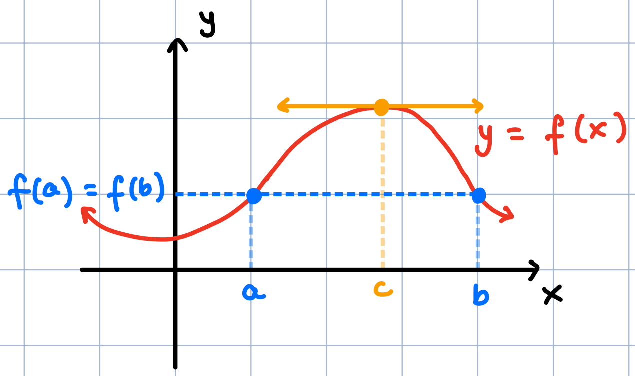
Sections 4.2-3

Mean Value Theorem and First Derivative Test

Learning Goals

Learning Goal	*Homework Problems*
4.2.1 Find the values that satisfy the conclusion of the Mean Value Theorem.	1-8.
4.2.2 Identify functions that satisfy the hypotheses of the Mean Value Theorem.	9-16.
4.2.3 Use Rolle's theorem to investigate the number of zeroes of a function on given intervals.	17-28.
4.2.4 Find functions and values of functions given the derivatives.	29-42.
4.2.5 Find the position of an object given its velocity or acceleration with appropriate initial value(s).	43-50.
4.2.6 Solve applications involving the Mean Value Theorem.	51-56.
4.2.7 Answer conceptual questions involving the Mean Value Theorem.	57-78.
Learning Goal	*Homework Problems*
4.3.1 Find intervals where the function is increasing or decreasing, and extrema given the function, its derivative, or the graph of either.	1-46, 67-68.
4.3.2 Find local and absolute extrema in given domains.	47-58, 69-70.
4.3.3 Discuss extreme-value behavior of functions by analyzing their first derivatives.	59-66, 75-85.
4.3.4 Create a function that has given extreme values.	71-74.
4.3.5 Show that given functions have inverses over their domains.	86-90.

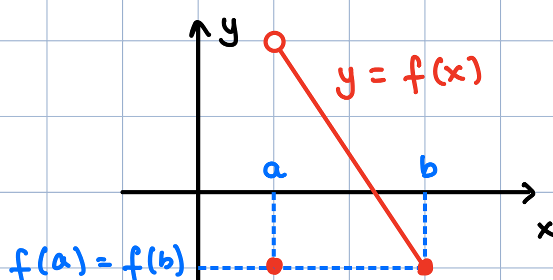
Rolle's Theorem : suppose f is continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) = f(b)$.



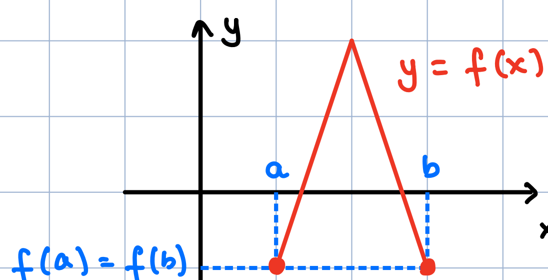
Then there exists at least one point c in (a,b) such that $\boxed{f'(c) = 0}$.

Explanation: since f is continuous on $[a,b]$, it has an absolute max. and min. on $[a,b]$ by the EVT. At a point c in (a,b) where f has an extremum, we have $f'(c) = 0$ by Fermat's Theorem ($f'(c)$ DNE is impossible since f is differentiable).

If the assumptions are not met, Rolle's Theorem can fail. Examples :



f is not continuous on $[a,b]$

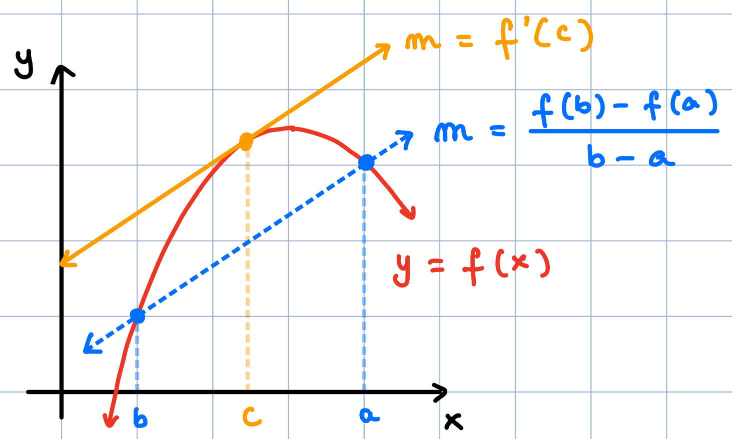


f is not differentiable on (a,b)

In these two examples, f' is never 0 on (a,b) even though $f(a) = f(b)$.

