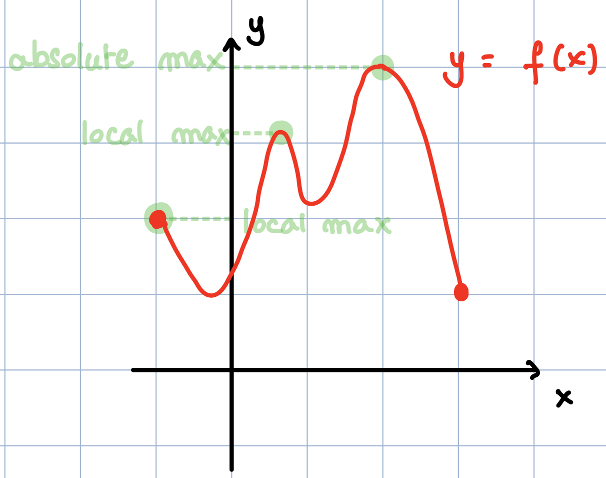
Section 4.1

Extreme Values

Learning Goals

Learning Goal	*Homework Problems*
4.1.1 Find extreme values and where they occur using the graph of the function.	1-20, 79-82.
4.1.2 Find absolute extrema and where they occur on finite closed intervals.	21-44.
4.1.3 Find critical points, local and absolute extrema, and domain endpoints for functions.	45-64.
4.1.4 Understand concepts related to extreme values of functions.	65-74, 76.
4.1.5 Solve applications involving extreme values.	75-78.

Definitions



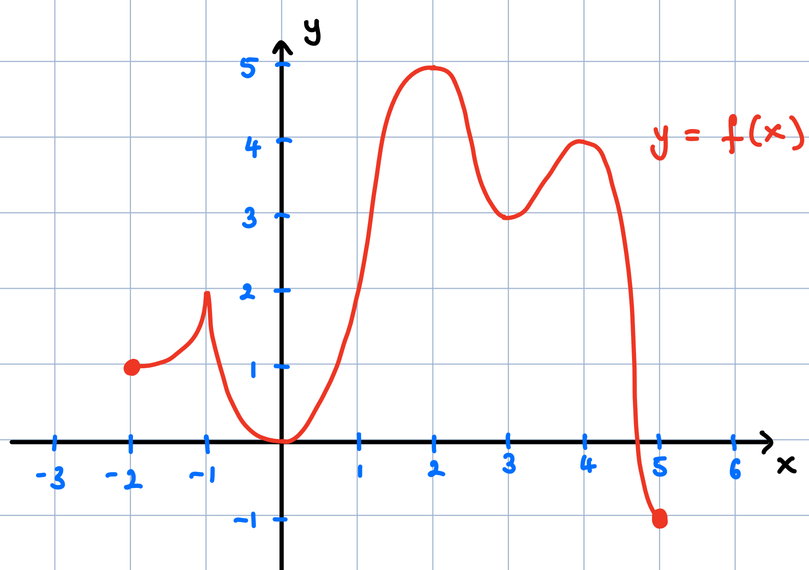
- **Local/relative maximum :**
  a y-value $f(c)$ such that
  $f(c) \geq f(x)$ for x near c.
  ↳ a " peak " on the graph.

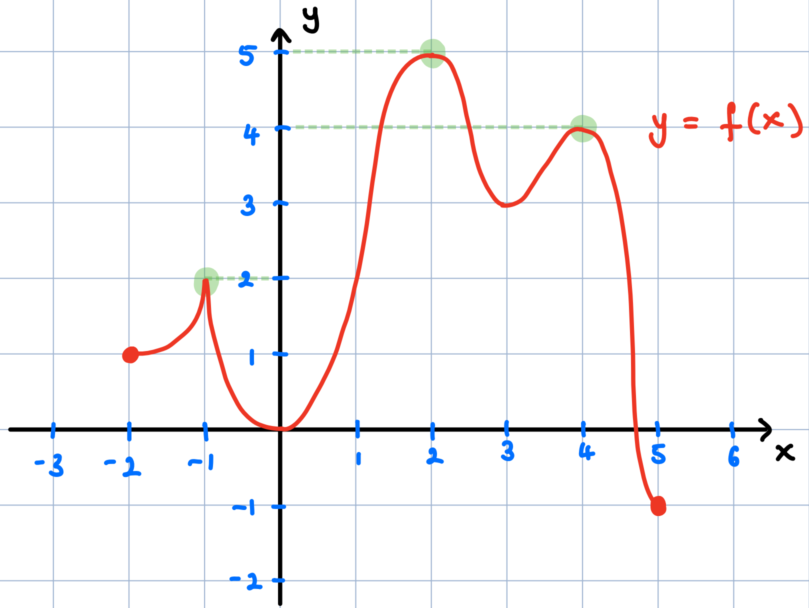
- **Absolute / global maximum on $[a,b]$:**
  a y-value $f(c)$ such that
  $f(c) \geq f(x)$ for all x in $[a,b]$.
  ↳ the " highest peak ".

