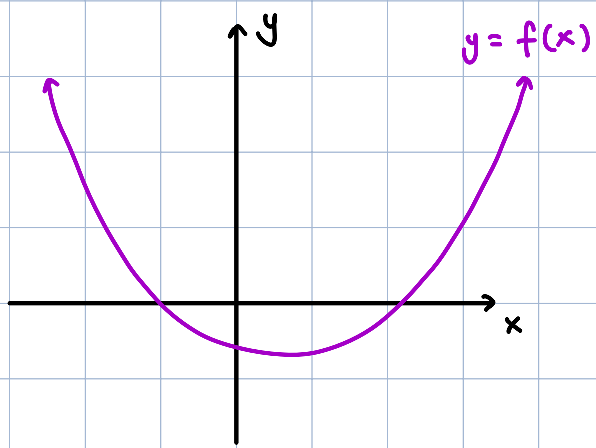
Section 4.4

Concavity and Curve Sketching

Learning Goals

Learning Goal	*Homework Problems*
4.4.1 Identify inflection points, local extrema, intervals of increasing/decreasing, and concavity from graphs.	1-8, 103, 109-116.
4.4.2 Graph functions and find any extrema and inflection points.	9-58, 129-134.
4.4.3 Graph $f(x)$ given information about the first and second derivatives, such as functions, graphs, or sign behavior.	59-102, 104-108.
4.4.4 Solve applications involving graphs of functions and their derivatives.	113-116.
4.4.5 Answer conceptual questions involving curve sketching using information about the function or its derivatives.	117-128.

What does the second derivative tell us?



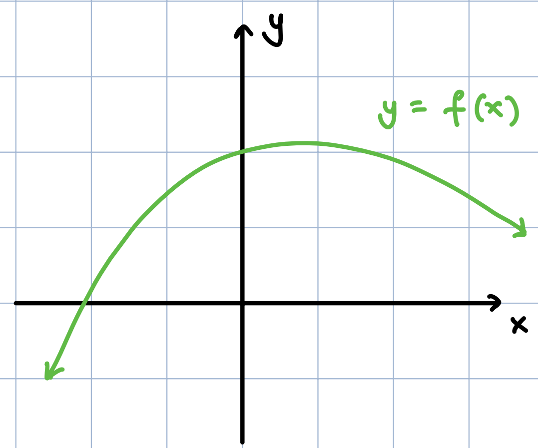
f is **CONCAVE UP**

f' is increasing

$f'' > 0$

Graph above tangent lines

" Bends up "



f is **CONCAVE DOWN**

f' is decreasing

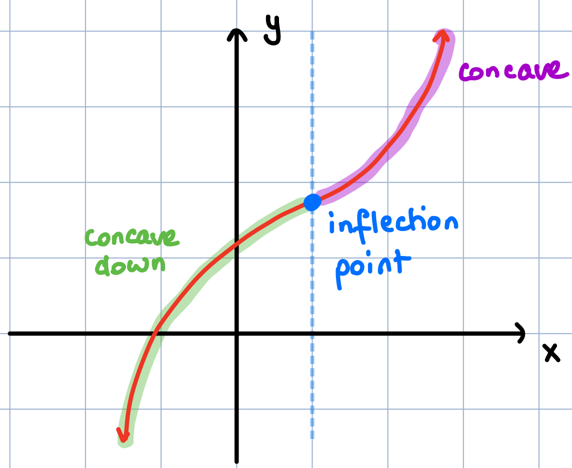
$f'' < 0$

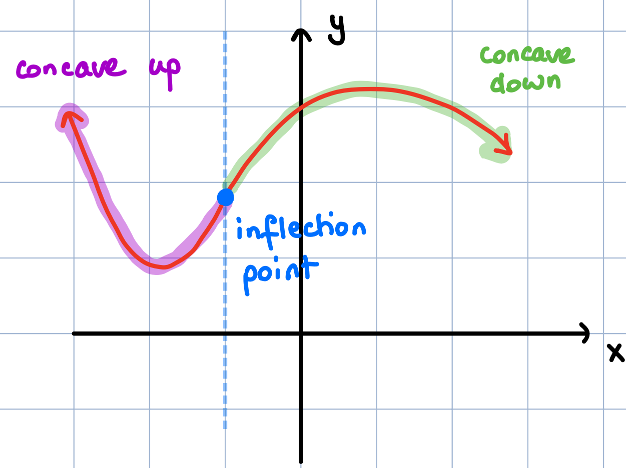
Graph below tangent lines

" Bends down "

