Section 4.8

Antiderivatives

Learning Goals

Learning Goal	*Homework Problems*
4.8.1 Compute general antiderivatives by inverting derivative rules. Verify that an antiderivative is correct by differentiating.	1-114, 119-131.
4.8.2 Use antiderivatives to solve initial value problems.	89-130.
4.8.3 Construct an initial value problem to model given information.	119-124, 126, 130.
4.8.4 Answer conceptual questions involving antiderivatives.	89, 90, 115-118, 131, 132.

Definition: a function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for x in I.

Examples: 1) Suppose $f(x) = x^2$.

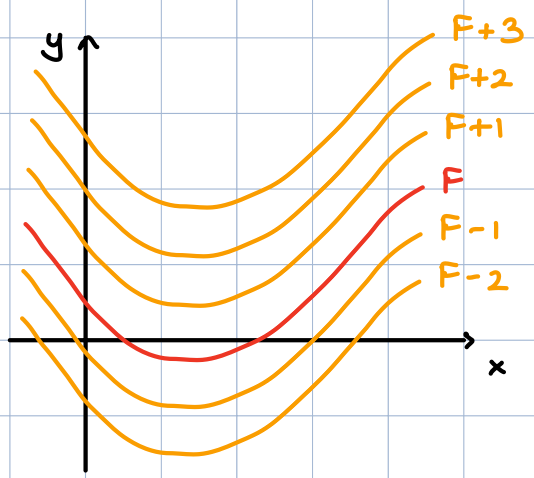
Then $F(x) = \frac{1}{3}x^3$ is an antiderivative of f since $F'(x) = \frac{1}{3}3x^2 = x^2$.

The functions $F_1(x) = \frac{1}{3}x^3 + 4$ and $F_2(x) = \frac{1}{3}x^3 - 11$ are also antiderivatives of f since $F_1'(x) = F_2'(x) = x^2$.

2) Suppose $f(x) = \sin(x)$.

Then $F_1(x) = -\cos(x)$ and $F_2(x) = -\cos(x) + 7$ are antiderivatives of f since $F_1'(x) = F_2'(x) = \sin(x)$.

Consequence of MVT: if $F(x)$ is an antiderivative of f on I, then the general form of all antiderivatives of f is $\underbrace{F(x) + C}_{\text{family of functions}}$, C constant.



(get one particular antiderivative by assigning a value to C.)

Notation: $\int f(x)\,dx$ = family of all antiderivatives of f.

↳ "(indefinite) integral of $f(x)$ with respect to x"

Terminology: $\int$ is the integral sign

$f(x)$ is called the integrand.

Examples : 1) We saw above that

$$\boxed{\int x^2 dx = \frac{1}{3}x^3 + C} \quad \text{and} \quad \boxed{\int \sin(x)\, dx = -\cos(x) + C}.$$

2) Other basic rules for antiderivatives.

$$\boxed{\int x^n dx = \frac{x^{n+1}}{n+1} + C} \quad (n \neq -1) \quad \text{since} \quad \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^{n+1-1}}{n+1} = x^n.$$

$$\boxed{\int \frac{1}{x} dx = \ln|x| + C} \quad \text{since} \quad \frac{d}{dx}(\ln|x|) = \frac{1}{x}.$$

$$\boxed{\int e^x dx = e^x + C} \quad \text{since} \quad \frac{d}{dx}(e^x) = e^x.$$

$$\boxed{\int \cos(x)\, dx = \sin(x) + C} \quad \text{since} \quad \frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\boxed{\int \sec^2(x)\, dx = \tan(x) + C} \quad \text{since} \quad \frac{d}{dx}(\tan(x)) = \sec^2(x)$$

$$\boxed{\int \sec(x)\tan(x)\, dx = \sec(x) + C} \quad \text{since} \quad \frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

$$\boxed{\begin{aligned}\int \frac{1}{1+x^2} dx &= \tan^{-1}(x) + C \\ &= \arctan(x) + C\end{aligned}} \quad \text{since} \quad \frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}.$$

$$\boxed{\begin{aligned}\int \frac{1}{\sqrt{1-x^2}} dx &= \sin^{-1}(x) + C \\ &= \arcsin(x) + C\end{aligned}} \quad \text{since} \quad \frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}.$$

↳ For each derivative rule, there is a corresponding antiderivative rule.

3) $\int e^{2x}\,dx \neq e^{2x} + C$ because $\frac{d}{dx}(e^{2x}) = 2e^{2x}$ (chain rule)

Instead, $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$ since $\frac{d}{dx}\left(\frac{1}{2}e^{2x}\right) = \frac{1}{2}2e^{2x} = e^{2x}$.

In general, $\boxed{\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C}$

and likewise, $\boxed{\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C}$ etc.

Similar formulas to memorize:

- Integrals involving inverse trig:

$$\boxed{\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

$$\boxed{\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C}$$

$(a>0)$

Indeed $\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{a}\cdot\frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a} = \frac{1}{a^2\left(1+\frac{x^2}{a^2}\right)} = \frac{1}{a^2+x^2}$

$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\cdot\frac{1}{a} = \frac{1}{\sqrt{\left(1-\frac{x^2}{a^2}\right)a^2}} = \frac{1}{\sqrt{a^2-x^2}}$.