We have similar definitions for local / relative minimum (" valleys " on the graph) and absolute / global minimum on $[a, b]$ (" lowest valley ")

Extremum or extreme values : maximum or minimum.
(plural : extrema, maxima, minima)

Examples : 1) Find the absolute and local extrema of the function sketched on $[-2, 5]$.



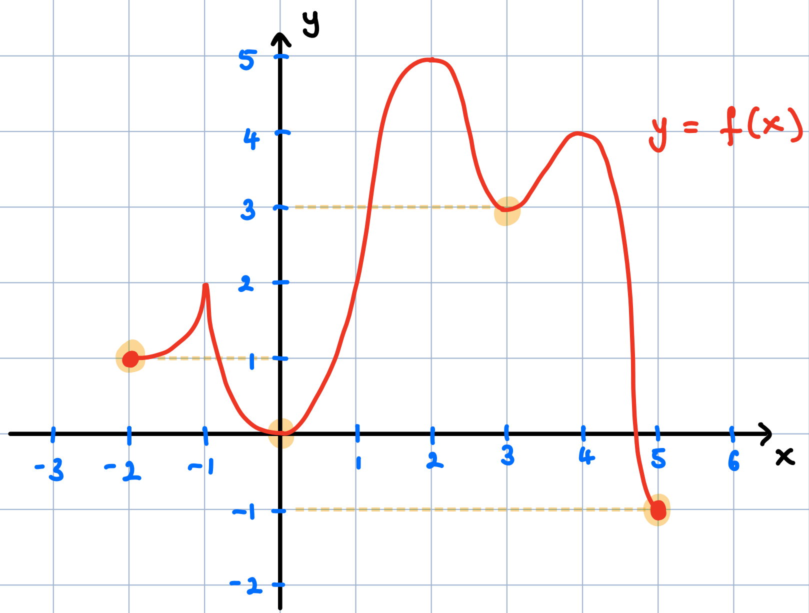


- Local maxima :

$\boxed{f(2) = 5, \quad f(4) = 4, \quad f(-1) = 2.}$

- Absolute maximum :

$\boxed{f(2) = 5}$.



- Local minima :

$\boxed{f(-2) = 1, \quad f(3) = 3, \quad f(0) = 0, \quad f(5) = -1}$

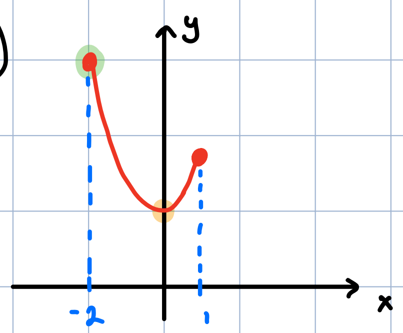
- Absolute minimum :

$\boxed{f(5) = -1}$.

2) Find the absolute extrema of $f(x) = x^2 + 1$ on

a) $[-2, 1]$ and b) $(-2, 1]$.

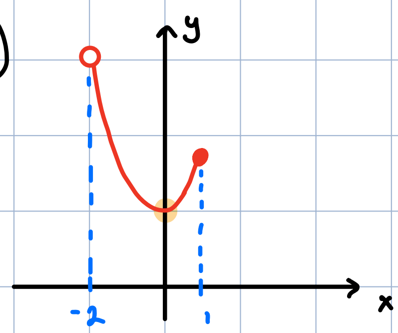
a)



Absolute max $= 5 = f(-2)$

Absolute min $= 1 = f(0)$

b)

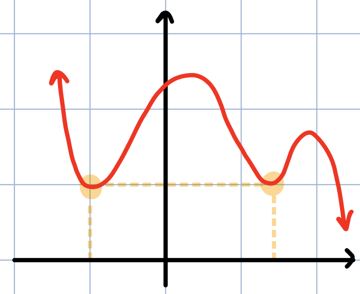


Absolute max : none

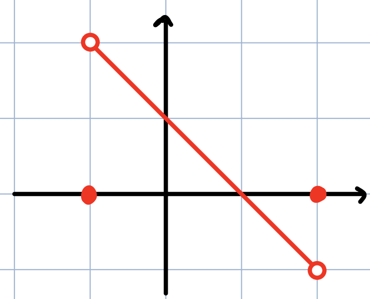
Absolute min $= 1 = f(0)$

Remarks :

- The same max/min value can occur at multiple x- values.