An <u>inflection point</u> is a point $(c, f(c))$ on the graph where:

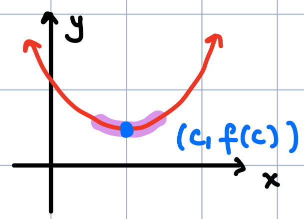
- f has a tangent line (i.e. $f'(c)$ exists)
- the concavity changes (i.e. f'' changes sign)



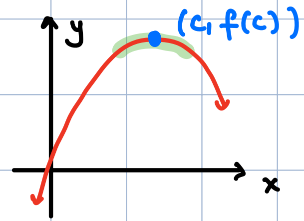


Second Derivative Test: we can use the second derivative to classify critical points c where $f'(c) = 0$.

- If $f''(c) > 0$, f has a local min. at $x = c$.



- If $f''(c) < 0$, f has a local max. at $x = c$.



- If $f''(c) = 0$: SDT inconclusive, use FDT.

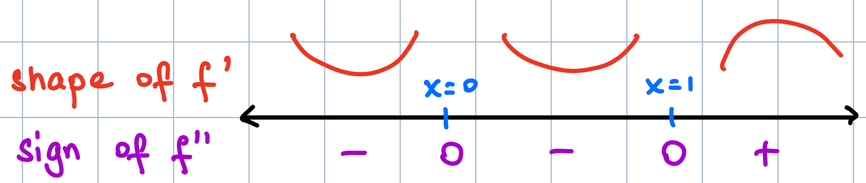
⚠ $f''(c) = 0$ does not mean there is no local extremum at $x = c$.

Examples: 1) For $f(x) = 3x^5 - 5x^4 + 10x - 3$, find the intervals where f is concave up, concave down and the inflection points of f.

$$f(x) = 3x^5 - 5x^4 + 10x - 3$$

$$\hookrightarrow f'(x) = 15x^4 - 20x^3 + 10$$

$$f''(x) = 60x^3 - 60x^2 = 60x^2(x-1).$$



So
$$\boxed{\begin{array}{l} f \text{ is concave up on } (-\infty, 1] \\ \quad \text{concave down on } [1, \infty) \\ (1, f(1)) = (1, 5) \text{ is a point of inflection} \end{array}}$$

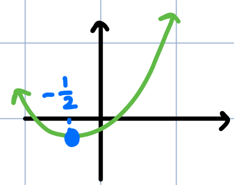
2) Find and classify the critical points of $f(x) = xe^{2x}$ and $g(x) = \sin(x) + \cos(x)$ on $[0, \frac{\pi}{2}]$.

For f: $f'(x) = e^{2x} + xe^{2x} \cdot 2 = e^{2x}(2x+1) \Rightarrow x = -\frac{1}{2}$ critical point.

SDT: $f''(x) = e^{2x} \cdot 2(2x+1) + e^{2x} \cdot 2$

$= 2e^{2x}(2x+2)$.

$f''(-\frac{1}{2}) = 2e^{-1}(-1+2) = 2e^{-1} > 0$.



So $\boxed{f \text{ has a local min. at } x = -\frac{1}{2}.}$

For g: $g'(x) = \cos(x) - \sin(x)$

$g'(x) = 0$ when $\cos(x) = \sin(x)$; this happens when $x = \frac{\pi}{4}$ in $[0, \frac{\pi}{2}]$.

