Section 4.6

Optimization

Learning Goals

Learning Goal	*Homework Problems*
4.6.1 Given a word problem about optimization, determine appropriate variables, a function to optimize, and the interval on which it should be optimized. Use these to solve the word problem.	1-49a, 50a, 51-56, 64, 66a, 67- 71a, 72a, 73, 74.
4.6.2 Decide whether a function has a max or min on an open, unbounded interval. If so, find it. Justify your conclusions.	8-11, 14, 15, 23, 28-30, 35, 36, 38-41, 43, 45, 52, 53, 73, 74.

<u>Goal</u> : solve applied problems involving finding min. and max. of functions.

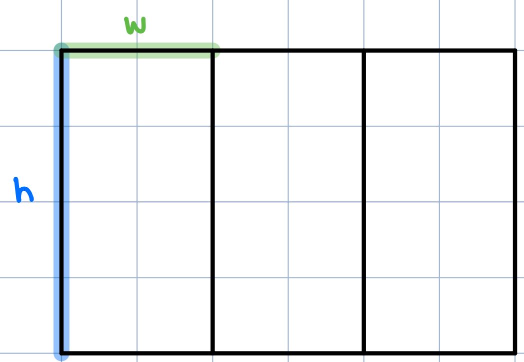
<u>Terminology</u> :

- <u>Objective function</u> : the quantity that we want to optimize. Neede to be expressed as a function of one single variable.

- <u>Feasible intervals</u> : the interval of possible values for the variable, taking physical limitations into account.

<u>Examples</u> : 1) A farmer has 240 ft of fencing to construct 3 adjacent rectangular pens. What dimensions will result in the largest total area?

<u>Step 1</u> : draw picture and name variables.



<u>Step 2</u> : find objective and constraints.

What do we need to maximize?

↳ Area $A = 3wh$

What are the constraints? → 240 ft of fencing

$6w + 4h = 240$.

<u>Step 3</u> : use constraint to express objective in terms of one variable only.

$6w + 4h = 240 \Rightarrow 4h = 240 - 6w = 6(40-w) \Rightarrow h = \frac{3}{2}(40-w)$.

So $A = 3wh = 3w\frac{3}{2}(40-w)$

$$A(w) = \frac{9}{2} w(40-w).$$

<u>Step 4</u> : find feasible interval.

Lengths cannot be negative, so $w \geq 0$

$$h \geq 0 \Rightarrow 240 - 6w \geq 0$$
$$\Rightarrow 40 \geq w.$$

<u>Step 5</u> : find max. of objective function on interval of interest.

$$A(w) = \frac{9}{2}(40w - w^2) \text{ on } [0, 40].$$

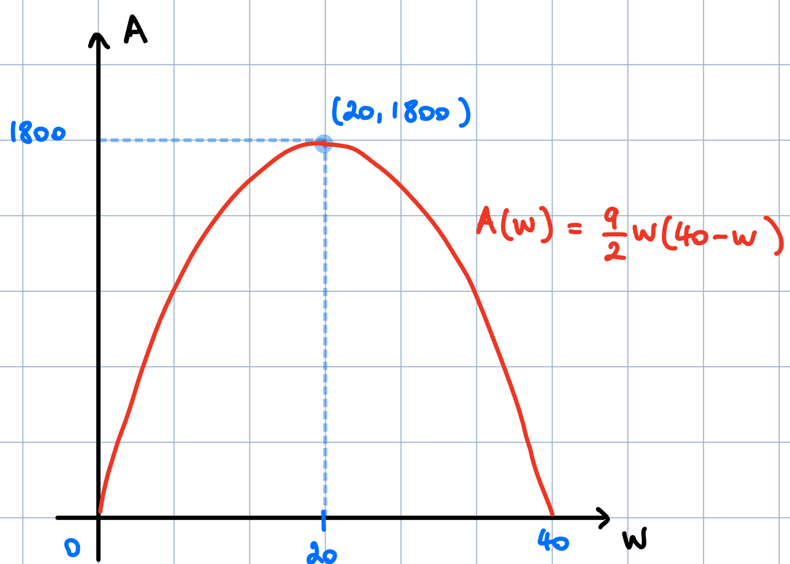
Critical points : $A'(w) = \frac{9}{2}(40 - 2w) = 9(20 - w)$

$A'(w) = 0 \Rightarrow w = 20.$

Evaluate $A(w)$ at critical points and endpoint:

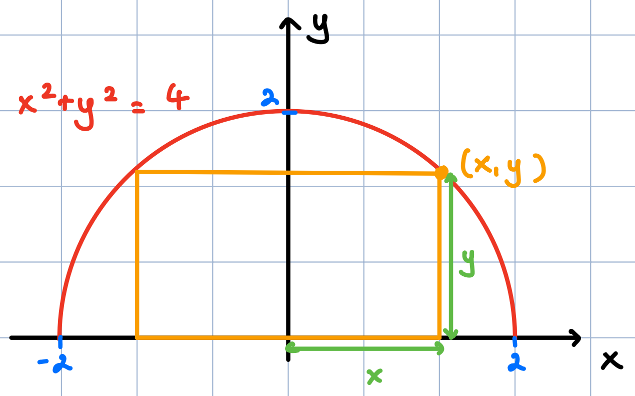
$$\left.\begin{aligned} A(0) &= 0 \\ A(20) &= 1800 \\ A(40) &= 0 \end{aligned}\right\} \Rightarrow$$

$$\boxed{\text{The largest area occurs for } w = 20 \text{ ft and } h = \tfrac{3}{2}(40-w) = 30 \text{ ft}}$$



2) Find the dimensions of the rectangle with largest area that can be inscribed in a semi-circle of radius 2.

Step 1: draw picture and name variables.



Step 2: find objective and constraints.

Objective: Area $A = 2xy$

Constraint: $x^2+y^2 = 4$

Step 3: use constraint to express objective in terms of one variable only.

$$x^2+y^2 = 4 \Rightarrow y^2 = 4-x^2$$

$$|y| = \sqrt{4-x^2}, \quad y \geq 0 \text{ for our purpose.}$$

$$y = \sqrt{4-x^2}$$

So $A(x) = 2x\sqrt{4-x^2}$.

Step 4: find feasible interval: $0 \leq x \leq 2$

Step 5: find max. of objective function on interval of interest.

$A(x) = 2x\sqrt{4-x^2}$ on $[0,2]$.

$$\hookrightarrow A'(x) = 2\left(\sqrt{4-x^2} + x\frac{1}{2\sqrt{4-x^2}}(-2x)\right) = 2\left(\sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}}\right)$$

$$= 2\,\frac{4-x^2-x^2}{\sqrt{4-x^2}} = 2\,\frac{4-2x^2}{\sqrt{4-x^2}} = 4\,\frac{2-x^2}{\sqrt{4-x^2}} = 4\,\frac{(\sqrt{2}-x)(\sqrt{2}+x)}{\sqrt{4-x^2}}.$$

Critical points: $A'(x) = 0 \Rightarrow x = \sqrt{2}, \cancel{-\sqrt{2}}$ not in interval

$A'(x)$ DNE $\Rightarrow 4-x^2 = 0 \Rightarrow x = 2, \cancel{-2}$ not in interval

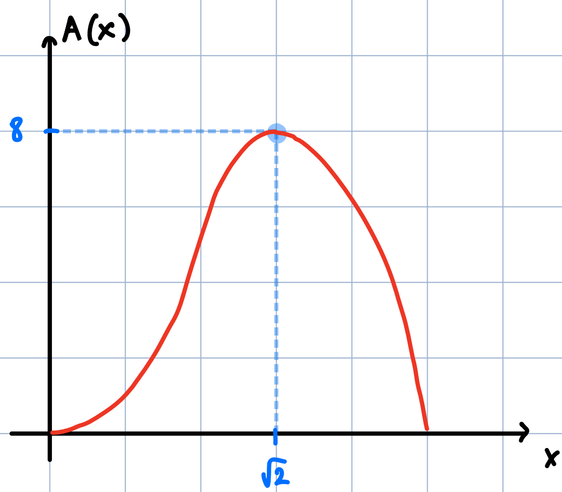
Evaluate $A(x)$ at critical points and endpoint:

$A(0) = 0$

$A(\sqrt{2}) = 2\sqrt{2}\sqrt{4-2} = 4$

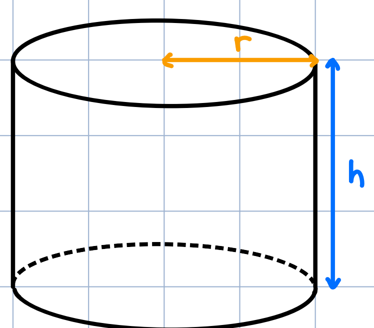
$A(2) = 0$

$\boxed{\text{The rectangle with largest area has width } 2x = 2\sqrt{2}, \text{ height } y = \sqrt{4-x^2} = \sqrt{2}}$



3) A cylindrical box with open top has a volume of 50π in^3.
The material used for the bottom of the box costs \$6 per in^2
The material used for the side of the box costs \$15 per in^2.
Find the dimensions of the cheapest box.