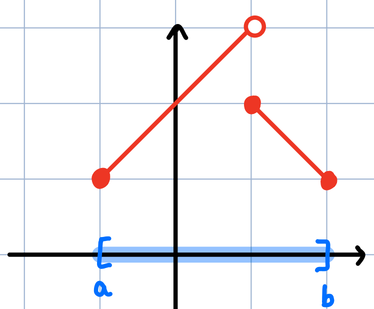
- A function may not have any extrema on an interval.



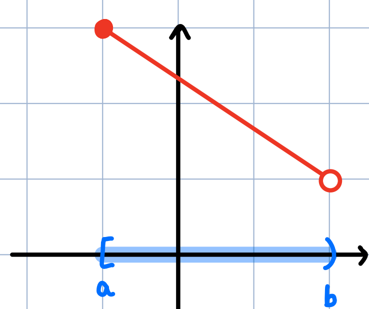
Extreme Value Theorem : if $f(x)$ is a continuous function on a closed and bounded interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.

All three conditions are necessary, otherwise the EVT may fail.

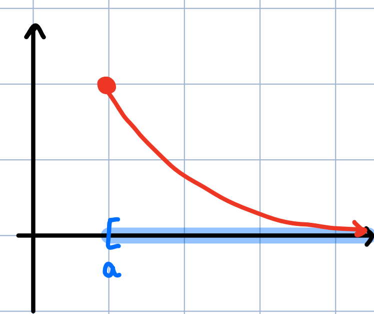
Examples :



not continuous
no absolute max.



$[a, b)$ not closed
no absolute min.



$[a, \infty)$ unbounded
no absolute min.

Finding extrema :

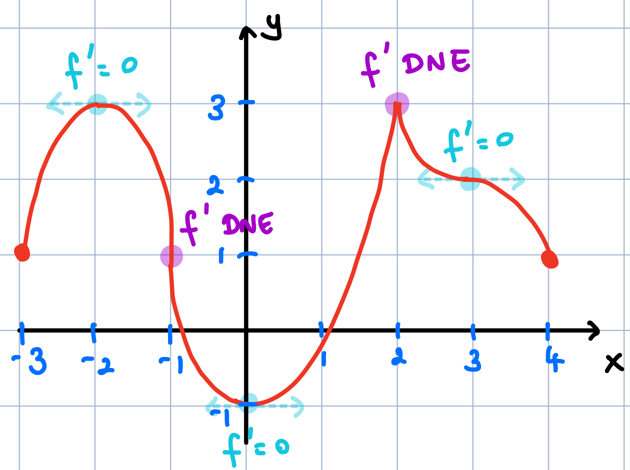
Fermat's Theorem : if f has a local extrema at $x=c$ and c is in the interior of the domain (not an endpoint), then

$$\underbrace{\boxed{f'(c) = 0 \quad \text{or} \quad f'(c) \text{ DNE.}}}$$

we say that $x=c$ is a critical point if c is in the interior of the domain of f and $f'(c)=0$ or $f'(c)$ DNE

Fermat's theorem tells us local extrema occur at critical points, but not every critical point will give a local extremum.

Example:



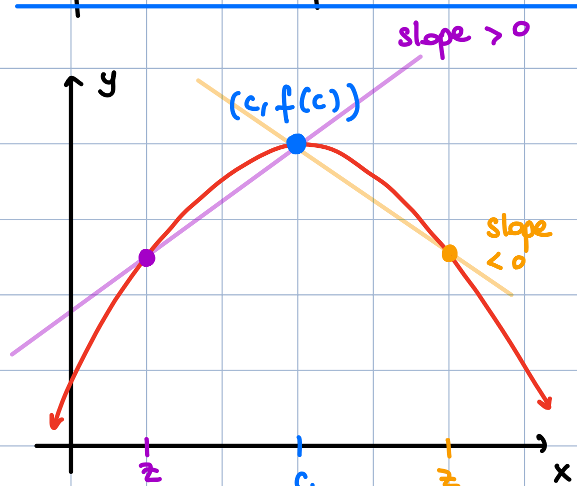
The critical points are :

$x=-2, 0, 3 \qquad (f'=0)$

$x=-1, 2 \qquad (f' \text{ DNE})$

But no extrema occur at $x=-1, 3$.

