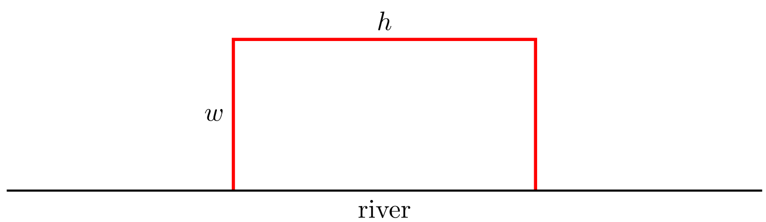
Rutgers University
Math 151

Section 4.6: Optimization - Worksheet Solutions

1. Farmer Brown wants to enclose rectangular pens for the animals on her farm. The three parts of this problem are independent.

   (a) Suppose that Farmer Brown wants to enclose a single pen alongside a river with 300 ft of fencing. The side of the pen alongside the river needs no fencing. What dimensions (length and width) would produce the pen with largest surface area?

   Solution. Call w the width and h the height of the pen, see figure below.



   The objective function is the area of the pen $A = wh$. To express this function in terms of a single variable, we use the constraint given by the fact that the amount of fencing is 300. This gives the equation $2w + h = 300$, so $h = 300 - 2w$. Therefore, the objective function in terms of the variable w is $A(w) = w(300 - 2w) = 300w - 2w^2$.

   To find the feasible interval, observe that lengths cannot be negative, so we need $w \geqslant 0$ and $h \geqslant 0$. This last inequality gives $300 - 2w \geqslant 0$, so $w \leqslant 150$. Therefore, the interval is $[0, 150]$.

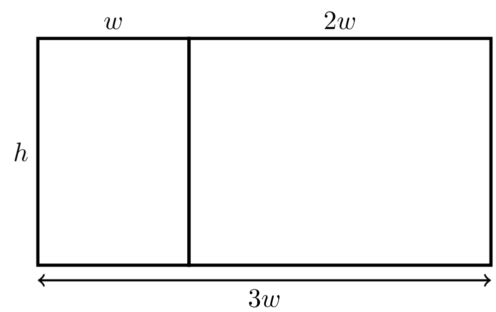
   We now use calculus to find the absolute maximum of $A(w) = 300w - 2w^2$ on $[0, 75]$. First, we find the critical points. We have $A'(w) = 300 - 4w$. The equation $A'(w) = 0$ gives the solution $w = 75$, which is the only critical point. To find the absolute maximum, we now evaluate $A(w)$ at the critical point and the endpoints.

   - $A(0) = 0$
   - $A(75) = 11250$
   - $A(150) = 0$

   Hence, the area of the pen is maximal when its width is $\boxed{w = 75 \text{ ft}}$ and its height is $h = 300 - 2w = \boxed{150 \text{ ft}}$.

   (b) Suppose that Farmer Brown has 360 ft of fencing to enclose 2 adjacent pens. Both pens have the same height, but the second one is twice as wide as the first. What is the largest total area that can be enclosed?

   Solution. Call h the height of the pens and w the width of the smaller one, see figure below.



The objective function is the total area of both pens $A = 3wh$. To express this function in terms of a single variable, we use the constraint given by the fact that the amount of fencing is 360. This gives the equation $6w + 3h = 360$, so $h = 120 - 2w$. Therefore, the objective function in terms of the variable w is $A(w) = 3w(120 - 2w) = 360w - 6w^2$.

To find the feasible interval, observe that lengths cannot be negative, so we need $w \geqslant 0$ and $h \geqslant 0$. This last inequality gives $120 - 2w \geqslant 0$, so $w \leqslant 60$. Therefore, the interval is $[0, 60]$.

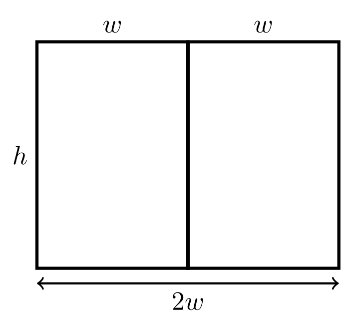
We now use calculus to find the absolute maximum of $A(w) = 360w - 6w^2$ on $[0, 60]$. First, we find the critical points. We have $A'(w) = 360 - 12w$. The equation $A'(w) = 0$ gives the solution $w = 30$, which is the only critical point. To find the absolute maximum, we now evaluate $A(w)$ at the critical point and the endpoints.

- $A(0) = 0$
- $A(75) = 5400$
- $A(60) = 0$

Therefore, the maximal total area that can be enclosed is $\boxed{5,400 \text{ ft}^2}$.