<u>Step 1</u>: draw picture and name variables.



<u>Step 2:</u> find objective and constraints.

Objective: cost C of the box

$C = \$6 \cdot (\text{surface of bottom}) + \$15 \cdot (\text{surface of side})$

$C = 6\pi r^2 + 15 \cdot 2\pi r h$

$C = 6\pi r^2 + 30\pi r h$

Constraints : volume is 50π $\Rightarrow$ $\pi r^2 h = 50\pi$

Step 3 : use constraint to express objective in terms of one variable only.

$$\pi r^2 h = 50\pi \Rightarrow h = \frac{50}{r^2}$$

So $C = 6\pi r^2 + 30\pi r \cdot \frac{50}{r^2} \Rightarrow C(r) = 6\pi r^2 + \frac{1500\pi}{r}$.

$$C(r) = 6\pi\left(r^2 + \frac{250}{r}\right).$$

Step 4 : find feasible interval.

Lengths cannot be negative : $r \geq 0$

$h \geq 0 \Rightarrow \frac{50}{r^2} \geq 0 \Rightarrow r \neq 0$.

So the feasible interval is $(0, \infty)$.

Step 5 : find max. of objective function on interval of interest.

$C(r) = 6\pi\left(r^2 + \frac{250}{r}\right)$ on $(0, \infty)$.

$$\Rightarrow C'(r) = 6\pi\left(2r - \frac{250}{r^2}\right) = 6\pi \frac{2r^3 - 250}{r^2} = 12\pi \frac{r^3 - 125}{r^2}$$

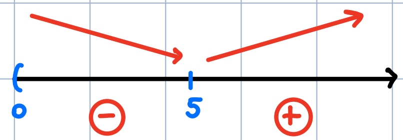
Critical points : $r^3 - 125 = 0 \Rightarrow r^3 = 125 \Rightarrow r = 5$.

⚠ Interval is not closed and bounded.

↳ We cannot find max/min by evaluating at endpoints and critical points.

↳ We use FDT or SDT to classify the critical point.

With FDT :



shape of f
sign of f'

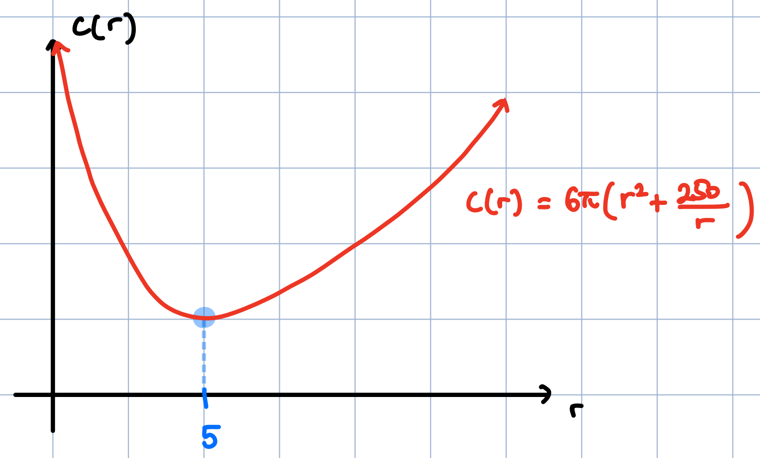
r	$C'(r)$
1	$\frac{(-)}{(+)} = (-)$
6	$\frac{(+)}{(+)} = (+)$

With SDT : $C''(r) = 6\pi\left(2 + \frac{500}{r^3}\right) > 0$ on $(0, \infty)$.

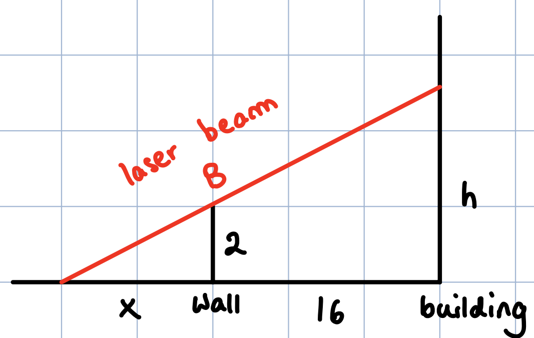
So C concave up on $(0, \infty)$

Either way, we conclude that $C(r)$ is minimal when

$$\boxed{r = 5 \text{ in} \quad \text{and} \quad h = \frac{50}{r^2} = 2 \text{ in}}$$



4) An 2ft wall stands 16ft away from an infinitely tall building. Find the length of the shortest straight beam that will reach the building from the ground outside the wall.



