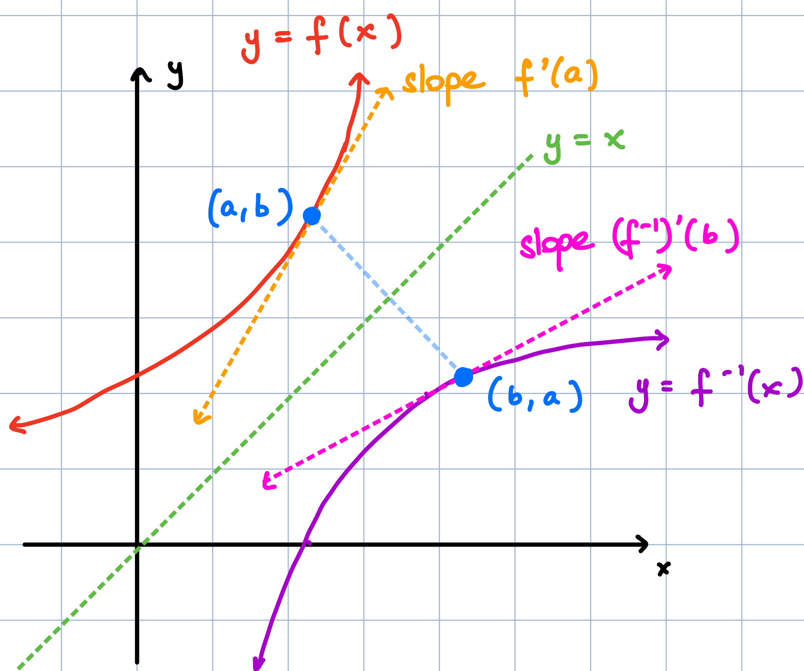
| *Learning Goal* | *Homework Problems* |
|---|---|
| 3.8.1 Understand how the derivatives of a function and its inverse behave graphically. Use Theorem 3 to compute the derivative of an inverse function, or to compute the derivative of the inverse function at a given point $x = f(a)$. | 1-10, 101, 105-114. |
| 3.8.2 Know the formulas for the derivatives of logarithmic and exponential functions of any base. Use these formulas to compute derivatives of related functions. | 11-40, 55-88, 95, 96, 98, 100. |
| 3.8.3 Use logarithmic differentiation to compute derivatives. Recognize when this technique is helpful, and when it is necessary. | 41-54, 89-100. |
| 3.8.4 Answer conceptual questions involving inverse functions and logarithms. | 9, 10, 104. |
| *Learning Goal* | *Homework Problems* |
| 3.9.1 Compute angles in a right triangle using inverse trigonometric functions. | 47-49. |
| 3.9.2 Use special values or information about the graphs of the six basic trigonometric functions to compute special values or limits of their inverses. | 1-20, 51-54. |
| 3.9.3 Use trigonometric identities and the methods of §3.8 to find formulas for the derivatives of the six basic trigonometric functions. | 55-58. |
| 3.9.4 Know the derivatives of the six basic trigonometric functions and use them to compute related derivatives. | 21-46. |
| 3.9.5 Answer conceptual questions involving the inverse trigonometric functions and their derivatives. | 49-54, 59, 60, 63-70. |

Conceptual introduction: suppose that $f$ is a one-to-one differentiable function. How can we find the derivative of the inverse function $f^{-1}$?

⚠ $f^{-1}(x) \neq \frac{1}{f(x)}$

The inverse function $f^{-1}$ is the "reverse assignment" of $f$, i.e: $f(a) = b \iff f^{-1}(b) = a$.

Reminder: the graph of $f^{-1}$ is the symmetric of the graph of $f$ about the line $y = x$.




The tangent line to $f^{-1}$ at $(b,a)$ is the symmetric to the tangent line to $f$ at $(a,b)$ about $y = x$.

$\Rightarrow$ The slopes are reciprocal ($x/y$ switched).