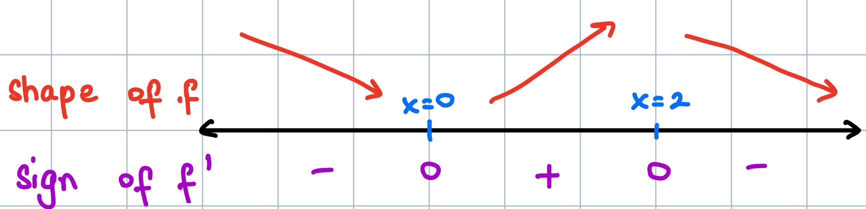
SDT: $f''(x) = -\sin(x) - \cos(x)$

$f''(\frac{\pi}{4}) = -\sin(\frac{\pi}{4}) - \cos(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2} < 0$.

So $\boxed{f \text{ has a local max. at } x = \frac{\pi}{4}}$

3) Sketch the graph of $f(x) = -x^3 + 3x^2 + 1$.

- End behavior: $\lim_{x \to \infty} f(x) = -\infty$, $\lim_{x \to -\infty} f(x) = \infty$.
- Vertical asymptotes: none.
- Info from f': $f'(x) = -3x^2 + 6x = -3x(x-2)$.



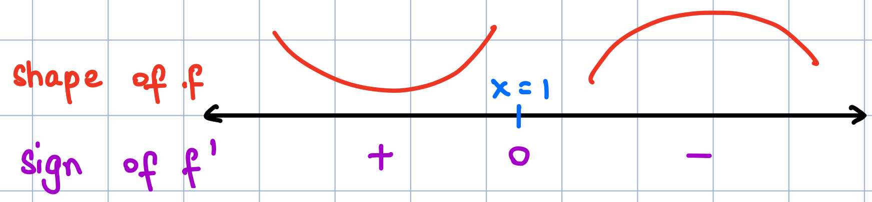
increasing on $[0, 2]$

decreasing on $(-\infty, 0]$, $[2, \infty)$

local min. at $x = 0$

local max. at $x = 2$.

- Info from f'' : $f''(x) = -6x+6 = -6(x-1)$.



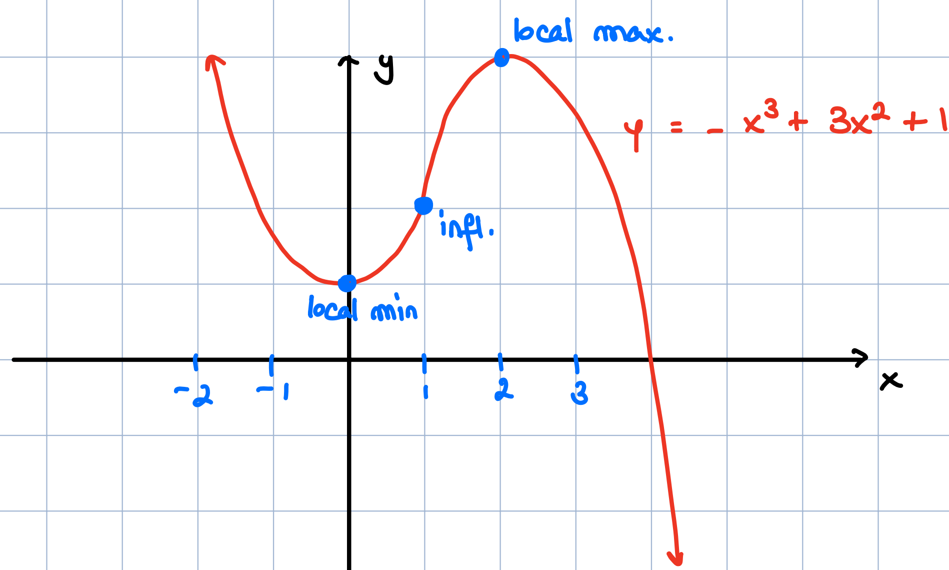
Concave up on $(-\infty, 1]$

concave down on $[1, \infty)$

inflection point at $x=1$.

