Rutgers University
Math 151

Sections 4.4: Concavity and Curve Sketching - Worksheet Solutions

1. Find the intervals where the functions below are concave up, concave down and find the inflection points.

(a) $f(x) = \dfrac{1}{x^2+12}$

Solution. We need to find the second derivative of f. We have

$$\begin{aligned} f'(x) &= -\frac{1}{(x^2+12)^2}(2x) = -2x(x^2+12)^{-2}, \\ f''(x) &= -2(x^2+12)^{-2} - 2x(-2)(x^2+12)^{-3}(2x) \\ &= -2(x^2+12)^{-3}\left((x^2+12-4x^2)\right) \\ &= -2\frac{12-3x^2}{(x^2+12)^3} \\ &= -6\frac{(2-x)(2+x)}{(x^2+12)^3}. \end{aligned}$$

We now use a sign analysis to determine the intervals on which $f''(x)$ is positive and negative.

- On $(-\infty,-2)$, the sign of $f''(x)$ is $(-)\frac{(+)(-)}{(+)} = (+)$.
- On $(-2,2)$, the sign of $f''(x)$ is $(-)\frac{(+)(+)}{(+)} = (-)$.
- On $(2,\infty)$, the sign of $f''(x)$ is $(-)\frac{(-)(+)}{(+)} = (+)$.

Therefore f is $\boxed{\text{concave up on } (-\infty,-2),(2,\infty)}$ and $\boxed{\text{concave down on } (-2,2)}$. The inflection points of f are $\boxed{\left(-2,\frac{1}{16}\right),\left(2,\frac{1}{16}\right)}$.

(b) $f(x) = x^4e^{-3x}$

Solution. We need to find the second derivative of f. We have

$$\begin{aligned} f'(x) &= 4x^3e^{-3x} - 3x^4e^{-3x} = e^{-3x}(4x^3-3x^4), \\ f''(x) &= e^{-3x}(12x^2-12x^3) - 3e^{-3x}(4x^3-3x^4) \\ &= e^{-3x}(12x^2-12x^3-12x^3+9x^4) \\ &= e^{-3x}(12x^2-24x^3+9x^4) \\ &= 3x^2e^{-3x}(4-8x+3x^2) \\ &= 3x^2e^{-3x}(3x-2)(x-2). \end{aligned}$$

We now use a sign analysis to determine the intervals on which $f''(x)$ is positive and negative.

- On $(-\infty,0)$, the sign of $f''(x)$ is $(+)$.

- On $\left(0, \frac{2}{3}\right)$, the sign of $f''(x)$ is $(+)$.
- On $\left(\frac{2}{3}, 2\right)$, the sign of $f''(x)$ is $(-)$.
- On $(2, \infty)$, the sign of $f''(x)$is $(+)$.

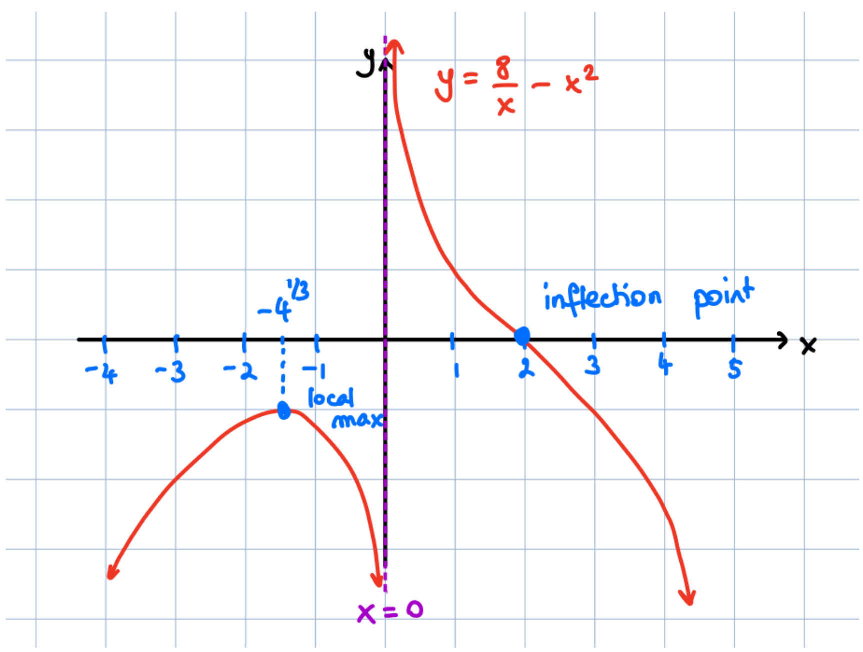
Therefore f is $\boxed{\text{concave up on } (-\infty, 0), \left(0, \frac{2}{3}\right), (2, \infty)}$ and $\boxed{\text{concave down on } \left(\frac{2}{3}, 2\right)}$. The inflection points of f are $\boxed{\left(\frac{2}{3}, \frac{16e^{-2}}{81}\right), (2, 16e^{-6})}$.

