Rutgers University
Math 151

Sections 3.8-9: Derivatives of Inverse Functions - Worksheet Solutions

1. Calculate the derivatives of the following functions.

(a) $f(x) = \sin^{-1}(4x)$

Solution.

$$f'(x) = \frac{1}{\sqrt{1-(4x)^2}} \cdot 4 = \boxed{\frac{4}{\sqrt{1-16x^2}}}.$$

(b) $f(x) = \ln(2\arctan(5x) + 1)$

Solution.

$$f'(x) = \frac{1}{2\arctan(5x)+1} \cdot 2\frac{1}{1+(5x)^2} \cdot 5 = \boxed{\frac{10}{(2\arctan(5x)+1)(1+25x^2)}}.$$

(c) $f(x) = x\sec^{-1}(7x)$

Solution.

$$f'(x) = \sec^{-1}(7x) + x\frac{1}{|7x|\sqrt{(7x)^2-1}} \cdot 7 = \boxed{\sec^{-1}(7x) + \frac{7x}{|7x|\sqrt{49x^2-1}}}.$$

(d) $f(x) = \ln(x)^2 + 8\arccos(-x)$

Solution.

$$f'(x) = 2\ln(x) \cdot \frac{1}{x} + 8 \cdot -\frac{1}{\sqrt{1-(-x)^2}} \cdot -1 = \boxed{\frac{2\ln(x)}{x} + \frac{8}{\sqrt{1-x^2}}}$$

(e) $f(x) = \cot^{-1}(e^{3x})$

Solution.

$$f'(x) = -\frac{1}{1+(e^{3x})^2} \cdot e^{3x} \cdot 3 = \boxed{-\frac{3e^{3x}}{1+e^{6x}}}.$$

(f) $f(x) = \cos(x)\log_7(\sec(x))$

Solution.

$$f'(x) = -\sin(x)\log_7(\sec(x)) + \cos(x)\cdot\frac{1}{\ln(7)\sec(x)}\cdot\sec(x)\tan(x) = \boxed{\sin(x)\log_7(\sec(x)) + \frac{\sin(x)}{\ln(7)}}.$$

(g) $f(x) = x^{3\tan^{-1}(2x)}$

Solution. With $y = x^{3\tan^{-1}(2x)}$ we have

$$\ln(y) = \ln\left(x^{3\tan^{-1}(2x)}\right) = 3\tan^{-1}(2x)\ln(x).$$

Differentiating with respect to x, we obtain

$$\begin{aligned} \frac{y'}{y} &= \frac{6\ln(x)}{1+4x^2} + \frac{3\tan^{-1}(2x)}{x} \\ \Rightarrow\ y' &= y\left(\frac{6\ln(x)}{1+4x^2} + \frac{3\tan^{-1}(2x)}{x}\right) \\ &= \boxed{x^{3\tan^{-1}(2x)}\left(\frac{6\ln(x)}{1+4x^2} + \frac{3\tan^{-1}(2x)}{x}\right)} \end{aligned}$$

(h) $f(x) = \cos(x)^{\ln(x)}$

Solution. With $y = \cos(x)^{\ln(x)}$ we have

$$\ln(y) = \ln\left(\cos(x)^{\ln(x)}\right) = \ln(x)\ln(\cos(x)).$$

Differentiating with respect to x, we obtain

$$\begin{aligned} \frac{y'}{y} &= \frac{\ln(\cos(x))}{x} + \ln(x)\frac{-\sin(x)}{\cos(x)} \\ &= \frac{\ln(\cos(x))}{x} - \ln(x)\tan(x) \\ \Rightarrow\ y' &= y\left(\frac{\ln(\cos(x))}{x} - \ln(x)\tan(x)\right) \\ &= \boxed{\cos(x)^{\ln(x)}\left(\frac{\ln(\cos(x))}{x} - \ln(x)\tan(x)\right)} \end{aligned}$$

(i) $f(x) = (1-5x)^{x^2}$

Solution. With $y = (1-5x)^{x^2}$ we have

$$\ln(y) = \ln\left((1-5x)^{x^2}\right) = x^2\ln(1-5x).$$

Differentiating with respect to x, we obtain

$$\begin{aligned} \frac{y'}{y} &= 2x\ln(1-5x) + x^2\frac{-5}{1-5x} \\ &= 2x\ln(1-5x) - \frac{5x^2}{1-5x} \\ \Rightarrow\ y' &= y\left(2x\ln(1-5x) - \frac{5x^2}{1-5x}\right) \\ &= \boxed{(1-5x)^{x^2}\left(2x\ln(1-5x) - \frac{5x^2}{1-5x}\right)} \end{aligned}$$