- Recap :

x	$-\infty$	0	1	2	∞
$f(x)$	∞	1	3	5	$-\infty$
inc.	↘	↗	↗	↘	↘
conc.	∪	∪	∩	∩	∩



4) Sketch the graph of $f(x) = \dfrac{x-3}{x(x-6)}$ using :

$$f'(x) = -\frac{(x-3)^2+9}{x^2(x-6)^2} \qquad f''(x) = \frac{2(x-3)\left((x-3)^2+27\right)}{x^3(x-6)^3}$$

- End behavior: $\displaystyle \lim_{x\to\infty} \frac{x-3}{x(x-6)} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1-\frac{3}{x}}{x-6} = \text{"}\frac{1}{\infty}\text{"} = 0 = \lim_{x\to-\infty} \frac{x-3}{x(x-6)}.$

So $y=0$ is a horizontal asymptote of f.

- Vertical asymptotes: $x = 0$ $\left(\text{gives } \frac{-3}{0}\right)$

  $x = 6$ $\left(\text{gives } \frac{3}{0}\right)$

- Info from f':

  Shape of f: ↘ | VA $x=0$ | ↘ | VA $x=6$ | ↘

  sign of f': $-$ | DNE | $-$ | DNE | $-$

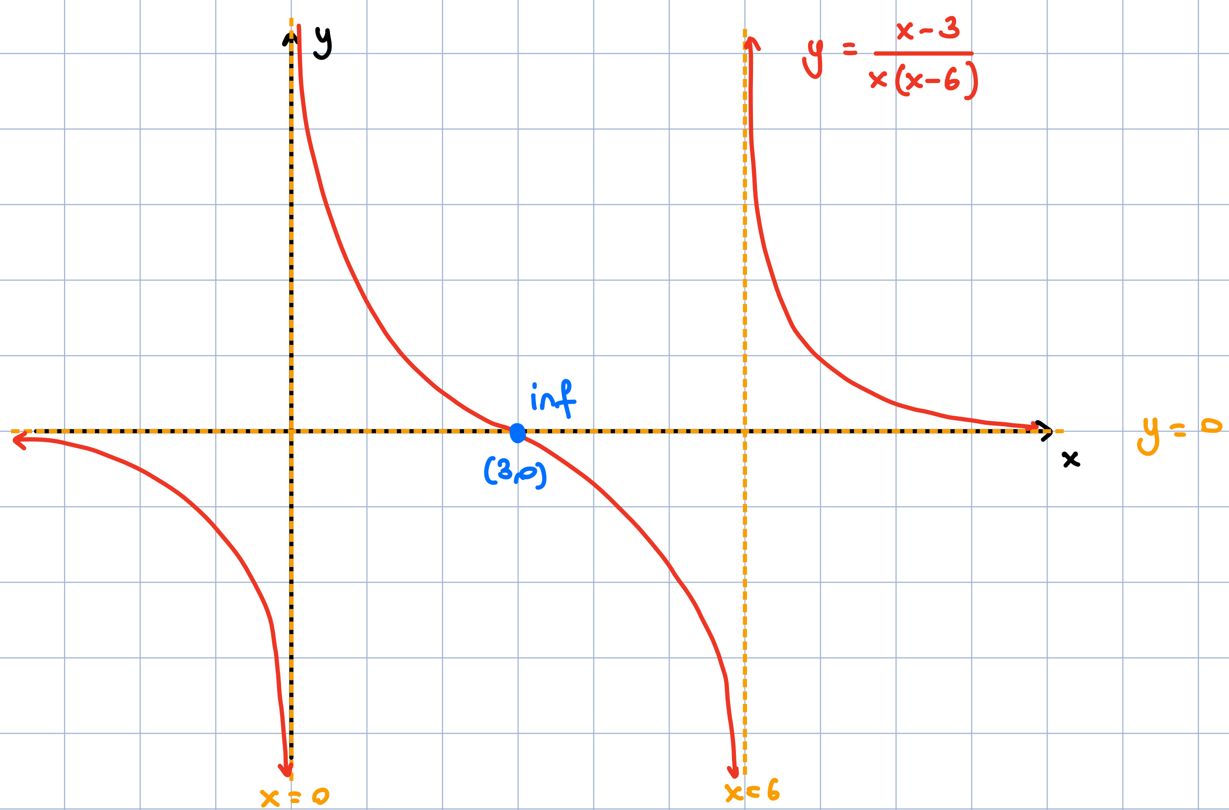
- Info from f'':