Mean Value Theorem (MVT) this is a slanted version of Rolle's Theorem.

Assume that f is continuous on $[a,b]$ and differentiable on (a,b).



Then there exists a point c in (a,b) such that

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

or $\boxed{f'(c)(b-a) = f(b)-f(a)}$

Interpretation: there is a point in (a,b) where the instantaneous rate of change is equal to the average rate of change on $[a,b]$.

Concrete example: if you drive 120 mi in 2 hours, at some point of the trip the instantaneous velocity was $\frac{120}{2} = 60$ mi/h.

Examples: 1) The function $f(x) = \sqrt{x+1}$ satisfies the assumptions of the MVT on $[3,8]$. Find the value of c such that satisfies $f'(c) = \frac{f(b)-f(a)}{b-a}$.

We have $\frac{f(b)-f(a)}{b-a} = \frac{f(8)-f(3)}{8-3} = \frac{3-2}{8-3} = \frac{1}{5}$.

and $f'(c) = \frac{d}{dx}\left(\sqrt{x+1}\right)\Big|_{x=c} = \frac{1}{2\sqrt{x+1}}\Big|_{x=c} = \frac{1}{2\sqrt{c+1}}.$

So we want to solve $\frac{1}{2\sqrt{c+1}} = \frac{1}{5} \Rightarrow \sqrt{c+1} = \frac{5}{2}$

$$c+1 = \frac{25}{4}$$

$$\boxed{c = \frac{21}{4}}.$$

2) Find the values of a, b so that the function

$$f(x) = \begin{cases} 3x+b & \text{if } x<0 \\ 7e^{ax} & \text{if } x \geqslant 0 \end{cases}$$ satisfies the assumptions

of the MVT on $[-1,1]$.

We need f to be continuous on $[-1,1]$ and differentiable on $(-1,1)$. The only point where one of these might fail is $x=0$.

Continuity at $x=0$: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$$b = 7e^{a\cdot 0} = 7$$

$$\Rightarrow \boxed{b=7}$$

Differentiability at $x=0$: $f'(x) = \begin{cases} 3 & \text{if } x<0 \\ 7ae^{ax} & \text{if } x>0 \end{cases}$

We need $3 = 7ae^{ax}\Big|_{x=0}$ so $3 = 7a$

$$\Rightarrow \boxed{a = \frac{3}{7}}$$

3) Suppose that f is continuous on $[1,5]$, differentiable on $(1,5)$. Assume that $f(1) = -2$ and $f'(x) \leq 3$. What is the largest possible value of $f(5)$?

MVT : $\dfrac{f(5)-f(1)}{5-1} = f'(c)$ for some c in $(1,5)$

$$\Rightarrow \quad \frac{f(5)-f(1)}{5-1} \leq 3.$$

$$\Rightarrow \quad f(5) - f(1) \leq 3(5-1) = 12$$

$$f(5) \leq 12 + f(1) = 10$$

The largest possible value of $f(5)$ is 10.

Concrete interpretation: if you leave at 1pm and walk at 3 mi/h at most, you'll have traveled at most 12 mi by 5pm.

Applications of MVT :

Ⓐ If $f'(x) = 0$ on I, then f is constant on I.
Indeed, $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0$ so $f(x_2) = f(x_1)$.
So $f(x) = C$ for a constant C.

If $f'(x) = g'(x)$ on I, then $(f-g)'(x) = 0$ so $f-g$ is constant, i.e. $f(x) = g(x) + C$.

Examples : 1)

If $f'(x) = 2x$, then $f(x) = x^2 + C$ because $\frac{d}{dx} x^2 = 2x$.

$g'(x) = \frac{1}{1+x^2}$, then $g(x) = \tan^{-1}(x) + C$ because $\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$

$h'(x) = x + \cos(x)$, then $h(x) = \frac{1}{2}x^2 + \sin(x) + C$

2) If an object moves with acceleration $a(t) = -2$, initial position $s(0) = 4$ and initial velocity $v(0) = -3$, find the position $s(t)$.

$v'(t) = a'(t) = -2 \Rightarrow v(t) = -2t + C$ to find C, use $v(0) = -3$

$-3 = -2 \cdot 0 + C$

$C = -3$

So $v(t) = -2t - 3 \Rightarrow s(t) = -t^2 - 3t + C$

$4 = -0^2 - 3 \cdot 0 + C \Rightarrow C = 4$

So $\boxed{s(t) = -t^2 - 3t + 4}$.