Explanation for Fermat's Theorem: assume f has a max. at $x=c$ and $f'(c)$ exists.



$$f'(c) = \lim_{z \to c^-} \frac{f(z)-f(c)}{\underbrace{z-c}_{\leq 0}} \geq 0$$

$$f'(c) = \lim_{z \to c^+} \frac{f(z)-f(c)}{\underbrace{z-c}_{\geq 0}} \leq 0$$

So $\boxed{f'(c) = 0}$.

Method to find absolute extrema of a continuous function f on a closed and bounded interval $[a,b]$.

1. Find the critical points of f in (a,b). These are the potential locations of extrema by Fermat's Theorem.

2. Evaluate f (NOT f') at the critical points and endpoints a, b

3. Largest value = absolute max.
   Least value = absolute min.

<u>Examples:</u> 1) Find the absolute extrema of $f(x) = x^3 - 3x + 4$ on $[0,3]$.

$f'(x) = 3x^2 - 3 = 3(x^2-1) = 3(x-1)(x+1)$

not in interval

Critical points : $f'(x) = 0 \Rightarrow 3(x-1)(x+1) = 0 \Rightarrow x = 1, \cancel{-1}$.

$f'(x)$ DNE $\Rightarrow$ none

CPs in $(0,3)$: $x=1$.

Now we evaluate f at the critical point in $(0,3)$ and the endpoints.

x	$f(x)$	
0	4	
1	2	least
3	22	largest

$\boxed{\begin{array}{l}\text{Absolute max. is } 22 \\ \text{Absolute min. is } 2\end{array}}$ (reached at $x=3$)
(reached at $x=1$)

2) Find the absolute extrema of $f(x) = 1 + (x^2-9)^{2/3}$ on $[-2,5]$.

$$f'(x) = \frac{2}{3}(x^2-9)^{-1/3}(2x) = \frac{4x}{3(x^2-9)^{1/3}}.$$

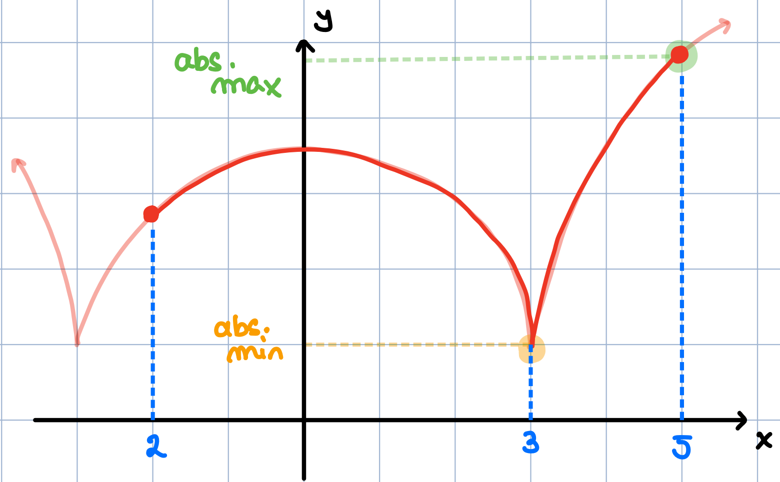
Critical points : $f'(x) = 0 \Rightarrow 4x = 0 \Rightarrow x = 0.$

$f'(x)$ DNE $\Rightarrow 3(x^2-9)^{1/3} = 0 \Rightarrow x^2 = 9 \Rightarrow x = 3, \cancel{-3}$ not in interval

The CPs in $(-2,5)$ are $x = 0, 3$.