2. Simplify each of the following. Your answer should not contain any trigonometric or inverse trigonometric functions.

   (a) $\cos\left(\sin^{-1}(x+1)\right)$

   Solution. We use the Pythagorean identity $\cos(\theta)^2+\sin(\theta)^2 = 1$ with $\theta = \sin^{-1}(x+1)$. By definition of $\sin^{-1}$, we know that $\sin(\theta) = x+1$ and θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We get

$$\begin{aligned} &\cos(\theta)^2 + (x+1)^2 = 1 \\ &\cos(\theta)^2 = 1-(x+1)^2 = -2x-x^2 \\ &\sqrt{\cos(\theta)^2} = \sqrt{-2x-x^2} \\ &|\cos(\theta)| = \sqrt{-2x-x^2} \\ &\cos(\theta) = \pm\sqrt{-2x-x^2} \end{aligned}$$

   To determine which sign is appropriate, recall that $\theta = \sin^{-1}(x+1)$ is an angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\cos(\theta) \geqslant 0$. Hence

$$\boxed{\cos(\sin^{-1}(x)) = \sqrt{-2x-x^2}}.$$

   (b) $\sin(2\cos^{-1}(3x))$

   Solution. We start by using the identity

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

   with $\theta = \cos^{-1}(3x)$. This means that $\cos(\theta) = 3x$ and that θ is in $[0,\pi]$. To find $\sin(\theta)$, we use the Pythagorean identity $\cos(\theta)^2+\sin(\theta)^2 = 1$, which gives

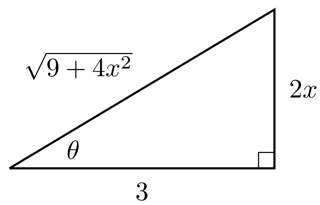
$$\begin{aligned} &(3x)^2 + \sin(\theta)^2 = 1 \\ &\sin(\theta)^2 = 1-9x^2 \\ &\sqrt{\sin(\theta)^2} = \sqrt{1-9x^2} \\ &|\sin(\theta)| = \sqrt{1-9x^2} \\ &\sin(\theta) = \sqrt{1-9x^2} \text{ since } \sin(\theta) > 0 \text{ as } 0 \leqslant \theta \leqslant \pi. \end{aligned}$$

   Therefore

$$\sin(2\cos^{-1}(3x)) = 2\sqrt{1-9x^2}(3x) = \boxed{6x\sqrt{1-9x^2}}.$$

(c) $\csc\left(\tan^{-1}\left(\frac{2x}{3}\right)\right)$

Solution. Let us solve this one with a right triangle. Consider a right triangle with base angle $\theta = \tan^{-1}\left(\frac{2x}{3}\right)$. Then $\tan(\theta) = \frac{2x}{3}$, so we can take the opposite side to be $2x$ and the adjacent to be 3. By the Pythagorean identity, the hypotenuse is $\sqrt{9+4x^2}$.



We get

$$\boxed{\csc\left(\tan^{-1}\left(\frac{2x}{3}\right)\right) = \frac{\sqrt{9+4x^2}}{2x}}.$$

Remark: in general, this method only yields the correct answer up to a sign. Here however, there is no sign issue as θ is an angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and csc and tan have the same sign on this interval, which is also that of x.

(d) $\sec(\theta)$ given that $\cot(\theta) = 5$ and $\sin(\theta) < 0$

Solution. We use the Pythagorean identity $\sec(\theta)^2 = 1 + \tan(\theta)^2$, which gives

$$\begin{aligned}
\sec(\theta)^2 &= 1 + \frac{1}{\cot(\theta)^2} = 1 + \frac{1}{25} = \frac{26}{25} \\
\sqrt{\sec(\theta)^2} &= \sqrt{\frac{26}{25}} \\
|\sec(\theta)| &= \frac{2\sqrt{6}}{25} \\
\sec(\theta) &= \pm\frac{2\sqrt{6}}{25}
\end{aligned}$$

To find the appropriate sign, observe that $\cot(\theta) > 0$ and $\sin(\theta) < 0$, which means that θ is an angle in quadrant III. Therefore, $\sec(\theta) < 0$. So

$$\boxed{\sec(\theta) = -\frac{2\sqrt{6}}{5}}.$$

3. Suppose that f is a one-to-one function and that the tangent line to the graph of $y = f(x)$ at $x = 3$ is $y = -4x + 5$. Find an equation of the tangent line to the graph of $y = f^{-1}(x)$ at $x = f(3)$.

Solution. We have $f(3) = -4 \cdot 3 + 5 = -7$ and $f'(3) = -4$. So

$$f^{-1}(-7) = 3, \qquad (f^{-1})'(-7) = \frac{1}{f'(f^{-1}(-7))}\frac{1}{f'(3)} = -\frac{1}{4}.$$

Hence, the tangent line has equation $\boxed{y = -\frac{1}{4}(x+7) - 3}$.