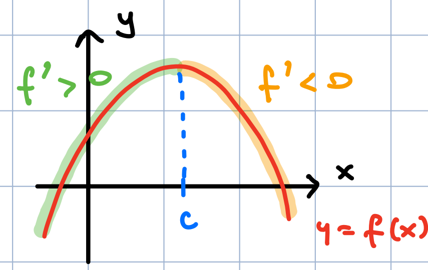
B If $f'(x) > 0$ on I, then f is increasing on I

Indeed, if $x_2 > x_1$, $f(x_2) - f(x_1) = \underbrace{(x_2 - x_1)}_{>0}\underbrace{f'(c)}_{>0} > 0$.

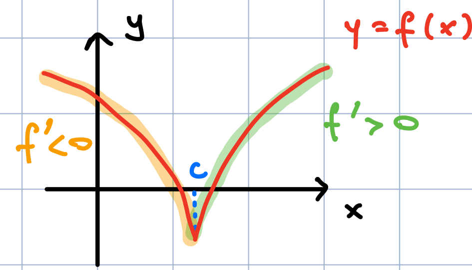
Likewise, if $f'(x) < 0$ on I, then f is decreasing on I.

Indeed, if $x_2 > x_2$, $f(x_2) - f(x_1) = \underbrace{(x_2 - x_1)}_{>0}\underbrace{f'(c)}_{<0} < 0$.

First Derivative Test: assume c is a critical point of f.

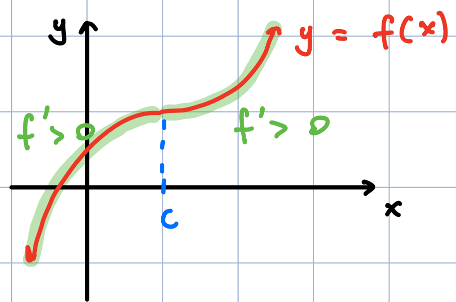


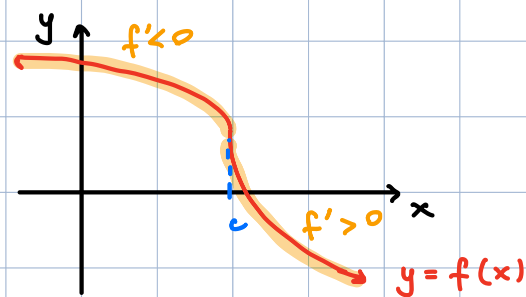
- If f' changes from $+$ to $-$ at c, then f has a local max. at $x = c$.



- If f' changes from $-$ to $+$ at c, then f has a local min. at $x = c$.

- If f' does not change sign at c, then f does not have a local extremum at $x = c$.





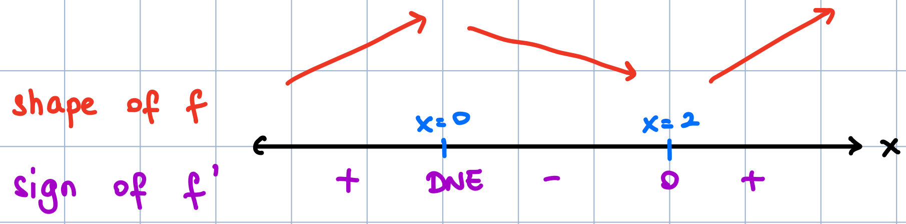
Examples: 1) For $f(x) = x^{2/3}(x-5)$, find the intervals where f is increasing, decreasing and the local extrema.

$$f(x) = x^{5/3} - 5x^{2/3} \Rightarrow f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5x^{-1/3}}{3}(x-2)$$
$$= \frac{5(x-2)}{3x^{1/3}}.$$

Critical points: $f'(x) = 0 \Rightarrow x = 2$

$f'(x)$ DNE $\Rightarrow x = 0$

We now use a sign chart for f' to answer the questions.