So for any point $(b,a) = (x, f^{-1}(x))$ on the graph of $f^{-1}$:

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

or

$$\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}$$

## Proof of formula with implicit differentiation:

$y = f^{-1}(x)$, want to find $\frac{dy}{dx} = y'$.

$$\hookrightarrow \quad f(y) = x$$

$$\frac{d}{dx}: \quad f'(y)\, y' = 1$$

$$y' = \frac{1}{f'(y)}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}}.$$

## Derivatives of common inverse functions to know:

(see below for full derivation)

- Logarithmic Functions:

$$\boxed{\frac{d}{dx}(\ln(x)) = \frac{1}{x}} \qquad \boxed{\frac{d}{dx}(\log_a(x)) = \frac{1}{\ln(a)\, x}}$$

- Inverse Trigonometric Functions:

$$\boxed{\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}} \qquad \boxed{\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}} \qquad \boxed{\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{|x|\sqrt{x^2-1}}}$$

$$\boxed{\frac{d}{dx}(\cot^{-1}(x)) = -\frac{1}{1+x^2}} \qquad \boxed{\frac{d}{dx}(\csc^{-1}(x)) = -\frac{1}{|x|\sqrt{x^2-1}}}$$

**Examples:** 1) Find the derivatives of $\ln(x)$ and $\log_a(x)$ for $a > 0$.

If $f(x) = e^x$, $f^{-1}(x) = \ln(x)$ and $f'(x) = e^x$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

So $\boxed{\frac{d}{dx}(\ln(x)) = \frac{1}{x}}$ * memorize

If $f(x) = a^x$, $f^{-1}(x) = \log_a(x)$ and $f'(x) = \ln(a)a^x$.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\ln(a)a^{\log_a(x)}} = \frac{1}{\ln(a)x}$$

So $\boxed{\frac{d}{dx}(\log_a(x)) = \frac{1}{\ln(a)x}}$ * memorize

**Remarks:** • the second formula also follows from $\log_a(x) = \frac{\ln(x)}{\ln(a)}$

$$\frac{d}{dx}(\log_a(x)) = \frac{d}{dx}\left(\frac{\ln(x)}{\ln(a)}\right) = \frac{1}{\ln(a)} \frac{d}{dx}(\ln(x)) = \frac{1}{\ln(a)} \cdot \frac{1}{x} = \frac{1}{\ln(a)x}.$$

• We can also use implicit differentiation to find these derivatives.

If $y = \ln(x)$, then $e^y = x$ and we want to find $\frac{dy}{dx}$.

$$\frac{d}{dx}\quad e^y = x$$

$$e^y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{e^y} \quad \text{and} \quad e^y = x$$

$$\boxed{\frac{d}{dx}(\ln(x)) = \frac{1}{x}}$$

2) Find the derivatives of $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, $\sec^{-1}$, $\cot^{-1}$ and $\csc^{-1}$.

- $\theta = \sin^{-1}(x) = \arcsin(x)$ is the angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin(\theta) = x$. We want to find $\frac{d\theta}{dx}$.

$$\sin(\theta) = x$$

$$\frac{d}{dx}: \quad \cos(\theta)\frac{d\theta}{dx} = 1 \Rightarrow \frac{d\theta}{dx} = \frac{1}{\cos(\theta)}$$

To express $\cos(\theta)$ in terms of $x$, use Pythagorean identity $\cos(\theta)^2 = 1 - \sin(\theta)^2 = 1 - x^2$ since $\sin(\theta) = x$.

$$\Rightarrow \cos(\theta) = \pm\sqrt{1-x^2} = \sqrt{1-x^2}$$

$\theta$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, so $\cos(\theta) \geq 0$.

So $\boxed{\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}}$.

* memorize

- $\theta = \cos^{-1}(x) = \arccos(x)$ is the angle in $[0, \pi]$ such that $\cos(\theta) = x$. We want to find $\frac{d\theta}{dx}$.

$$\cos(\theta) = x$$

$$\frac{d}{dx}: \quad -\sin(\theta)\frac{d\theta}{dx} = 1 \Rightarrow \frac{d\theta}{dx} = -\frac{1}{\sin(\theta)}$$

To express $\sin(\theta)$ in terms of $x$, we use the Pythagorean identity $\sin(\theta)^2 = 1 - \cos(\theta)^2 = 1 - x^2$ since $\cos(\theta) = x$.

$$\Rightarrow \sin(\theta) = \pm\sqrt{1-x^2} = \sqrt{1-x^2}.$$

$\theta$ in $[0, \pi]$ so $\cos(\theta) \geq 0$

So $\boxed{\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}}$

* memorize

• $\theta = \tan^{-1}(x) = \arctan(x)$ is the angle in $(-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\tan(\theta) = x$. We want to find $\frac{d\theta}{dx}$.

$$\tan(\theta) = x$$

$$\frac{d}{dx} \qquad \sec(\theta)^2 \frac{d\theta}{dx} = 1 \Rightarrow \frac{d\theta}{dx} = \frac{1}{\sec(\theta)^2}$$

To express $\sec(\theta)^2$ in terms of $x$, use Pythagorean identity $\sec(\theta)^2 = 1 + \tan(\theta)^2 = 1 + x^2$ since $\tan(\theta) = x$.

