Rutgers University
Math 151

Section 4.5: L'Hôpital's Rule - Worksheet Solutions

1. Evaluate the following limits. **Note:** L'Hôpital's Rule is not possible/necessary for every limit.

   (a) $\displaystyle\lim_{x\to 8} \frac{\sqrt[3]{x}-2}{64-x^2}$

   Solution. We can compute this limit using L'Hôpital's Rule twice with the indeterminate form $\frac{0}{0}$. This gives

$$\begin{aligned}\lim_{x\to 8} \frac{\sqrt[3]{x}-2}{64-x^2} &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to 8} \frac{\frac{1}{3}x^{-2/3}}{-2x} \\ &= \frac{\frac{1}{3}\cdot 8^{-2/3}}{-16} \\ &= \boxed{-\frac{1}{192}}.\end{aligned}$$

   (b) $\displaystyle\lim_{x\to\infty} \frac{\ln(x)^2}{\sqrt{x}}$

   Solution. We can compute this limit using L'Hôpital's Rule twice with the indeterminate form $\frac{\infty}{\infty}$. This gives

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln(x)^2}{\sqrt{x}} &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{2\ln(x)\frac{1}{x}}{\frac{1}{2\sqrt{x}}} \\ &= \lim_{x\to\infty} \frac{4\ln(x)}{\sqrt{x}} \\ &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{\frac{4}{x}}{\frac{1}{2\sqrt{x}}} \\ &= \lim_{x\to\infty} \frac{8}{\sqrt{x}} \\ &= \boxed{0}.\end{aligned}$$

   (c) $\displaystyle\lim_{x\to 0} \frac{5^x-3^x}{\sin(2x)}$

   Solution. This limit is an indeterminate form $\frac{0}{0}$. We can resolve the indeterminate form using L'Hôpital's Rule, remembering that for a positive constant a, we have

$$\frac{d}{dx}a^x = \ln(a)a^x.$$

We obtain

$$\begin{aligned}
\lim_{x\to 0}\frac{5^x-3^x}{\sin(2x)} &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to 0}\frac{\ln(5)5^x-\ln(3)3^x}{2\cos(2x)} \\
&= \frac{\ln(5)5^0-\ln(3)^0}{2\cos(2\cdot 0)} \\
&= \boxed{\frac{\ln(5)-\ln(3)}{2}}.
\end{aligned}$$

(d) $\displaystyle\lim_{\theta\to\frac{\pi}{2}}\frac{1-\csc(\theta)}{1-\sec(4\theta)}$

Solution. Solution. We can compute this limit using L'Hôpital's Rule twice with the indeterminate form $\frac{0}{0}$. This gives

$$\begin{aligned}
\lim_{\theta\to\frac{\pi}{2}}\frac{1-\csc(\theta)}{1-\sec(4\theta)} &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{\theta\to\frac{\pi}{2}}\frac{\csc(\theta)\cot(\theta)}{-4\sec(4\theta)\tan(4\theta)} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{\theta\to\frac{\pi}{2}}\frac{-\csc(\theta)\cot(\theta)\cot(\theta)+\csc(\theta)(-\csc^2(\theta))}{-16\sec(4\theta)\tan(4\theta)\tan(4\theta)-16\sec(4\theta)\sec^2(4\theta)} \\
&= \lim_{\theta\to\frac{\pi}{2}}\frac{-\csc(\theta)\cot^2(\theta)-\csc^3(\theta)}{-16\sec(4\theta)\tan^2(4\theta)-16\sec^3(4\theta)} \\
&= \frac{-1\cdot 0^2-1^3}{-16\cdot 1\cdot 0^2-16\cdot 1^3} \\
&= \boxed{\frac{1}{16}}.
\end{aligned}$$

(e) $\displaystyle\lim_{x\to\infty}\ln(5x+1)-\ln(x)$

Solution. This limit is an indeterminate form $\infty-\infty$. It can be evaluated by combining the logarithms and evaluating the limit of the inside. This gives

$$\begin{aligned}
\lim_{x\to\infty}\ln(5x+1)-\ln(x) &= \lim_{x\to\infty}\ln\left(\frac{5x+1}{x}\right) \\
&= \lim_{x\to\infty}\ln\left(5+\frac{1}{x}\right) \\
&= \ln(5+0) \\
&= \boxed{\ln(5)}.
\end{aligned}$$

