Rutgers University
Math 151

Sections 4.2-3: Mean Value Theorem and First Derivative Test - Worksheet Solutions

1. Find the values of the constants A, B for which the following function satisfies the assumptions of the Mean Value Theorem on the interval $[-2, 2]$.

$$f(x) = \begin{cases} e^{5x+B} & \text{if } x \geqslant 0 \\ \arctan(Ax+1) & \text{if } x < 0 \end{cases}$$

Solution. To satisfy the assumptions of the MVT, f must be continuous on $[-2, 2]$ and differentiable on $(-2, 2)$. Each piece of f is differentiable (therefore also continuous) so we only need to check for continuity and differentiability at $x = 0$. For continuity, we will want

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0).$$

This gives $e^B = \arctan(1) = \frac{\pi}{4}$, so $\boxed{B = \ln\left(\frac{\pi}{4}\right)}$.

For differentiability at $x = 0$, we start by computing the derivative of each piece of f:

$$f'(x) = \begin{cases} 5e^{5x+B} & \text{if } x > 0 \\ \dfrac{A}{1+(Ax+1)^2} & \text{if } x < 0 \end{cases}$$

For f to be differentiable at 0, we need $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x)$, which gives $5e^B = \frac{A}{2}$. Therefore, $A = 10e^B = 10\frac{\pi}{4} = \boxed{\frac{5\pi}{2}}$.

2. Suppose that f is continuous on $[-2, 4]$, that $f(4) = 1$ and that $f'(x) \geqslant 3$ for x in $(-2, 4)$. Find the largest possible value of $f(-2)$.

Solution. We use the MVT: there exists a point c in $(-2, 4)$ such that

$$\frac{f(4) - f(-2)}{4 - (-2)} = f'(c).$$

This gives $\frac{1 - f(-2)}{6} = f'(c)$, which is $f(-2) = 1 - 6f'(c)$. By assumption, we have $f'(c) \geqslant 3$, so $6f'(c) \geqslant 18$ and $f'(2) = 1 - 6f'(c) \leqslant -17$. So the largest possible value of $f'(2)$ is $\boxed{-17}$.

3. Find and classify the critical points of the following functions.

(a) $f(x) = x^{4/7}(72 - x^2)$

Solution. We have

$$f'(x) = \frac{4(72-x^2)}{7x^{3/7}} - 2x^{11/7} = \frac{4(72-x^2) - 14x^2}{7x^{3/7}} = \frac{288 - 18x^2}{7x^{3/7}} = \frac{18(4-x)(4+x)}{7x^{3/7}}.$$

The critical points of f are $x = 4, -4$ (where f' is 0) and $x = 0$ (where f' is undefined). We now test the sign of f' between each critical point.

- On $(-\infty, -4)$, f' is $\dfrac{(+)(-)}{(-)} = (+)$.
- On $(-4, 0)$, f' is $\dfrac{(+)(+)}{(-)} = (-)$.
- On $(0, 4)$, f' is $\dfrac{(+)(+)}{(+)} = (+)$.
- On $(-\infty, -4)$, f' is $\dfrac{(-)(+)}{(+)} = (-)$.

So f has local maxima at $x = -4, 4$ (f' changes from + to -) and a local minimum at $x = 0$ (f' changes from - to +).

(b) $f(x) = x^5 \ln(x)$

Solution. Note that the domain of f is $(0, \infty)$. We have

$$f'(x) = 5x^4 \ln(x) + x^5 \cdot \frac{1}{x} = 5x^4 \ln(x) + x^4 = x^4(5\ln(x) + 1).$$

The critical point of f is $x = e^{-1/5}$ (where f' is 0). We now test the sign of f' on either side of the critical point.

- On $(0, e^{-1/5})$, f' is negative.
- On $(e^{-1/5}, \infty)$, f' is positive.

So f has a local minimum at $x = e^{-1/5}$ (f' changes from - to +).

(c) $f(x) = x + \cos(2x)$ on $\left[0, \dfrac{\pi}{2}\right]$

Solution. We have

$$f'(x) = 1 - 2\sin(2x).$$

Solving $f'(x) = 0$ gives $\sin(2x) = \frac{1}{2}$, which gives the solutions $x = \frac{\pi}{12}, \frac{5\pi}{12}$ on the interval $\left[0, \dfrac{\pi}{2}\right]$. We now test the sign of f' between each critical point.

- On $\left(0, \frac{\pi}{12}\right)$, f' is positive since $\sin(2x) < \frac{1}{2}$.
- On $\left(\frac{\pi}{12}, \frac{5\pi}{12}\right)$, f' is negative since$\sin(x) > \frac{1}{2}$.
- On $\left(\frac{5\pi}{12}, \frac{\pi}{2}\right)$, f' is positive since $\sin(2x) < \frac{1}{2}$.