So $\boxed{\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}}$.

* memorize

• $\theta = \sec^{-1}(x) = \text{arcsec}(x)$ is the angle in $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ such that $\sec(\theta) = x$. We want to find $\frac{d\theta}{dx}$.

$$\sec(\theta) = x$$

$$\frac{d}{dx} \qquad \sec(\theta)\tan(\theta) \frac{d\theta}{dx} = 1 \Rightarrow \frac{d\theta}{dx} = \frac{1}{\sec(\theta)\tan(\theta)}$$

We know $\sec(\theta) = x$. To express $\tan(\theta)$ in terms of $x$, we use the Pythagorean identity $\tan(\theta)^2 = \sec(\theta)^2 - 1 = x^2 - 1$

So $\tan(\theta) = \pm\sqrt{x^2-1}$

when $\theta$ in $[0, \frac{\pi}{2})$, $\tan(\theta) \geq 0$ so $+$

when $\theta$ in $(\frac{\pi}{2}, \pi]$, $\tan(\theta) \leq 0$ so $-$

So $$\frac{d\theta}{dx} = \begin{cases} \frac{1}{x\sqrt{x^2-1}} & \text{if } \theta \text{ in } [0, \frac{\pi}{2}), \text{ i.e. } \sec(\theta) = x > 0. \\ -\frac{1}{x\sqrt{x^2-1}} & \text{if } \theta \text{ in } (\frac{\pi}{2}, \pi], \text{ i.e. } \sec(\theta) = x < 0. \end{cases}$$

So $\boxed{\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{|x|\sqrt{x^2-1}}}$.

* memorize

- $\theta = \cot^{-1}(x) = \operatorname{arccot}(x)$ is the angle in $(0, \pi)$ such that $\cot(\theta) = x$. We want to find $\frac{d\theta}{dx}$.

$$\frac{d}{dx}\Big( \quad \cot(\theta) = x$$
$$-\csc(\theta)^2 \frac{d\theta}{dx} = 1 \Rightarrow \frac{d\theta}{dx} = -\frac{1}{\csc(\theta)^2}$$

To express $\csc(\theta)^2$ in terms of $x$, use Pythagorean identity $\csc(\theta)^2 = 1 + \cot(\theta)^2 = 1 + x^2$ since $\cot(\theta) = x$.

So $\boxed{\frac{d}{dx}\left(\cot^{-1}(x)\right) = -\frac{1}{1+x^2}}$.

* memorize

- $\theta = \csc^{-1}(x) = \operatorname{arccsc}(x)$ is the angle in $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ such that $\csc(\theta) = x$. We want to find $\frac{d\theta}{dx}$.

$$\frac{d}{dx}\Big( \quad \csc(\theta) = x$$
$$-\csc(\theta)\cot(\theta)\frac{d\theta}{dx} = 1 \Rightarrow \frac{d\theta}{dx} = -\frac{1}{\csc(\theta)\cot(\theta)}$$

We know $\csc(\theta) = x$. To express $\cot(\theta)$ in terms of $x$, we use the Pythagorean identity $\cot(\theta)^2 = \csc(\theta)^2 - 1 = x^2 - 1$

So $\cot(\theta) = \pm\sqrt{x^2-1}$

when $\theta$ in $(0, \frac{\pi}{2}]$, $\cot(\theta) \geq 0$ so $+$

when $\theta$ in $[-\frac{\pi}{2}, 0)$, $\cot(\theta) \leq 0$ so $-$

So $$\frac{d\theta}{dx} = \begin{cases} \frac{1}{x\sqrt{x^2-1}} & \text{if } \theta \text{ in } [-\frac{\pi}{2}, 0), \text{ i.e. } \csc(\theta) = x < 0 \\ -\frac{1}{x\sqrt{x^2-1}} & \text{if } \theta \text{ in } (0, \frac{\pi}{2}], \text{ i.e. } \csc(\theta) = x > 0 \end{cases}$$

So $\boxed{\frac{d}{dx}\left(\csc^{-1}(x)\right) = -\frac{1}{|x|\sqrt{x^2-1}}}$.

* memorize

3) Let $f(x) = x^3 + 2x + 7$. Find an equation of the tangent line to $y = f^{-1}(x)$ at $x = 7$.