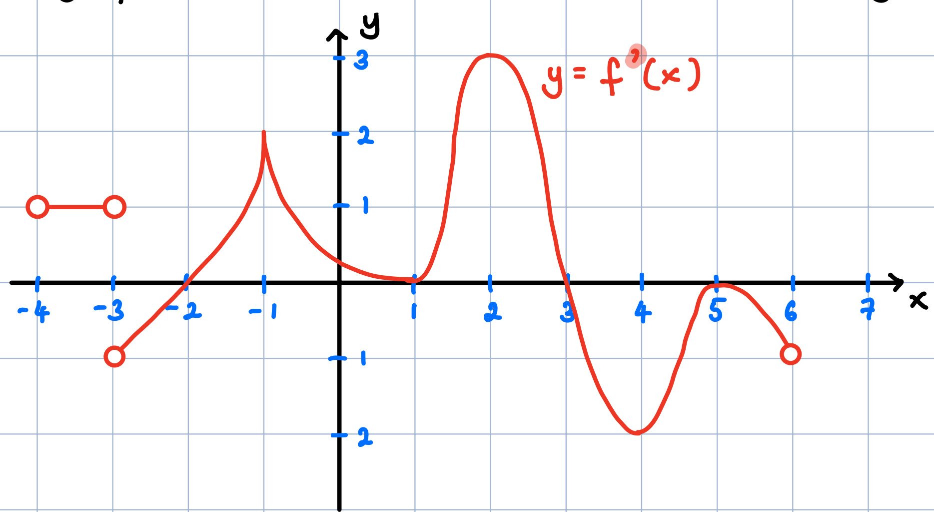


x	$f'(x)$
-1	$\frac{(-)}{(-)} = (+)$
1	$\frac{(-)}{(+)} = (-)$
3	$\frac{(+)}{(+)} = (+)$

Conclusion:

$\boxed{\begin{array}{l} f \text{ is increasing on } (-\infty, 0], [2, \infty) \\ f \text{ is decreasing on } [0, 2] \\ f \text{ has a local max. at } x = 0 \\ f \text{ has a local min. at } x = 2 \end{array}}$

2) Suppose that f is differentiable on $(-4, 6)$. Below is the graph of the derivative of f, $y = f'(x)$.



Critical points of f: $x = -2, 1, 3, 5$ $(f' = 0)$

$x = -3$ $(f'$ DNE$)$

Intervals of increase of f: $(-4, -3], [-2, 3]$ $(f' > 0)$

Intervals of decrease of f: $[-3, -2], [3, 6)$ $(f' < 0)$

Location of local max. of f: $x = 3$ $(f'$ + to $-)$

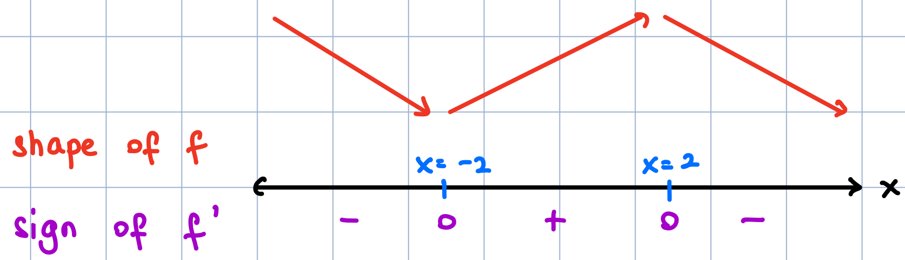
Location of local min. of f: $x = -2$ $(f'$ $-$ to $+)$

3) For $f(x) = -x^3 + 12x + 5$, find and classify the critical points of f.

$f'(x) = -3x^2 + 12 = -3(x^2 - 4) = -3(x-2)(x+2)$

Critical points: $f'(x) = 0 \Rightarrow x = 2, -2$

$f'(x)$ DNE : none.



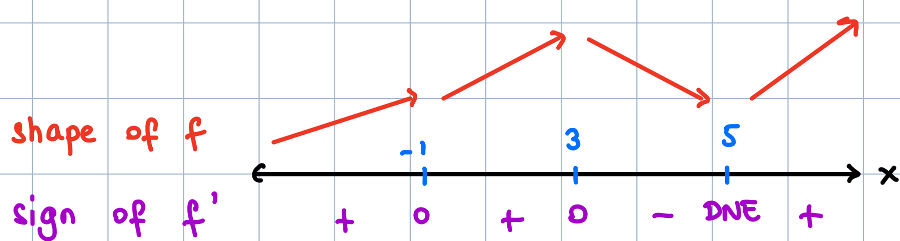
x	$f'(x)$
-3	$(-)(-)(-) = (-)$
0	$(-)(-)(+) = (+)$
3	$(-)(+)(+) = (-)$

So $\boxed{\begin{array}{l} f \text{ has a local max. at } x = 2 \\ \text{and a local min. at } x = -2 \end{array}}$

4) Assume that f is continuous on $(-\infty, \infty)$ and $f'(x) = \dfrac{(x-3)(x+1)^2}{(x-5)^{1/3}}$. Find and classify the critical points of f.

Critical points: $f'(x) = 0 \Rightarrow x = -1, 3.$

$f'(x)$ DNE $\Rightarrow x = 5$



x	$f'(x)$
-2	$\frac{(-)(+)}{(-)} = (+)$
0	$\frac{(-)(+)}{(-)} = (+)$
4	$\frac{(+)(+)}{(-)} = (-)$
6	$\frac{(+)(+)}{(+)} = (+)$

So $\boxed{\begin{array}{l} f \text{ has a local max. at } x = 3 \\ \text{and a local min. at } x = 5 \end{array}}$