Rutgers University
Math 151

Section 4.1: Extreme Values - Worksheet Solutions

1. Find the absolute extrema of the following functions on the given interval.

   (a) $f(x) = 2x^3 + 3x^2 - 12x + 1$ on $[-1, 2]$.

   Solution. First, we find the critical points of f in $[-1, 2]$. We have $f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1)$.

   - $f'(x) = 0$ gives $x = -2, 1$.
   - $f'(x)$ undefined: no x-values.

   So the critical point in $[-1, 2]$ is $x = 1$. Now, we evaluate $f(x)$ at the endpoints and the critical point.

x	-1	1	2
$f(x)$	14	-6	5

   Therefore, the absolute maximum of $f(x)$ on $[-1, 2]$ is $\boxed{14}$ (reached at $x = -1$) and the absolute minimum is $\boxed{-6}$ (reached at $x = -1$).

   (b) $f(x) = x(7 - x)^{2/5}$ on $[1, 6]$.

   Solution. First, we find the critical points of f in $[1, 6]$. We have

   $$f'(x) = (7 - x)^{2/5} - \frac{2x}{5(7 - x)^{3/5}} = \frac{5(7 - x) - 2x}{5(7 - x)^{3/5}} = \frac{35 - 7x}{5(7 - x)^{3/5}}.$$

   - $f'(x) = 0$ gives $35 - 7x = 0$, so $x = 5$.
   - $f'(x)$ undefined gives $x = 7$.

   So the critical point in $[1, 6]$ is $x = 5$. Now, we evaluate $f(x)$ at the endpoints and the critical point.

x	1	5	6
$f(x)$	$6^{2/5}$	$5 \cdot 4^{1/5}$	6

   We need to determine which of these is the largest and which is the least. First, observe that $6 > 6^{2/5}$ since $\frac{2}{5} < 1$. Next, we have $6 < 5 \cdot 4^{1/5}$. To see this, we can compare the 5th power of these numbers to see that $6^5 = 7776 < 5^5 \cdot 4 = 12500$. Therefore, the absolute maximum of $f(x)$ on $[1, 6]$ is $\boxed{5 \cdot 4^{1/5}}$ (reached at $x = 5$) and the absolute minimum is $\boxed{6^{2/5}}$ (reached at $x = 1$).

   (c) $f(x) = 3x^4 - 10x^3 + 6x^2 - 7$ on $[-2, 1]$.

   Solution. First, we find the critical points of f in $[-2, 1]$. We have $f'(x) = 12x^3 - 30x + 12x = 6x(2x - 1)(x - 2)$.

   - $f'(x) = 0$ gives $x = 0, \frac{1}{2}, 2$.

- $f'(x)$ undefined: no x-values.

So the critical points in $[-2,1]$ are $x = 0, \frac{1}{2}$. Now, we evaluate $f(x)$ at the endpoints and the critical point.

x	-2	0	$\frac{1}{2}$	1
$f(x)$	145	-7	$-\frac{105}{16}$	-8

Therefore, the absolute maximum of $f(x)$ on $[-2,1]$ is $\boxed{145}$ (reached at $x = -2$) and the absolute minimum is $\boxed{-8}$ (reached at $x = 1$).

(d) $f(x) = (e^x - 2)^{4/7}$ on $[0, \ln(3)]$.

Solution. First, we find the critical points of f in $[0, \ln(3)]$. We have $f'(x) = \dfrac{4e^x}{7(e^x-2)^{3/7}}$.

- $f'(x) = 0$ gives $4e^x = 0$, which has no solution.
- $f'(x)$ undefined gives $e^x - 2 = 0$, so $x = \ln(2)$.

So the critical point in $[0, \ln(3)]$ is $x = \ln(2)$. Now, we evaluate $f(x)$ at the endpoints and the critical point.

x	0	$\ln(2)$	$\ln(3)$
$f(x)$	1	0	1

Therefore, the absolute maximum of $f(x)$ on $[0, \ln(3)]$ is $\boxed{1}$ (reached at $x = 0$ and $x = \ln(3)$) and the absolute minimum is $\boxed{0}$ (reached at $x = \ln(2)$).

(e) $f(x) = \dfrac{\ln(x)}{\sqrt{x}}$ on $[1, e^4]$.

Solution. First, we find the critical points of f in $[1, e^4]$. We have $f'(x) = \dfrac{1}{x^{3/2}} - \dfrac{\ln(x)}{2x^{3/2}} = \dfrac{2-\ln(x)}{2x^{3/2}}$.

- $f'(x) = 0$ gives $2 - \ln(x) = 0$, so $x = e^2$.
- $f'(x)$ undefined gives no solution in the domain of f, which is $(0, \infty)$.

So the critical point in $[1, e^4]$ is $x = e^2$. Now, we evaluate $f(x)$ at the endpoints and the critical point.

x	1	e^2	2^4
$f(x)$	0	$\frac{2}{e}$	$\frac{4}{e^2}$

It is clear that the smallest of these values is $\boxed{0}$, which is the absolute minimum of f on $[1, e^4]$. To find the largest value, observe that $\frac{4}{e^2} = \left(\frac{2}{e}\right)^2$. Since $\frac{2}{e} < 1$, $\left(\frac{2}{e}\right)^2 < \frac{2}{e}$. Therefore, the absolute maximum of $f(x)$ on $[1, e^4]$ is $\boxed{\dfrac{2}{e}}$.

(f) **[Advanced]** $f(x) = 2\arctan(3x) - 3x$ on $\left[0, \frac{1}{\sqrt{3}}\right]$. *(Hint: use the approximations $\pi \simeq 3.1$ and $\sqrt{3} \simeq 1.7$).)*

Solution. First, we find the critical points of f in $\left[0, \frac{1}{\sqrt{3}}\right]$. We have $f'(x) = \dfrac{6}{1+9x^2} - 3 = \dfrac{3-27x^2}{1+9x^2}$.

- $f'(x) = 0$ gives $3 - 27x^2 = 0$, so $x = \frac{1}{3}, -\frac{1}{3}$.
- $f'(x)$ undefined gives $1 + 9x^2 = 0$, which has no solution.

So the critical point in $\left[0, \frac{1}{\sqrt{3}}\right]$ is $x = \frac{1}{3}$. Now, we evaluate $f(x)$ at the endpoints and the critical point.

x	0	$\frac{1}{3}$	$\frac{1}{\sqrt{3}}$
$f(x)$	0	$\frac{\pi}{2} - 1$	$\frac{2\pi}{3} - \sqrt{3}$

It is clear that the smallest of these values is $\boxed{0}$, which is the absolute minimum of f on $\left[0, \frac{1}{\sqrt{3}}\right]$. To find the largest value, observe that $\frac{\pi}{2} - 1 \simeq 0.5$ using the approximation $\pi \simeq 3.1$, and $\frac{2\pi}{3} - \sqrt{3} \simeq 2 - 1.7 = 0.3$. So the absolute maximum of f on $\left[0, \frac{1}{\sqrt{3}}\right]$ is $\boxed{\dfrac{\pi}{2} - 1}$.