Objective : length of beam. For easier differentiation, we will optimize the square of the length.

$$B = h^2 + (x+16)^2.$$

Constraint : similar triangles $\frac{h}{2} = \frac{x+16}{x} \Rightarrow h = 2\,\frac{x+16}{x}$

So objective is $B(x) = \left(2\,\frac{x+16}{x}\right)^2 + (x+16)^2 = 4\left(\frac{x+16}{x}\right)^2 + (x+16)^2$

Feasable interval : $x > 0$

We find the min. of $B(x) = 4\left(\frac{x+16}{x}\right)^2 + (x+16)^2$ on $[0, \infty)$.

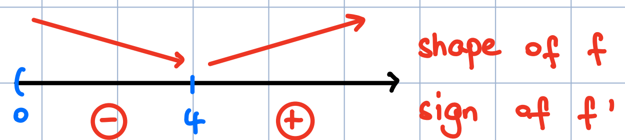
$$B'(x) = 4 \cdot \underbrace{2\,\frac{x+16}{x}}_{\text{outside}} \cdot \underbrace{\frac{x-(x+16)}{x^2}}_{\text{inside}} + 2(x+16) = -\frac{128(x+16)}{x^3} + 2(x+16)$$

$$= -2(x+16)\left(\frac{64}{x^3} - 1\right) = -2(x+16)\,\frac{64-x^3}{x^3}.$$

Critical points : $B'(x) = 0 \Rightarrow x+16 = 0 \Rightarrow \cancel{x = -16}$ not in interval

$64 - x^3 = 0 \Rightarrow x^3 = 64 \Rightarrow x = 4$

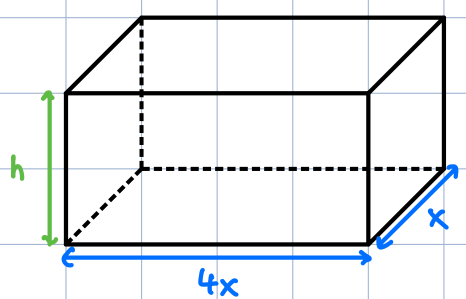
We use FDT to classify the critical point $x = 4$.



x	$B'(x)$
1	$(-)(+)\frac{(+)}{(+)} = (-)$
6	$(-)(+)\frac{(-)}{(+)} = (+)$

So the shortest beam is obtained when $x = 4$ft. It has length $L = \sqrt{2\left(\frac{4+16}{4}\right)^2 + (4+16)^2} = \boxed{20\sqrt{2}\text{ ft}}$.

5) A rectangular box has total surface area 216 in^2 and its length is 4 times its width. Find the dimensions of such a box with largest volume.



Objective : volume $V = 4x^2h$

Constraints : $S = 216$

$2xh + 8xh + 8x^2 = 216$

$10xh + 8x^2 = 216$

So $10xh = 216 - 8x^2 \Rightarrow h = \dfrac{216-8x^2}{10x} = \dfrac{108-4x^2}{5x}$

So the objective is $V(x) = 4x^2 \dfrac{108-4x^2}{5x}$

$$V(x) = \frac{4}{5}x(108-4x^2) = \frac{4}{5}(108x - 4x^3)$$

Feasible interval: lengths must be $\geqslant 0$

$x \geqslant 0$

$h \geqslant 0 \Rightarrow \dfrac{108-4x^2}{5x} > 0 \Rightarrow \begin{array}{l} x^2 \leqslant \frac{108}{4} = 27 \Rightarrow x \leqslant \sqrt{27}. \\ x \neq 0 \end{array}$

So the interval of interest is $(0, \sqrt{27}]$.

We find the absolute maximum of $V(x) = \frac{4}{5}(108x - 4x^3)$ on $(0, \sqrt{27}]$.

$V'(x) = \frac{4}{5}(108 - 12x^2)$.

Critical points: $108 - 12x^2 = 0$

$x^2 = 9$

$x = 3, \cancel{-3}$ not in interval

We use the SDT to classify the critical point $x = 3$.

$V''(x) = \frac{4}{5}(-24x) < 0$ on $(0, \sqrt{27}]$.

So V is concave down on $(0, \sqrt{27}]$. So V is maximal when

$$\boxed{\begin{array}{l} x = 3 \text{ in} \\ h = \dfrac{108-4x^2}{5x} = \dfrac{24}{5} \text{ in} \end{array}}$$