  Shape of f: ∩ | VA $x=0$ | ∪ | $x=3$ | ∩ | VA $x=6$ | ∪

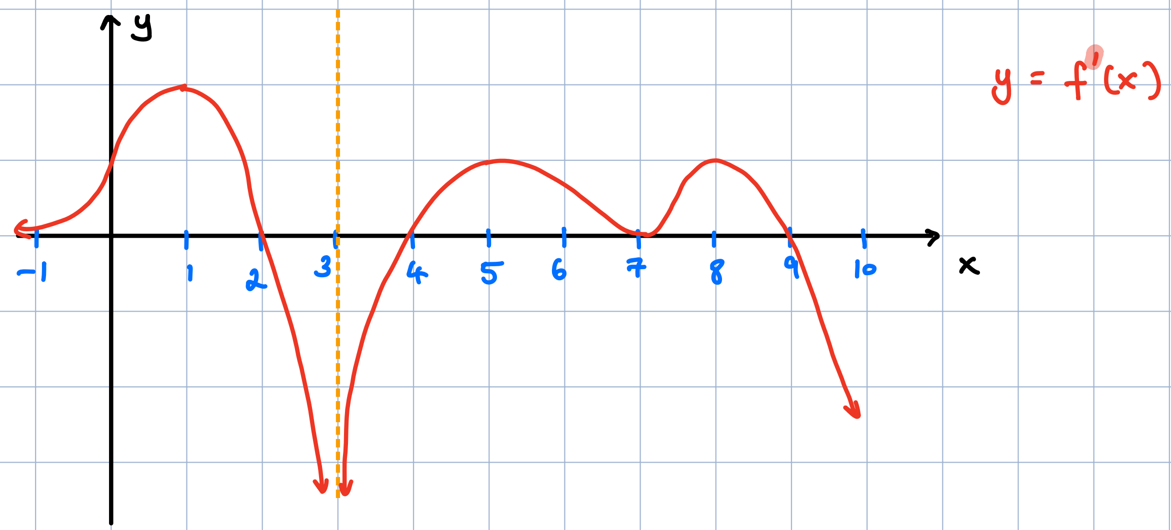
  sign of f': $-$ | DNE | $+$ | 0 | $-$ | DNE | $+$

- Recap

x	$-\infty$		0		3		6		∞
$f(x)$	0		VA		0		VA		0
inc.		↘		↘		↘		↘	
conc.		∩		∪		∩		∩	



5) Suppose f is continuous on $(-\infty, \infty)$. Below is the graph of the <u>derivative</u> of f, $y = f'(x)$.



Find the following: critical points of f, intervals of increase and decrease of f, location of local extrema, intervals where f is concave up / down, location of inflection points.