First, find $f^{-1}(7)$. For this we need to solve $f(x) = 7$

$$x^3 + 2x + 7 = 7$$

$$x^3 + 2x = 0$$

$$x(x^2 + 2) = 0$$

$x = 0$ or $x^2 + 2 = 0$
no solution

$\Rightarrow f^{-1}(7) = 0$

Next, find $(f^{-1})'(7)$. We know that:

$$(f^{-1})'(7) = \frac{1}{f'(f^{-1}(7))}$$

$f'(x) = 3x^2 + 2$
$f^{-1}(7) = 0$

$$= \frac{1}{f'(0)} = \frac{1}{2}.$$

So the tangent line to $y = f^{-1}(x)$ at $x = 7$:

$\begin{cases} \text{passes through } (7,0) \\ \text{has slope } \frac{1}{2} \end{cases} \Rightarrow$ equation $\boxed{y = \frac{1}{2}(x - 7)}$

4) Suppose that the tangent line to $y = f(x)$ at $x = 2$ has equation $y = -3x + 7$. Find an equation of the tangent line to $y = f^{-1}(x)$ at $x = f(2)$.

First find the point $(f(2), f^{-1}(f(2))) = (f(2), 2)$

$f(2) = -3(2) + 7 = -6 + 7 = 1.$

So the tangent line passes through $(1,2)$.

Slope: $(f^{-1})'(f(2)) = \dfrac{1}{f'(2)} = -\dfrac{1}{3}.$

So the equation is $\boxed{y = -\frac{1}{3}(x-1) + 2}$

5) Calculate the derivatives of the following functions.

a) $f(x) = \ln\left(\tan^{-1}(5x)\right)$

b) $g(x) = \sec^{-1}\left(\frac{2}{x}\right)$

c) $h(x) = \sin^{-1}(7-x)^2 + 5\sqrt{\ln(x)}$

d) $k(x) = \dfrac{1}{x\ln(x)} + \ln(11).$

a) $f'(x) = \dfrac{d}{dx}\left(\ln\left(\tan^{-1}(5x)\right)\right)$

$$= \frac{1}{\tan^{-1}(5x)} \cdot \frac{1}{1+(5x)^2} \cdot 5 = \boxed{\frac{5}{\tan^{-1}(5x)(1+25x^2)}}$$

b) $g'(x) = \dfrac{d}{dx}\left(\sec^{-1}\left(\frac{2}{x}\right)\right) = \dfrac{1}{\left|\frac{2}{x}\right|\sqrt{\left(\frac{2}{x}\right)^2-1}} \cdot \left(-\dfrac{2}{x^2}\right) = -\dfrac{1}{|x|\sqrt{\frac{4}{x^2}-1}}$

$$= \boxed{-\frac{1}{\sqrt{4-x^2}}}$$

c) $h'(x) = \dfrac{d}{dx}\left(\sin^{-1}(7-x)^2 + 5\sqrt{\ln(x)}\right)$

$$= \boxed{2\sin^{-1}(7-x)\frac{1}{\sqrt{1-(7-x)^2}}(-1) + \frac{5}{2\sqrt{\ln(x)}\,x}}.$$

d) $k'(x) = \dfrac{d}{dx}\left(\dfrac{1}{x\ln(x)} + \overbrace{\ln(11)}^{\text{constant}}\right) = -\dfrac{\frac{d}{dx}(x\ln(x))}{(x\ln(x))^2}$

$$= -\frac{\ln(x) + x\frac{1}{x}}{(x\ln(x))^2} = \boxed{-\frac{\ln(x)+1}{(x\ln(x))^2}}.$$

**Logarithmic Differentiation:** method to compute derivatives for functions involving many factors or exponents, or with base and exponents both depending on $x$.

Basic example: $y = x^x$, calculate $\frac{dy}{dx}$.

⚠ This is neither an exponential (base depends on $x$) nor a power (exponent depends on $x$). So we cannot use any of the basic rules.

