Section 4.5

L'Hôpital's Rule

Learning Goals

Learning Goal	*Homework Problems*
4.5.1 Decide whether L'Hôpital's Rule can be used to evaluate a limit. Apply the rule when appropriate and decide how many times to apply it.	1-77, 79-84, 87, 88.
4.5.2 Identify and classify indeterminate forms. Use limit laws and algebraic manipulations, including logarithms, to make an indeterminate limit suitable for evaluation by L'Hôpital's Rule.	25, 26, 37-42, 46, 51-66, 77, 81, 82, 84-87a, 88.
4.5.3 Answer conceptual questions involving L'Hôpital's Rule.	75-79, 81, 82.

Conceptual introduction: indeterminate forms are expressions that require more work when evaluating limits.

These are:

$\frac{0}{0}$, $\frac{\infty}{\infty}$ $\rightarrow$ L'Hôpital's Rule (L'H) applies

$\infty \cdot 0$, $\infty - \infty$, 1^∞, ∞^0, 0^0 $\rightarrow$ algebra to rewrite in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then L'H.

L'Hôpital's Rule: if substitution in $\lim_{x \to a} \frac{f(x)}{g(x)}$ gives $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

provided the new limit exists or is infinite.

This works also if $a = \pm\infty$ or for one-sided limits.

This is NOT the quotient rule: we are taking the derivative of f and g separately.

Quotient rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.

Examples: 1) Evaluate $\lim_{x \to 0} \frac{e^{2x} - 1}{\ln(x+1)}$ and $\lim_{x \to 5} \frac{\sqrt{x+4} - 3}{x^2 - 4x - 5}$

$\lim_{x \to 0} \frac{e^{2x} - 1}{\ln(x+1)}$ substitution gives $\frac{0}{0}$, can apply L'H.

$$\overset{L'H}{\underset{\frac{0}{0}}{=}} \lim_{x \to 0} \frac{2e^{2x}}{\frac{1}{x+1}} = \frac{2e^0}{\frac{1}{0+1}} = \boxed{2}.$$

$\lim_{x \to 5} \frac{\sqrt{x+4} - 3}{x^2 - 4x - 5}$ substitution gives $\frac{0}{0}$, can apply L'H.

$$\overset{L'H}{\underset{0/0}{=}} \lim_{x\to 5} \frac{\frac{1}{2\sqrt{x+4}}}{2x-4} = \frac{\frac{1}{2\sqrt{5+4}}}{2\cdot 5-4} = \boxed{\frac{1}{36}}.$$

2) Evaluate $\displaystyle\lim_{x\to\infty} \frac{\ln(x)}{3\sqrt{x}}$ and $\displaystyle\lim_{x\to\infty} \frac{e^{1+3x}}{2x-5}$.

$\displaystyle\lim_{x\to\infty} \frac{\ln(x)}{3\sqrt{x}}$ substitution gives $\frac{\infty}{\infty}$: can use L'H.

$$\overset{L'H}{\underset{\infty/\infty}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{3}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{1}{x}\cdot\frac{2\sqrt{x}}{3} = \lim_{x\to\infty} \frac{2}{3\sqrt{x}} = \boxed{0}.$$

$\displaystyle\lim_{x\to\infty} \frac{e^{1+3x}}{2x-5}$ substitution gives $\frac{\infty}{\infty}$: can use L'H.

$$\overset{L'H}{\underset{\infty/\infty}{=}} \lim_{x\to\infty} \frac{3e^{1+3x}}{2} = \text{"}\frac{\infty}{2}\text{"} = \boxed{\infty}.$$