(f) $\displaystyle\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

Solution. This limit is an indeterminate power 1^∞. **Warning:** limits of the form 1^∞ need not be equal to 1! This is because the base is not equal to 1, it is *approaching* 1. We can resolve the indeterminate form by rewriting the power with an exponential using the formula

$$a^b = e^{b\ln(a)}$$

and applying L'Hôpital's Rule in the resulting exponent. This gives

$$\begin{aligned}
\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x} &= \lim_{x\to\infty} e^{x\ln\left(1+\frac{2}{x}\right)} \\
&= e^{\lim_{x\to\infty}\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} e^{\lim_{x\to\infty}\frac{-\frac{2}{x^2}\cdot\frac{1}{1+\frac{2}{x}}}{-\frac{1}{x^2}}} \\
&= e^{\lim_{x\to\infty}\frac{2}{1+\frac{2}{x}}} \\
&= \boxed{e^2}.
\end{aligned}$$

(g) $\displaystyle\lim_{x\to 0}\frac{2^{\sin(x)}-1}{\sin^{-1}(5x)}$

Solution. This limit is a $\frac{0}{0}$ indeterminate form, which we can evaluate using L'Hôpital's Rule. We obtain

$$\begin{aligned}
\lim_{x\to 0}\frac{2^{\sin(x)}-1}{\sin^{-1}(5x)} &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to 0}\frac{\ln(2)2^{\sin(x)}\cos(x)}{\frac{5}{\sqrt{1-(5x)^2}}} \\
&= \boxed{\frac{\ln(2)}{5}}.
\end{aligned}$$

(h) $\displaystyle\lim_{x\to -\infty}\frac{2x+3\cos(x)}{5x}$

Solution. This limit is an indeterminate form $\frac{\infty}{\infty}$. However, we **cannot use L'Hôpital's Rule** here. This is because L'Hôpital's Rule only applies if the resulting limit exists or is infinite, but here, the resulting limit

$$\lim_{x\to -\infty}\frac{2-3\sin(x)}{5}$$

does not exist. The Squeeze (or Sandwich) Theorem will work for this limit. Since $-1 \leqslant \cos(x) \leqslant 1$ for all x, we have

$$\frac{2x-3}{5x} \leqslant \frac{2x+3\cos(x)}{5x} \leqslant \frac{2x+3}{5x}$$

for any $x \neq 0$. Furthermore, we have

$$\begin{aligned}
\lim_{x\to -\infty}\frac{2x-3}{5x} &= \lim_{x\to\infty}\frac{2}{5}-\frac{3}{5x} = \frac{2}{5}, \\
\lim_{x\to -\infty}\frac{2x+3}{5x} &= \lim_{x\to\infty}\frac{2}{5}+\frac{3}{5x} = \frac{2}{5}.
\end{aligned}$$

Since the two limits are equal, we conclude that

$$\boxed{\lim_{x\to -\infty}\frac{2x+3\cos(x)}{5x} = \frac{2}{5}}.$$

(i) $\lim_{x\to\infty} x^{1/x}$

Solution. This limit is an indeterminate power ∞^0. **Warning:** limits of the form ∞^0 need not be equal to 1! We can resolve the indeterminate form by rewriting the power with an exponential using the formula

$$a^b = e^{b\ln(a)}$$

and applying L'Hôpital's Rule in the resulting exponent. This gives

$$\begin{aligned}
\lim_{x\to\infty} x^{1/x} &= \lim_{x\to\infty} e^{\frac{\ln(x)}{x}} \\
&= e^{\lim_{x\to\infty} \frac{\ln(x)}{x}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} e^{\lim_{x\to\infty} \frac{1/x}{1}} \\
&= e^0 \\
&= \boxed{1}.
\end{aligned}$$

(j) $\lim_{x\to-\infty} x^3 e^{5x+2}$

Solution. This limit is an indeterminate form $\infty \cdot 0$. We can resolve the indeterminate form by rewriting the expression as a fraction of the form $\frac{\infty}{\infty}$ and applying L'Hôpital's Rule 3 times. This gives

$$\begin{aligned}
\lim_{x\to-\infty} x^3 e^{5x+2} &= \lim_{x\to-\infty} \frac{x^3}{e^{-5x-2}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to-\infty} \frac{3x^2}{-5e^{-5x-2}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to-\infty} \frac{6x}{25e^{-5x-2}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to-\infty} \frac{6}{-125e^{-5x-2}} \\
&= \boxed{0}.
\end{aligned}$$

(k) $\lim_{x\to0^+} \sqrt[3]{x}\log_2(x)$

Solution. This limit is an indeterminate form $0 \cdot \infty$. We can resolve the indeterminate form by rewriting the expression as a fraction of the form $\frac{\infty}{\infty}$ and applying L'Hôpital's Rule. This gives

$$\begin{aligned}
\lim_{x\to0^+} \sqrt[3]{x}\log_2(x) &= \lim_{x\to0^+} \frac{\log_2(x)}{x^{-1/3}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to0^+} \frac{\frac{1}{\ln(2)x}}{-\frac{1}{3}x^{-4/3}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to0^+} \frac{-3x^{1/3}}{\ln(2)} \\
&= \boxed{0}.
\end{aligned}$$