2. Sketch the graphs of the following functions. Your graph should clearly show any asymptotes, local extrema and inflection points of the functions.

(a) $f(x) = \frac{8}{x} - x^2$

Solution.

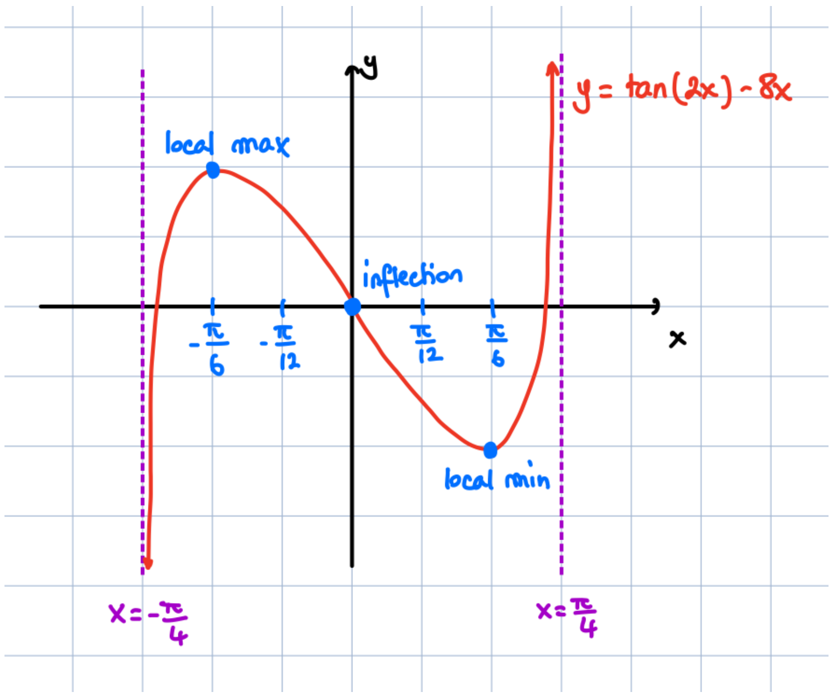
- Horizontal asymptotes: none since $\lim_{x\to\infty} f(x) = -\infty = \lim_{x\to-\infty} f(x)$.
- Vertical asymptotes: $x = 0$.
- Info from first derivative: $f'(x) = -\frac{8}{x^2} - 2x = -2\frac{4+x^3}{x^2}$. The critical point of f is $x = -4^{1/3}$, and $f'(x)$ is positive on $(-\infty, -4^{1/3})$ and negative on $(-4^{1/3}, 0)$, $(0, \infty)$. So f is increasing on $(-\infty, -4^{1/3})$ and decreasing on $(-4^{1/3}, 0)$, $(0, \infty)$. Therefore, f has a local maximum at $x = -4^{1/3}$.
- Info from second derivative: $f''(x) = \frac{16}{x^3} - 2 = 2\frac{8-x^3}{x^3}$. The sign of $f''(x)$ is positive on $(0, 2)$ and negative on $(-\infty, 0)$, $(2, \infty)$. Therefore, f is concave up on $(0, 2)$ and concave down on $(-\infty, 0)$, $(2, \infty)$, and f has an inflection point at $x = 2$.



(b) $f(x) = \tan(2x) - 8x$ on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

Solution.

- Horizontal asymptotes: none since we are graphing on a bounded interval.
- Vertical asymptotes: $x = \frac{\pi}{4}$, $x = -\frac{\pi}{4}$ (since $\tan(x)$ has infinite discontinuities at $x = \pm\frac{\pi}{2}$).
- Info from first derivative: $f'(x) = 2\sec^2(2x) - 8 = 2(\sec(2x) - 2)(\sec(2x) + 2)$. To find the critical points of f, we need to solve $f'(x) = 0$, which gives $\sec(2x) = -2$ (no solution in the interval) and $\sec(2x) = 2$ (solutions in the interval are $x = \pm\frac{\pi}{6}$). The sign of $f'(x)$ is positive (so f is increasing) on $\left(-\frac{\pi}{4}, -\frac{\pi}{6}\right)$ and $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$, and negative (so f is decreasing) on $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$. So f has a local maximum at $x = -\frac{\pi}{6}$ and a local minimum at $x = -\frac{\pi}{6}$.
- Info from second derivative: $f''(x) = 4\sec(2x)\sec(2x)\tan(2x)(2) = 8\sec^2(2x)\tan(2x)$. We see that $f''(x) > 0$ (so f is concave up) on $\left(-\frac{\pi}{4}, 0\right)$ and $f''(x) < 0$ (so f is concave down) on $\left(0, \frac{\pi}{4}\right)$. Therefore, f has an inflection point at $x = 0$.