3) Evaluate $\displaystyle\lim_{x\to 0} \frac{x^3}{\sin(2x)-2x}$ and $\displaystyle\lim_{x\to 0} \frac{1-\cos(5x)}{x^4+3x^2}$

$$\lim_{x\to 0} \frac{x^3}{\sin(2x)-2x} \overset{L'H}{\underset{0/0}{=}} \lim_{x\to 0} \frac{3x^2}{2\cos(2x)-2}$$

$$\overset{L'H}{\underset{0/0}{=}} \lim_{x\to 0} \frac{6x}{-4\sin(2x)}$$

$$\overset{L'H}{\underset{0/0}{=}} \lim_{x\to 0} \frac{6}{-8\cos(2x)}$$ not $\frac{0}{0}$ anymore: stop L'H

$$= -\frac{6}{8} = \boxed{-\frac{3}{4}}.$$

$$\lim_{x\to 0} \frac{1-\cos(5x)}{x^4+3x^2} \overset{L'H}{\underset{\frac{0}{0}}{=}} \lim_{x\to 0} \frac{5\sin(5x)}{4x^3+6x}$$

$$\overset{L'H}{\underset{\frac{0}{0}}{=}} \lim_{x\to 0} \frac{25\cos(5x)}{12x^2+6}$$ not $\frac{0}{0}$ anymore: stop L'H

$$= \boxed{\frac{25}{6}}.$$

4) Evaluate $\lim_{x\to 0^+} x^4 \ln(x)$ and $\lim_{x\to -\infty} x^2 e^{6x}$.

$\lim_{x\to 0^+} x^4\ln(x)$ $0\cdot\infty$: rewrite as a fraction and use L'H.

$$= \lim_{x\to 0^+} \frac{\ln(x)}{x^{-4}} \overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-4x^{-5}} = \lim_{x\to 0^+} \frac{x^5}{-4x} = \lim_{x\to 0^+} \frac{x^4}{-4} = \boxed{0}.$$

$\lim_{x\to -\infty} x^2 e^{6x}$ $\infty\cdot 0$: rewrite as a fraction and use L'H.

$$= \lim_{x\to -\infty} \frac{x^2}{e^{-6x}} \overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to -\infty} \frac{2x}{-6e^{-6x}} \overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to -\infty} \frac{2}{36e^{-6x}} = \lim_{x\to -\infty} \frac{e^{6x}}{18}$$

$$= \boxed{\infty}.$$

5) Evaluate $\lim_{x\to 0} \frac{1}{\sin(x)} - \frac{1}{x}$.

$\lim_{x\to 0^+} \frac{1}{\sin(x)} - \frac{1}{x}$ $\infty - \infty$: combine into a fraction and use L'H.

$$= \lim_{x\to 0^+} \frac{x-\sin(x)}{x\sin(x)} \overset{L'H}{\underset{\frac{0}{0}}{=}} \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)+x\cos(x)} \overset{L'H}{\underset{\frac{0}{0}}{=}} \lim_{x\to 0} \frac{\sin(x)}{\cos(x)+\cos(x)-x\sin(x)}$$

$$= \lim_{x\to 0^+} \frac{\sin(x)}{2\cos(x)-x\sin(x)} = \frac{0}{2+0} = \boxed{0}.$$

6) Evaluate $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ and $\lim\limits_{x\to\infty}\left(1-\frac{5}{x}\right)^x$.

$L = \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ $\quad$ 1^∞ indeterminate power : apply ln to both sides.

$\ln(L) = \ln\left(\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right) \underset{\uparrow}{=} \lim\limits_{x\to\infty}\ln\left(\left(1+\frac{1}{x}\right)^x\right)$

ln continuous

$\ln(L) = \lim\limits_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)$ $\quad$ $\infty\cdot 0$: rewrite as fraction and L'H

$= \lim\limits_{x\to\infty}\dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \overset{L'H}{\underset{\frac{0}{0}}{=}} \lim\limits_{x\to\infty}\dfrac{\frac{1}{1+\frac{1}{x}}\cdot\cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}} = \lim\limits_{x\to\infty}\dfrac{1}{1+\frac{1}{x}} = 1.$

⚠ this is ln(L), not the original limit

$\ln(L) = 1 \Rightarrow L = e^1 = \boxed{e}$.

$L = \lim\limits_{x\to\infty}\left(1-\frac{5}{x}\right)^x$ $\quad$ 1^∞ indeterminate power : apply ln to both sides.