4. Consider the one-to-one function $f(x) = 3xe^{x^2-4}$. Calculate $f(2)$ and find an equation of the tangent line to the graph of $y = f^{-1}(x)$ at $x = f(2)$.

Solution. We have
$$f(2) = 3 \cdot 2e^{2^2-4} = \boxed{6}.$$
So $f^{-1}(6) = 2$. To find $(f^{-1})'(6)$, we will need $f'(2)$. We have
$$f'(2) = \left[3e^{x^2-4} + 3xe^{x^2-4}2x\right]_{|x=2} = 27.$$
So
$$f^{-1}(6) = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(2)} = \frac{1}{27}.$$
Hence the tangent line has equation $\boxed{y = \frac{1}{27}(x-6)+2}$.

5. Suppose that f and g are differentiable functions such that

$$\begin{array}{lll} f(-1) = 9, & f(0) = 2, & f(1) = 4, \\ f'(-1) = 3, & f'(0) = -5, & f'(1) = 8, \\ g(-1) = 2, & g(0) = 3, & g(1) = -2, \\ g'(-1) = 7, & g'(0) = -4, & g'(1) = 6. \end{array}$$

(a) For $F(x) = \ln\big(f(x^2) + g(x)\big)$, evaluate $F'(-1)$.

Solution. We have
$$F'(x) = \frac{1}{f(x^2)+g(x)} \cdot (f'(x^2)2x + g'(x)) = \frac{2f'(x^2)x + g'(x)}{f(x^2)+g(x)}.$$
So
$$F'(-1) = \frac{2f'((-1)^2)(-1) + g'(-1)}{f((-1)^2) + g(-1)} = \frac{-2f'(1) + g'(-1)}{f(1) + g(-1)} = \frac{-2 \cdot 8 + 7}{4+2} = \boxed{-\frac{3}{2}}.$$

(b) For $G(x) = \arctan\left(3\sqrt{f(x)}\right)$, evaluate $G'(1)$.

Solution. We have
$$G'(x) = \frac{1}{1 + \left(3\sqrt{f(x)}\right)^2} \cdot \frac{3}{2\sqrt{f(x)}} \cdot f'(x) = \frac{3f'(x)}{2\sqrt{f(x)}(1+9f(x))}.$$
So
$$G'(1) = \frac{3f'(1)}{2\sqrt{f(1)}(1+9f(1))} = \frac{3 \cdot 8}{2\sqrt{4}(1+9 \cdot 4)} = \boxed{\frac{6}{37}}.$$

(c) For $H(x) = 2^{f(x)} g(3x+1)$, evaluate $H'(0)$.

Solution. We have

$$H'(x) = \ln(2)2^{f(x)} f'(x) g(3x+1) + 2^{f(x)} g'(3x+1) \cdot 3 = 2^{f(x)} \left(\ln(2) f'(x) g(3x+1) + 3 f(x) g'(3x+1) \right).$$

So

$$H'(0) = 2^{f(0)} \left(\ln(2) f'(0) g(1) + 3 f(0) g'(1) \right) = 2^2 \left(\ln(2)(-5)(-2) + 3 \cdot 2 \cdot 6 \right) = \boxed{8\left(5\ln(2) + 18\right)}.$$

(d) **[Advanced]** For $K(x) = f(2x)^{g(x)}$, evaluate $K'(0)$.

Solution. We have

$$\ln(K(x)) = \ln\left(f(2x)^{g(x)} \right) = g(x) \ln(f(2x)).$$

Taking derivatives with respect to x, we get

$$\begin{aligned} \frac{K'(x)}{K(x)} &= g'(x) \ln(f(2x)) + g(x) \frac{1}{f(2x)} \cdot f'(2x) \cdot 2 \\ &= g'(x) \ln(f(2x)) + \frac{2 g(x) f'(2x)}{f(2x)} \\ \Rightarrow \; K'(x) &= K(x) \left(g'(x) \ln(f(2x)) + \frac{2 g(x) f'(2x)}{f(2x)} \right) \\ &= f(2x)^{g(x)} \left(g'(x) \ln(f(2x)) + \frac{2 g(x) f'(2x)}{f(2x)} \right). \end{aligned}$$

So

$$K'(0) = f(0)^{g(0)} \left(g'(0) \ln(f(0)) + \frac{2 g(0) f'(0)}{f(0)} \right) = 2^3 \left(-4\ln(2) + \frac{2 \cdot 3(-5)}{2} \right) = \boxed{-8(4\ln(2) + 15)}.$$