(c) Suppose that Farmer Brown wants to enclose a total of 2,400 ft^2 in two adjacent pens having the same dimensions. What is the minimal amount of fencing needed?

Solution. Call h the height and w the width of the pens, see figure below.



The objective function is the amount of fencing used (or perimeter of the figure), $P = 4w + 3h$. To express this function in terms of a single variable, we use the constraint given by the fact that the total area is 2,400 ft^2. This gives $2wh = 2400$, so $h = \frac{1200}{w}$. Therefore, the objective function in terms of the variable w is $P(w) = 4w + \frac{3600}{w}$.

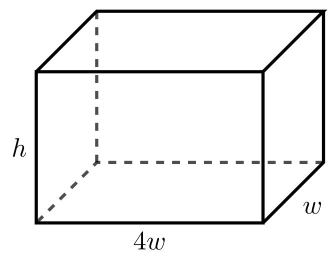
To find the feasible interval, observe that lengths cannot be negative, so we need $w \geqslant 0$ and $h \geqslant 0$. This last inequality gives $\frac{1200}{w} \geqslant 0$, so $w > 0$. Therefore, the interval is $(0, \infty)$.

We now use calculus to find the absolute minimum of $P(w) = 4w + \frac{1200}{w}$ on $(0, \infty)$. First, we find the critical points. We have $P'(w) = 4 - \frac{3600}{w^2}$. The equation $P'(w) = 0$ gives $w^2 = 900$, so $w = \pm 30$. The only critical point in the feasible interval is $w = 30$.

To determine if $w = 30$ gives a local maximum or minimum of $P(w)$, we use the SDT. We have $P''(w) = \frac{7200}{w^3}$. Since $P''(w) > 0$ on $(0, \infty)$, $P(w)$ is concave up on $0, \infty$, and therefore $w = 30$ gives a local minimum of $P(w)$. Hence, the minimal amount of fencing needed is $P(30) = \boxed{240 \text{ ft}}$.

2. A rectangular box has total surface area 216 in^2, and the length of its base is 4 times its width. Find the dimensions of such a box with largest volume.

Solution. Call w the width of the box and h its height, see figure below.



The objective function is the volume of the box $V = h(4w)w = 4hw^2$. The constraint equation is given by the surface area being 216, which gives $2h(4w) + 2w(4w) + 2hw = 216$, or $2(5wh + 4w^2) = 216$. Solving this for h gives $5wh = 108 - 4w^2$, or $h = \frac{108-4w^2}{5w}$. Therefore, the objective function in terms of the variable w is $V(w) = 4\frac{108-4w^2}{5w}w^2 = \frac{16}{5}(27w - w^3)$.

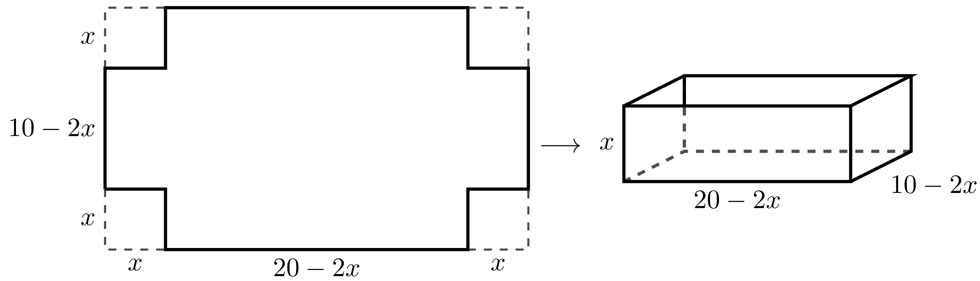
To find the feasible interval, we observe that lengths cannot be negative, so we need $w \geqslant 0$ and $h \geqslant 0$. This last inequality gives $\frac{108-4w^2}{5w} \geqslant 0$, which gives $0 < w \leqslant \sqrt{27}$. So the interval is $(0, \sqrt{27}]$.

We now use calculus to find the absolute maximum of $V(w) = 4\frac{108-4w^2}{5w}w^2 = \frac{16}{5}(27w - w^3)$ on the interval $(0, \sqrt{27}]$. We have $V'(w) = \frac{16}{5}(27 - 3w^2)$. The equation $V'(w) = 0$ gives $w^2 = 9$, so $w = \pm 3$. The only critical point in $(0, \sqrt{27}]$ is $w = 3$.

Let us use the SDT to determine whether $V(w)$ has a maximum or a minimum at $w = 3$. We have $V''(w) = \frac{16}{5}(-6w)$. Since $V''(w) < 0$ on the interval $(0, \sqrt{27}]$, $V(w)$ is concave down in $(0, \sqrt{27}]$, and therefore reaches its absolute maximum at $w = 3$. Hence, the box with largest volume has width $\boxed{3 \text{ ft}}$, height $h = \frac{108-4w^2}{5w} = \boxed{\frac{25}{4} \text{ ft}}$ and length $4w = \boxed{12 \text{ ft}}$.