(l) $\displaystyle\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+4}}$

Solution. This limit is an indeterminate form $\frac{\infty}{\infty}$, but using L'Hôpital's Rule would result in an infinite loop and would not help evaluate the limit. Instead, we use algebra to cancel out the highest powers of x. We have

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+4}} &= \lim_{x\to-\infty} \frac{x}{\sqrt{x^2\left(1+\frac{4}{x^2}\right)}} \\
&= \lim_{x\to-\infty} \frac{x}{|x|\sqrt{1+\frac{4}{x^2}}} \\
&= \lim_{x\to-\infty} \frac{x}{-x\sqrt{1+\frac{4}{x^2}}} \qquad (x<0) \\
&= \lim_{x\to-\infty} \frac{1}{-\sqrt{1+\frac{4}{x^2}}} \\
&= \boxed{-1}.
\end{aligned}$$

(m) $\displaystyle\lim_{x\to0} \cos(3x)^{1/x^2}$

Solution. This limit is an indeterminate power 1^∞. **Warning:** limits of the form 1^∞ need not be equal to 1! This is because the base is not equal to 1, it is *approaching* 1. We can resolve the indeterminate form by rewriting the power with an exponential using the formula

$$a^b = e^{b\ln(a)}$$

and applying L'Hôpital's Rule in the resulting exponent. This gives

$$\lim_{x\to0} \cos(3x)^{1/x^2} = \lim_{x\to0} e^{\ln(\cos(3x))/x^2}$$

Now we calculate the limit of the exponent using L'Hôpital's Rule and we obtain

$$\begin{aligned}
\lim_{x\to0} \frac{\ln(\cos(3x))}{x^2} &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to0} \frac{\frac{1}{\cos(3x)}(-\sin(3x))3}{2x} \\
&= \lim_{x\to0} \frac{-3\tan(3x)}{2x} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to0} \frac{-9\sec^2(3x)}{2} \\
&= -\frac{9}{2}.
\end{aligned}$$

Going back to the original limit, we obtain

$$\lim_{x\to0} \cos(3x)^{1/x^2} = \lim_{x\to0} e^{\ln(\cos(3x))/x^2} = \boxed{e^{-9/2}}.$$

(n) $\lim_{x\to\infty}\left(\frac{x+5}{x+3}\right)^{4x}$

Solution. This limit is an indeterminate power 1^∞. **Warning:** limits of the form 1^∞ need not be equal to 1! This is because the base is not equal to 1, it is *approaching* 1. We can resolve the indeterminate form by rewriting the power with an exponential using the formula

$$a^b = e^{b\ln(a)}$$

and applying L'Hôpital's Rule in the resulting exponent. This gives

$$\lim_{x\to\infty}\left(\frac{x+5}{x+3}\right)^{4x} = \lim_{x\to\infty} e^{4x\ln\left(\frac{x+5}{x+3}\right)}.$$

We now compute the limit of the exponent using L'Hôpital's Rule:

$$\begin{aligned}
\lim_{x\to\infty} 4x\ln\left(\frac{x+5}{x+3}\right) &= \lim_{x\to\infty} 4\frac{\ln(x+5)-\ln(x+3)}{\frac{1}{x}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} 4\frac{\frac{1}{x+5}-\frac{1}{x+3}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} -4x^2\frac{(x+3)-(x+5)}{(x+5)(x+3)} \\
&= \lim_{x\to\infty} \frac{8x^2}{(x+5)(x+3)}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{8}{(1+5/x)(1+3/x)} \\
&= 8.
\end{aligned}$$

So

$$\lim_{x\to\infty} e^{4x\ln\left(\frac{x+5}{x+3}\right)} = \boxed{e^8}.$$

(o) $\lim_{x\to\infty} x^{1/\ln(x+1)}$

Solution. This limit is an indeterminate power ∞^0. **Warning:** limits of the form ∞^0 need not be equal to 1! We can resolve the indeterminate form by rewriting the power with an exponential using the formula

$$a^b = e^{b\ln(a)}$$

and applying L'Hôpital's Rule in the resulting exponent. This gives

$$\lim_{x\to\infty} x^{1/\ln(x+1)} = \lim_{x\to\infty} e^{\frac{\ln(x)}{\ln(x+1)}}$$

We now compute the limit of the exponent using L'Hôpital's Rule, an we obtain

$$\begin{aligned}
\lim_{x\to\infty}\frac{\ln(x)}{\ln(x+1)} &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{x+1}} \\
&= \lim_{x\to\infty}\frac{x+1}{x} \\
&= 1.
\end{aligned}$$

Therefore

$$\lim_{x\to\infty} x^{1/\ln(x+1)} = \lim_{x\to\infty} e^{\frac{\ln(x)}{\ln(x+1)}} = e^1 = \boxed{e}.$$