- Critical points of f: $x = 2, 4, 7, 9$ $(f' = 0)$
  $x = 3$ $(f'$ DNE$)$
- f increasing: $(-\infty, 2]$, $[4, 9]$ $(f' > 0)$
- f decreasing: $[2, 4]$, $[9, \infty)$ $(f' < 0)$
- Local max. at $x = 2, 9$ $(f'$ + to $-)$
- Local min. at $x = 4$ $(f'$ − to $+)$
- f concave up on $(-\infty, 1]$, $[3, 5]$, $[7, 8]$ $(f' \nearrow)$
- f concave down on $[1, 3]$, $[5, 7]$, $[8, \infty)$ $(f' \searrow)$

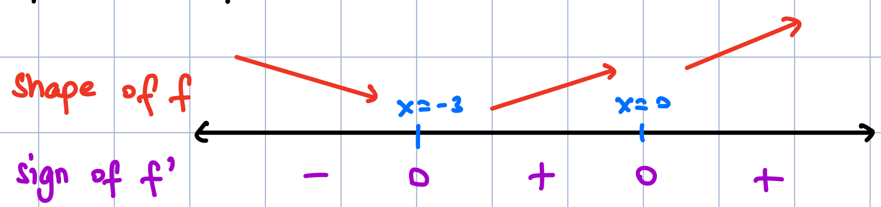
Practice: sketch the graph of the following functions.

1) $f(x) = x^4 + 4x^3 - 1$

2) $f(x) = \cos(2x) - x$ on $[0, \pi]$.

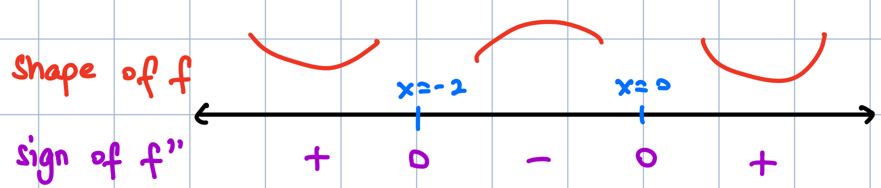
1) $f(x) = x^4 + 4x^3 - 1$

- End behavior: $\lim_{x\to\infty} f(x) = \infty = \lim_{x\to -\infty} f(x)$.
- No VAs.
- Info from f': $f'(x) = 4x^3 + 12x^2 = 4x^2(x+3)$



local min at $x=-3$

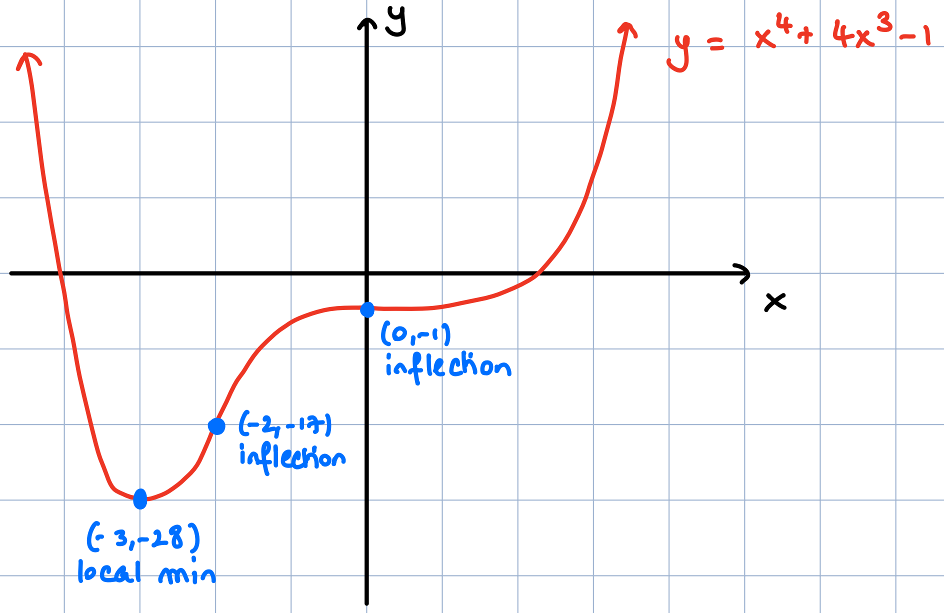
- Info from f'': $f''(x) = 12x^2 + 24x = 12x(x+2)$



inflection at $x=-2, 0$.

- Recap

x	$-\infty$	-3	-2	0	∞
f(x)	∞	-28	-17	-1	∞
inc.					
conc.					