x	$f(x)$	
-2	$1+5^{2/3}$	
0	$1+9^{2/3}$	
3	1	least
5	$1+16^{2/3}$	largest

Absolute max. is $1+16^{2/3}$
Absolute min. is 1



3) Find the absolute extrema of $f(x) = x^2 - 6|x| - 2$ on $[-4,1]$

$$f(x) = \begin{cases} x^2-6x-2 & \text{if } x \geq 0 \\ x^2+6x-2 & \text{if } x < 0 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x-6 & \text{if } x \geq 0 \\ 2x+6 & \text{if } x < 0 \end{cases}$$

$f'(0)$ DNE since $\lim_{x\to 0^+} 2x-6 = -6$, $\lim_{x\to 0^-} 2x+6 = 6$ slope different on left and right of $x=0$ (corner)

Critical points : $f'(x) = 0 \Rightarrow \begin{cases} 2x-6 = 0 \Rightarrow \cancel{x=3} & \text{not in interval} \\ 2x+6 = 0 \Rightarrow x = -3 \end{cases}$

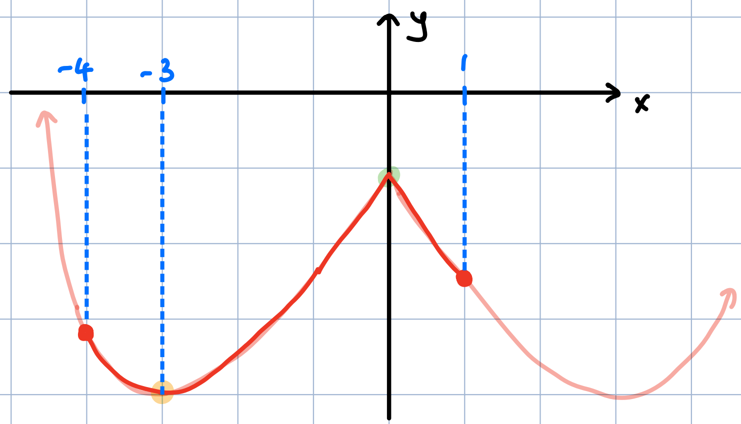
$f'(x)$ DNE $\Rightarrow$ $x = 0$.

So the CPs in $(-4, 1)$ are $x = -3, 0$.

x	$f(x)$	
-4	-10	
-3	-11	least
0	-2	largest
1	-7	

Absolute max. is -2

Absolute min. is -11



4) Find the absolute extrema of $f(x) = \frac{1}{x^2} + 2\ln(x)$ on $\left[\frac{1}{\sqrt{e}}, \sqrt{e}\right]$.

$$f'(x) = -\frac{2}{x^3} + \frac{2}{x} = \frac{2(x^2-1)}{x^3} = \frac{2(x-1)(x+1)}{x^3}.$$

Critical points : $f'(x) = 0 \Rightarrow 2(x-1)(x+1) = 0 \Rightarrow x = 1, \cancel{-1}$ not in interval

$f'(x)$ DNE $\Rightarrow x^3 = 0 \Rightarrow x = \cancel{0}$ not in interval

CP in $\left(\frac{1}{\sqrt{e}}, \sqrt{e}\right)$: $x = 1$

x	$f(x)$	
$1/\sqrt{e}$	$e-1$	largest
1	1	least
$\sqrt{e}$	$\frac{1}{e}+1$	

Absolute max. is $e-1$

Absolute min. is 1

$e \simeq 2.7 > 2$ so $e - 1 \simeq 1.7$

and $\frac{1}{e} < 0.5$ so $\frac{1}{e} + 1 < 1.5$

5) Find the absolute extrema of $f(x) = e^{-x}\sin(x)$ on $[0, 2\pi]$.

$f'(x) = -e^{-x}\sin(x) + e^{-x}\cos(x) = e^{-x}(\cos(x) - \sin(x))$.

Critical points : $f'(x) = 0 \Rightarrow$ $e^{-x} = 0$ no solution; $\cos(x) - \sin(x) = 0$, $\cos(x) = \sin(x)$, $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

$f'(x)$ DNE $\Rightarrow$ no solution.

CPs in $[0, 2\pi]$ are $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

x	$f(x)$
0	0
$\pi/4$	$e^{-\pi/4}\frac{\sqrt{2}}{2}$
$5\pi/4$	$-e^{-5\pi/4}\frac{\sqrt{2}}{2}$
2π	0

largest (only positive)

least (only negative)

Absolute max. is $e^{-\pi/4}\frac{\sqrt{2}}{2}$
Absolute min. is $e^{-5\pi/4}\frac{\sqrt{2}}{2}$