3. Suppose that f is continuous on $(-\infty, \infty)$, that $f'(x) = \dfrac{x}{(x+4)^{1/3}}$ and that $f''(x) = \dfrac{2x+12}{3(x+4)^{4/3}}$.

(a) Find the critical points of f.

Solution.

- $f'(x) = 0$ when $x = 0$.
- $f'(x)$ is undefined when $x = -4$.

Therefore, the critical points of f are $\boxed{x = -4, 0}$.

(b) Find the intervals where f is increasing and the intervals where f is decreasing.

Solution. $f'(x) > 0$ on $(-\infty, -4)$, $(0, \infty)$, and $f'(x) < 0$ on $(-4, 0)$. Therefore, f is increasing on $\boxed{(-\infty, -4], [0, \infty)}$ and decreasing on $\boxed{[-4, 0]}$.

(c) Find the location of the local extrema of f.

Solution. Based on our previous answer, f has a local maximum at $\boxed{x = -4}$ and a local minimum at $\boxed{x = 0}$.

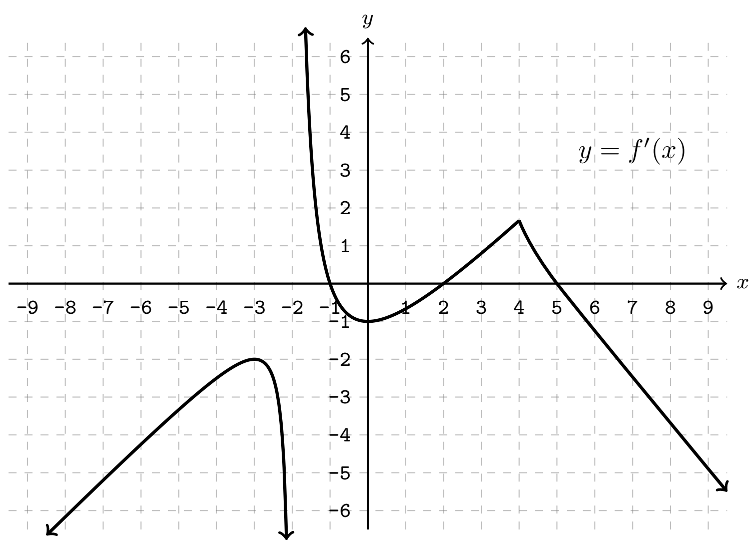
(d) Find the intervals where f is concave up and the intervals where f is concave down.

Solution. $f''(x) > 0$ on $(-6, \infty)$ and $f''(x) < 0$ on $(-\infty, -6)$. So f is concave up on $\boxed{[-6, \infty)}$ and concave down on $\boxed{(-\infty, -6]}$.

(e) Find the x-coordinates of the inflection points of f.

Solution. The only place where f changes concavity is $\boxed{x = -6}$.

4. Suppose that f is a differentiable function. The graph of the **derivative** of f, $y = f'(x)$, is sketched below.



(a) Find the critical points of f.

Solution.

- $f'(x) = 0$ when $x = -1, 2, 5$.
- $f'(x)$ is undefined when $x = -2$.

Therefore, the critical points of f are $\boxed{x = -2, -1, 2, 5}$.

(b) Find the intervals where f is increasing and the intervals where f is decreasing.

Solution. $f'(x) > 0$ is positive on $(-2, -1)$ and $(2, 5)$, so f is increasing on $\boxed{[-2, -1], [2, 5]}$. $f'(x) < 0$ on $(-\infty, -2), (-1, 2)$ and $(5, \infty)$, so f is decreasing on $\boxed{(-\infty, -2], [-1, 2], [5, \infty)}$.

(c) Find the location of the local extrema of f.

Solution. f has local maxima at $\boxed{x = -1, 5}$ (f' changes from positive to negative) and local minima at $\boxed{x = -2, 2}$ (f' changes from negative to positive).

(d) Find the intervals where f is concave up and the intervals where f is concave down.

Solution. f is concave up when f' is increasing, which happens on $\boxed{(-\infty, -3], [0, 4]}$. f is concave down when f' is decreasing, which happens on $\boxed{[-3, -2], [-2, 0], [4, \infty)}$.

(e) Find the x-coordinates of the inflection points of f.

Solution. f has inflection points at $\boxed{x = -3, 0, 4}$.