So f has a local maximum at $x = \frac{\pi}{12}$ (f' changes from + to -) and a local minimum at $x = \frac{5\pi}{12}$ (f' changes from - to +).

(d) $f(x) = \sin^{-1}\left(e^{-x^2}\right)$

Solution. We have

$$f'(x) = \frac{1}{\sqrt{1-(e^{-x^2})^2}} e^{-x^2}(-2x) = \frac{-2xe^{-x^2}}{\sqrt{1-e^{-2x^2}}}.$$

The critical point of f is $x = 0$ (where f' is undefined). We now test the sign of f' on either side of the critical point.

- On $(-\infty, 0)$, f' is $\dfrac{(+)(+)}{(+)} = (+)$.
- On $(0, \infty)$, f' is $\dfrac{(-)(+)}{(+)} = (-)$.

So f has a local maximum at $x = 0$ (f' changes from + to -).

4. Suppose that f is continuous on $(-\infty, \infty)$ and that $f'(x) = \dfrac{(x+3)(x-5)^2}{x^{2/3}(x-1)^{1/5}}$.

(a) Find the critical points of f.

Solution. We have $f'(x) = 0$ when $x = -3, 5$. We have $f'(x)$ undefined when $x = 0, -$. Therefore, the critical points of f are $\boxed{x = -3, 0, 1, 5}$.

(b) Find the intervals where f is increasing and the intervals where f is decreasing.

Solution. We test for the sign of f' between the critical points.

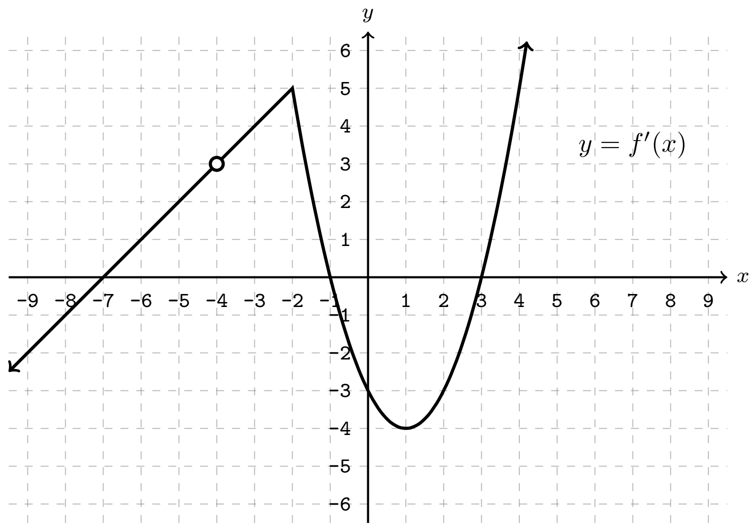
- On $(-\infty, -3)$: $\dfrac{(-)(+)}{(+)(-)} = (+)$
- On $(-3, 0)$: $\dfrac{(+)(+)}{(+)(-)} = (-)$
- On $(0, 1)$: $\dfrac{(+)(+)}{(+)(-)} = (-)$
- On $(1, 5)$: $\dfrac{(+)(+)}{(+)(+)} = (+)$
- On $(5, \infty)$: $\dfrac{(+)(+)}{(+)(+)} = (+)$

So f is increasing on $(-\infty, -3]$ and $[1, \infty)$, and decreasing on $[-3, 1]$.

(c) Find the location of the local extrema of f.

Solution. Based on our previous analysis, we can conclude that f has a local maximum at $x = -3$ and a local minimum at $x = 1$.

5. Suppose that f is a differentiable function. The graph of the **derivative** of f, $y = f'(x)$, is sketched below.



(a) Find the critical points of f.

Solution. Using the graph, we see that $f'(x) = 0$ when $x = -7, -1, 3$ and $f'(x)$ is undefined when $x = -4$. Therefore, the critical points of f are $\boxed{x = -7, -4, -1, 3}$.

(b) Find the intervals where f is increasing and the intervals where f is decreasing.

Solution. Using the graph, we see that $f'(x) > 0$ on $(-7, -4), (-4, -1)$ and $(3, \infty)$. So f is increasing on $[-7, -1]$ and $[3, \infty)$. Likewise, $f'(x) < 0$ on $(-\infty, -7)$ and $(-1, 3)$. So f is decreasing on $(-\infty, -7]$ and $[-1, 3]$.

(c) Find the location of the local extrema of f.

Solution. f has a local maximum at $x = -1$ (f' changes from + to -) and local minima at $x = -7, 3$ (f' changes from - to +).