3. A rectangular box is created by cutting equal size squares from the corners of a 10 in by 20 in cardboard rectangle and folding the sides. What size should the cut squares be for the resulting box to have the largest possible volume?

Solution. Call x the side length of the squares cut from the corners of the rectangle. The resulting box has height x, length $20-2x$ and width $10-2x$, see figure below.



The objective function is the volume of the box $V(x) = x(10-2x)(20-2x) = 4(x^3 - 15x^2 + 50x)$. To find the interval of interest, observe that all lengths must be postive, so we need $x \geqslant 0$, $10-2x \geqslant 0$ and $20-2x \geqslant 0$. This gives the interval $[0,5]$. We now use calculus to find the absolute maximum of $V(x) = 4(x^3 - 15x^2 + 50x)$ on $[0,5]$.

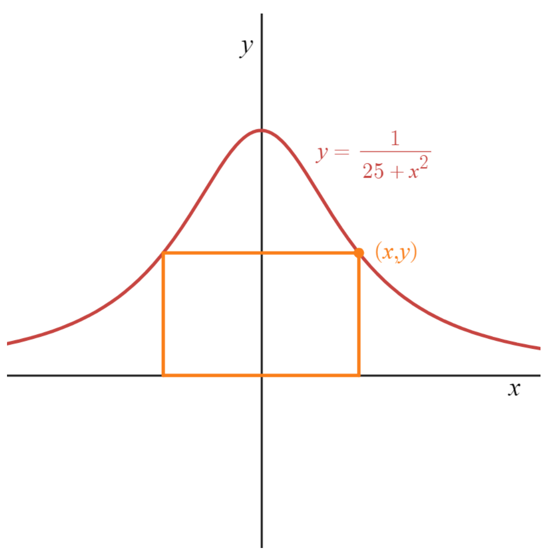
First, we find the critical points of $V(x)$ we have $V'(x) = 4(3x^2 - 30x + 50)$. Using the quadratic formula, the solutions of $V'(x) = 0$ are $x = \frac{30 \pm \sqrt{300}}{6} = 5 \pm \frac{5\sqrt{3}}{3}$. The only critical point in the interval of interest is $x = 5 - \frac{5\sqrt{3}}{3}$. We now evaluate $V(x)$ at the critical point and the endpoint.

- $V(0) = 0$
- $V\left(5 - \frac{5\sqrt{3}}{3}\right)$ is some positive value.
- $V(5) = 0$

Therefore, the volume is maximal when the square cut off from the the corners has base $x = \boxed{5 - \dfrac{5\sqrt{3}}{3} \text{ in}}$.

4. A rectangle has base on the x-axis and its two other vertices on the graph of $y = \frac{1}{25+x^2}$. Find the dimensions of such a rectangle with largest possible area.

   Solution. Call (x,y) the vertex of the rectangle on the graph in the first quadrant, see figure below.



The rectangle has base $2x$ and height y. The objective function is the area $A = 2xy$. The constraint is given by the fact that (x, y) is a point on the graph, which gives the equation $y = \frac{1}{25+x^2}$. Therefore, the objective function in terms of the variable x only is $A = \frac{2x}{25+x^2}$.

To find the feasible interval, observe that (x, y) can be any point on the graph in the first quadrant, so we have $x \geqslant 0$. This gives the interval $[0, \infty)$.

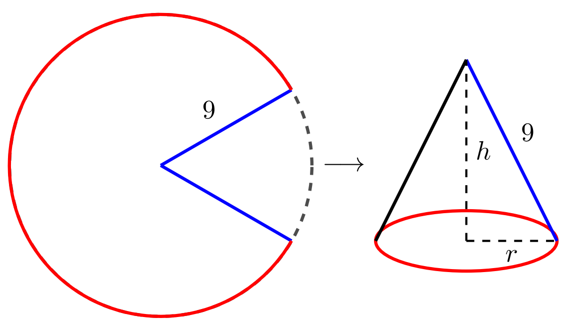
We now use calculus to find the absolute maximum of $A(x) = \frac{2x}{25+x^2}$ on the interval $[0, \infty)$. First, we find the critical points. We have

$$A'(x) = \frac{2(25+x^2) - 2x(2x)}{(25+x^2)^2} = \frac{50-2x^2}{(25+x^2)^2} = \frac{2(5-x)(5+x)}{(25+x^2)^2}.$$

The equation $A'(x) = 0$ gives the solutions $x = 5, -5$, and there are no values of x for which $A'(x)$ is undefined. Therefore, the only critical point in $[0, \infty)$ is $x = 5$. We can use the FDT to classify the critical point. When $0 \leq x < 5$, we have $A'(x) > 0$, so $A(x)$ is increasing on $[0, 5]$. When $x > 5$, we have $A'(x) < 50$, so $A(x)$ is decreasing on $[5, \infty)$. Therefore, we can conclude that the absolute maximum of $A(x)$ occurs when $x = 5$. For this value of x, the rectangle has width $2x = \boxed{10}$ and height $y = \dfrac{1}{25+x^2} = \boxed{\dfrac{1}{50}}$.