2) $f(x) = \cos(2x) - x$ on $[0, \pi]$.

- Info from f' : $f'(x) = -2\sin(2x) - 1$

	0		$7\pi/12$		$11\pi/12$		π		
Shape of f		$\searrow$		$\nearrow$		$\searrow$			
sign of f'		$-$	0	$+$	0	$-$			

local min at $x = \frac{7\pi}{12}$

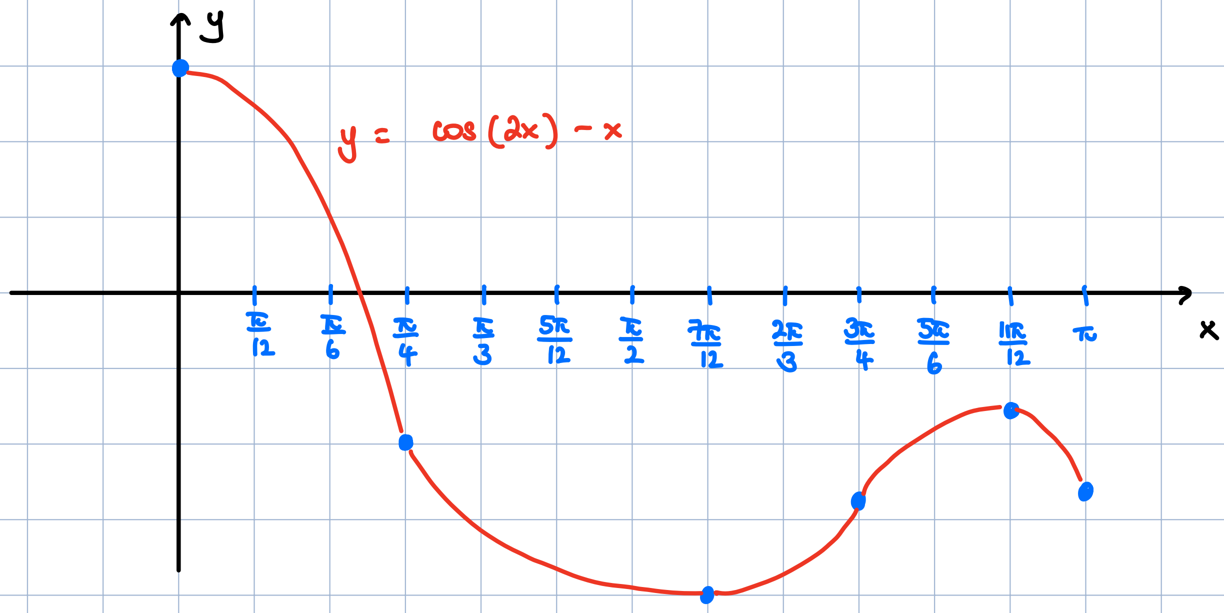
local max at $x = \frac{11\pi}{12}$.

- Info from f'' : $f''(x) = -4\cos(2x)$

	0		$\pi/4$		$3\pi/4$		π
Shape of f		$\frown$		$\smile$		$\frown$	
sign of f''		$-$	0	$+$	0	$-$	

- Recap

x	0	$\frac{\pi}{4}$	$\frac{7\pi}{12}$	$\frac{3\pi}{4}$	$\frac{11\pi}{12}$	π
f(x)	1	$-\frac{\pi}{4}$	$-\frac{\sqrt{3}}{2} - \frac{7\pi}{12}$	$-\frac{3\pi}{4}$	$\frac{\sqrt{3}}{2} - \frac{11\pi}{12}$	$1-\pi$
inc.	$\searrow$	$\searrow$		$\nearrow$		$\searrow$
conc.	$\frown$		$\smile$		$\frown$	



y
x
y = cos(2x) − x
π/12
π/6
π/4
π/3
5π/12
π/2
7π/12
2π/3
3π/4
5π/6
11π/12
π