$\ln(L) = \ln\left(\lim\limits_{x\to\infty}\left(1-\frac{5}{x}\right)^x\right) \underset{\uparrow}{=} \lim\limits_{x\to\infty}\ln\left(\left(1-\frac{5}{x}\right)^x\right)$

ln continuous

$\ln(L) = \lim\limits_{x\to\infty} x\ln\left(1-\frac{5}{x}\right)$ $\quad$ $\infty\cdot 0$: rewrite as fraction and L'H

$= \lim\limits_{x\to\infty}\dfrac{\ln\left(1-\frac{5}{x}\right)}{\frac{1}{x}} \overset{L'H}{\underset{\frac{0}{0}}{=}} \lim\limits_{x\to\infty}\dfrac{\frac{1}{1-\frac{5}{x}}\cdot\cancel{\frac{-1}{x^2}}(-5)}{\cancel{\frac{-1}{x^2}}} = \lim\limits_{x\to\infty}\dfrac{-5}{1-\frac{5}{x}} = -5$

⚠ this is ln(L), not the original limit

$\ln(L) = 1 \Rightarrow L = \boxed{e^{-5}}$.

7) Evaluate $\lim\limits_{x\to\infty} x^{1/x}$ and $\lim\limits_{x\to\infty}\left(2^x+1\right)^{1/x}$.

$L = \lim\limits_{x\to\infty} x^{1/x}$ $\quad$ ∞^0 indeterminate power : apply ln to both sides.

$$\ln(L) = \lim_{x\to\infty} \ln\left(x^{1/x}\right) = \lim_{x\to\infty} \frac{1}{x}\ln(x) = \lim_{x\to\infty} \frac{\ln(x)}{x}$$

$$\overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$ ⚠ this is ln(L), not the original limit

$$\ln(L) = 0 \Rightarrow L = e^0 = \boxed{1}$$

$$L = \lim_{x\to\infty} \left(2^x+1\right)^{1/x}$$ ∞^0 indeterminate power : apply ln to both sides.

$$\ln(L) = \lim_{x\to\infty} \ln\left(\left(2^x+1\right)^{1/x}\right) = \lim_{x\to\infty} \frac{1}{x}\ln(2^x+1) = \lim_{x\to\infty} \frac{\ln(2^x+1)}{x}$$

$$\overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{\frac{1}{2^x+1}\ln(2)2^x}{1} = \lim_{x\to\infty} \frac{\ln(2)2^x}{2^x+1}\cdot\frac{\frac{1}{2^x}}{\frac{1}{2^x}} = \lim_{x\to\infty} \frac{\ln(2)}{1+\frac{1}{2^x}}$$

$$= \ln(2)$$ ⚠ This is ln(L), not the original limit.

$$\ln(L) = \ln(2) \Rightarrow L = e^{\ln(2)} = \boxed{2}.$$

⚠ Warning : L'Hôpital's Rule does not work for every limit !

Some examples :

- $$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} \overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{\frac{2x}{2\sqrt{x^2+1}}}{1} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{1}{\frac{2x}{2\sqrt{x^2+1}}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$$: back to the original limit !

L'H gives a repeating cycle and never yields a determinate form.

But we can calculate this limit with algebra.

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to\infty} \frac{\sqrt{x^2(1+\frac{1}{x^2})}}{x} = \lim_{x\to\infty} \frac{|x|\sqrt{1+\frac{1}{x^2}}}{x} = \lim_{x\to\infty} \sqrt{1+\frac{1}{x^2}} = 1.$$

$x>0$
so $|x| = x$

- $\lim_{x\to\infty} \dfrac{x+\cos(x)}{x}$ $\quad \frac{\infty}{\infty}$

L'H would yield : $\lim_{x\to\infty} \dfrac{1+\cos(x)}{1} = \lim_{x\to\infty} 1+\cos(x)$ DNE so L'H does not apply.

Instead, we can calculate this limit with the Squeeze Theorem.

Since $-1 \leq \cos(x) \leq 1$, we have $\dfrac{x-1}{x} \leq \dfrac{x+\cos(x)}{x} \leq \dfrac{x+1}{x}$

and $\lim_{x\to\infty} \dfrac{x-1}{x} = \lim_{x\to\infty} \dfrac{x+1}{x} = 1$

So $\lim_{x\to\infty} \dfrac{x+\cos(x)}{x} = 1$.