With logarithmic differentiation:

$$y = x^x$$

Step 1: take ln

$$\ln(y) = \ln(x^x)$$

Step 2: simplify

$$\ln(y) = x\ln(x)$$

$$\frac{1}{y}y' = \ln(x) + x\frac{1}{x} = \ln(x)+1.$$

Step 3: solve for $y'$

$$y' = y(\ln(x)+1)$$

Step 4: replace $y$

$$\boxed{y' = x^x(\ln(x)+1)}$$

**Remark:** we could also use properties of logs to write $y = e^{x\ln(x)}$ and use the chain rule.

**Examples:** compute the derivatives of the following functions using logarithmic differentiation.

1) $y = (3x-1)^{\sqrt{x}}$

2) $y = \dfrac{(x^2-1)^{17}\sqrt{x-3}}{(x+1)^{44}}$

3) $y = (1-5x)^{\sin^{-1}(3x)}$

4) $y = \dfrac{x\csc^{-1}(x)}{\cot(7x)^2}$.

1) $y = (3x-1)^{\sqrt{x}}$

$$\frac{d}{dx}\Big( \ln(y) = \ln\left((3x-1)^{\sqrt{x}}\right) = \sqrt{x}\ln(3x-1)$$

$$\frac{1}{y}y' = \frac{1}{2\sqrt{x}}\ln(3x-1) + \sqrt{x}\cdot\frac{3}{3x-1} = \frac{\ln(3x-1)}{2\sqrt{x}} + \frac{3\sqrt{x}}{3x-1}$$

$$y' = y\left(\frac{\ln(3x-1)}{2\sqrt{x}} + \frac{3\sqrt{x}}{3x-1}\right) = \boxed{(3x-1)^{\sqrt{x}}\left(\frac{\ln(3x-1)}{2\sqrt{x}} + \frac{3\sqrt{x}}{3x-1}\right)}.$$

2) $y = \dfrac{(x^2-1)^{17}\sqrt{x-3}}{(x+1)^{44}}$

$$\frac{d}{dx}\Big( \ln(y) = \ln\left(\frac{(x^2-1)^{17}\sqrt{x-3}}{(x+1)^{44}}\right) = 17\ln(x^2-1) + \frac{1}{2}\ln(x-3) - 44\ln(x+1)$$

$$\frac{1}{y}y' = \frac{34x}{x^2-1} + \frac{1}{2(x-3)} - \frac{44}{x+1}$$

$$y' = y\left(\frac{34x}{x^2-1} + \frac{1}{2(x-3)} - \frac{44}{x+1}\right)$$

$$= \boxed{\frac{(x^2-1)^{17}\sqrt{x-3}}{(x+1)^{44}}\left(\frac{34x}{x^2-1} + \frac{1}{2(x-3)} - \frac{44}{x+1}\right)}$$

3) $y = (1-5x)^{\sin^{-1}(3x)}$

$$\frac{d}{dx}\Big( \ln(y) = \ln\left((1-5x)^{\sin^{-1}(3x)}\right) = \sin^{-1}(3x)\ln(1-5x)$$

$$\frac{1}{y}y' = \frac{3}{\sqrt{1-9x^2}}\ln(1-5x) + \sin^{-1}(3x)\frac{-5}{1-5x} = \frac{3\ln(1-5x)}{\sqrt{1-9x^2}} - \frac{5\sin^{-1}(3x)}{1-5x}$$

$$y' = y\left(\frac{3\ln(1-5x)}{\sqrt{1-9x^2}} - \frac{5\sin^{-1}(3x)}{1-5x}\right)$$

$$= \boxed{(1-5x)^{\sin^{-1}(3x)}\left(\frac{3\ln(1-5x)}{\sqrt{1-9x^2}} - \frac{5\sin^{-1}(3x)}{1-5x}\right)}$$

4) $$y = \frac{x\csc^{-1}(x)}{\cot(7x)^2}.$$

$$\ln(y) = \ln\left(\frac{x\csc^{-1}(x)}{\cot(7x)^2}\right) = \ln(x) + \ln(\csc^{-1}(x)) - 2\ln(\cot(7x))$$

$\frac{d}{dx}$

$$\frac{1}{y}y' = \frac{1}{x} - \frac{1}{|x|\sqrt{x^2-1}\,\csc^{-1}(x)} + \frac{14\csc(7x)^2}{\cot(7x)}$$

$$y' = y\left(\frac{1}{x} - \frac{1}{|x|\sqrt{x^2-1}\,\csc^{-1}(x)} + \frac{14\csc(7x)^2}{\cot(7x)}\right)$$

$$= \boxed{\frac{x\csc^{-1}(x)}{\cot(7x)^2}\left(\frac{1}{x} - \frac{1}{|x|\sqrt{x^2-1}\,\csc^{-1}(x)} + \frac{14\csc(7x)^2}{\cot(7x)}\right)}.$$