5. A circular cone is created by cutting a circular sector from a disk of radius 9in and sealing the resulting open wedge together. What is the largest possible volume of such a cone?

Solution. Call h and r the height and radius of the resulting cone, see figure below.



The objective function is the volume of the cone $V = \frac{1}{3}\pi r^r h$. The constraint is given by the fact that the slant height of the cone is 9 - the radius of the original disk. Therefore, $r^2 + h^2 = 81$, which gives $r^2 = 81 - h^2$. Hence, the volume in terms of the variable h only is $V(h) = \frac{1}{3}\pi(81 - h^2)h = \frac{1}{3}\pi(81h - h^3)$.

To find the feasible interval, observe that the height can take any value between 0 (which occurs when we don't cut anything from the disk and get a flat cone) and 9 (which occurs when we cut off the entire disk and get just a line segment as cone).Therefore, the interval is $[0, 9]$.

We now use calculus the find the absolute maximum of $V(h) = \frac{1}{3}\pi(81h - h^3)$ on $[0, 9]$. First, we find the critical points. We have

$$V'(h) = \frac{1}{3}\pi\left(81 - 3h^2\right).$$

So $V'(h) = 0$ when $3h^2 = 81$, which gives the solutions $h = \pm\sqrt{27}$. The only solution in the feasible interval is $h = \sqrt{27}$. We now evaluate $V(h)$ at the endpoints and the critical points.

- $V(0) = 0$
- $V(\sqrt{27}) = \frac{1}{3}\pi\left(81\sqrt{27} - (\sqrt{27})^3\right) = 54\pi\sqrt{3}$
- $V(9) = 0$

Hence, the largest possible value of the volume of such a cone is $\boxed{54\pi\sqrt{3} \text{ in}^3}$.

6. The parts of this problem are independent.

   (a) Find the point on the line $2x + y = 5$ that is closest to the origin.

   Solution. We find the absolute minimum of the square of the distance between the origin and a point (x, y) on the line. The objective function is therefore $F = x^2 + y^2$, subject to the constraint $2x + y = 5$. The constraint gives $y = 5 - 2x$, so the objective function in terms of x only is $F(x) = x^2 + (5 - 2x)^2$. The feasible interval is $(-\infty, \infty)$ as the point can be anywhere on the line.

   We now use calculus to find the absolute minimum of $F(x) = x^2 + (5 - 2x)^2$ on $(-\infty, \infty)$. We have $F'(x) = 2x + 2(5 - 2x)(-2) = 10x - 20$. Therefore, the only critical point of $F(x)$ is $x = 2$. Since $F''(x) = 10 > 0$, $F(x)$ is concave up on $(-\infty, \infty)$, and thus $F(x)$ reaches its absolute minimum at $x = 2$. For this value of x, we have $y = 5 - 2x = 1$. Hence, the point on the line $2x + y = 5$ closest to the origin is $\boxed{(2, 1)}$.

   (b) Find the point on the graph of $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

   Solution. We find the absolute minimum of the square of the distance between the point $(3, 0)$ and a point (x, y) on the curve. The objective function is therefore $F = (x - 3)^2 + y^2$, subject to the constraint $y = \sqrt{x}$. Using the constraint, the objective function in terms of x only is $F(x) = (x - 3)^2 + x$. The feasible interval is $[0, \infty)$ as the point can be anywhere on the graph of $y = \sqrt{x}$.

   We now use calculus to find the absolute minimum of $F(x) = (x - 3)^2 + x$ on $[0, \infty)$. We have $F'(x) = 2(x - 3) + 1 = 2x - 5$. Therefore, the only critical point of $F(x)$ is $x = \frac{5}{2}$. Since $F''(x) = 2 > 0$, $F(x)$ is concave up on $[0, \infty)$, and thus $F(x)$ reaches its absolute minimum at $x = \frac{5}{2}$. For this value of x, we have $y = \sqrt{\frac{5}{2}}$. Hence, the point on the curve $y = \sqrt{x}$ closest to $(3, 0)$ is $\boxed{\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)}$.