- Integrals involving exponentials:

$$\boxed{\int a^x\,dx = \frac{1}{\ln(a)}a^x + C}$$

Indeed, $\frac{d}{dx}\left(\frac{1}{\ln(a)}a^x\right) = \frac{1}{\ln(a)}\cdot\ln(a)\,a^x = a^x$.

Linearity rules for antiderivatives :

$$\boxed{\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx}$$

$$\boxed{\int kf(x)\,dx = k\int f(x)\,dx}$$ if k is a constant.

Examples : evaluate the following antiderivatives

1) $\int \left(e^{3x} - 4\cos(5x)\right)dx$

$= \int e^{3x}\,dx - 4\int \cos(5x)\,dx$ (linearity)

$= \boxed{\frac{1}{3}e^{3x} - \frac{4}{5}\sin(5x) + C}$

2) $\int \sqrt{x}\left(3x - \frac{1}{x}\right)dx$

$= \int \left(3x\sqrt{x} - \frac{\sqrt{x}}{x}\right)dx = \int \left(3x^{3/2} - x^{-1/2}\right)dx$

$= 3\int x^{3/2}\,dx - \int x^{-1/2}\,dx = 3\,\frac{x^{3/2+1}}{3/2+1} - \frac{x^{-1/2+1}}{-1/2+1} + C$

$= 3\,\frac{x^{5/2}}{5/2} - \frac{x^{1/2}}{1/2} + C = \boxed{\frac{6}{5}x^{5/2} - 2x^{1/2} + C}$.

3) $\int \frac{t^2 - 5t + 8}{t}\,dt$

$= \int \left(\frac{t^2}{t} - \frac{5t}{t} + \frac{8}{t}\right)dt$

$$= \int \left(t - 5 + \frac{8}{t}\right) dt$$

$$= \boxed{\frac{1}{2}t^2 - 5t + 8\ln|t| + C}.$$

4) $\int \sec(4\theta)(\sec(4\theta) + \tan(4\theta))\, d\theta$

$$= \int \left(\sec^2(4\theta) + \sec(4\theta)\tan(4\theta)\right) d\theta$$

$$= \boxed{\frac{1}{4}\tan(4\theta) + \frac{1}{4}\sec(4\theta) + C}.$$

5) $\int \left(\sqrt[3]{x} + \frac{4}{1+x^2}\right) dx$

$$= \int \left(x^{1/3} + \frac{4}{1+x^2}\right) dx$$

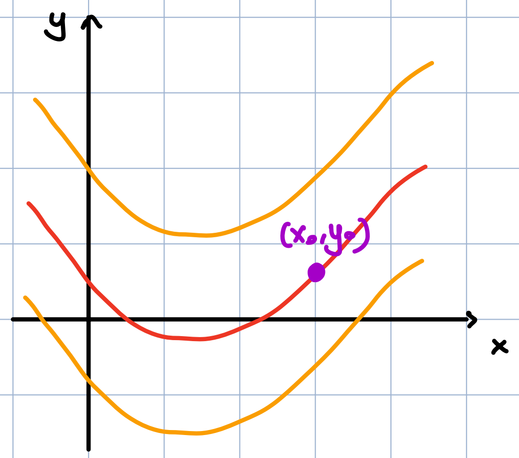
$$= \frac{x^{1/3+1}}{1/3+1} + 4\tan^{-1}(x) + C$$

$$= \boxed{\frac{3x^{4/3}}{4} + 4\tan^{-1}(x) + C}.$$

6) $\int \frac{dx}{\sqrt{25-x^2}}$ use $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C.$

$$= \int \frac{dx}{\sqrt{5^2 - x^2}} = \boxed{\sin^{-1}\left(\frac{x}{5}\right) + C}.$$

Initial Value Problems: if we know $f'(x)$ and one point on the graph of $y = f(x)$, we can identify a specific antiderivative and find $f(x)$.



There is a single antiderivative passing through $\underbrace{(x_0, y_0)}_{\text{initial condition}}$.

Examples: 1) Solve the initial value problem

$$\frac{dy}{dx} = \sec^2(2x) \quad \text{and} \quad y\left(\frac{\pi}{6}\right) = 4.$$

General antiderivative: $y(x) = \int \sec^2(2x)\,dx = \frac{1}{2}\tan(2x) + C.$

Find C using initial condition $y\left(\frac{\pi}{6}\right) = 4$

$$\frac{1}{2}\tan\left(\frac{\pi}{3}\right) + C = 4$$

$$\frac{\sqrt{3}}{2} + C = 4 \Rightarrow C = 4 - \frac{\sqrt{3}}{2}.$$

So the solution is $\boxed{y(x) = \frac{1}{2}\tan(2x) + 4 - \frac{\sqrt{3}}{2}}$

2) Solve the initial value problem $\frac{dy}{dx} = \frac{1}{9+x^2}$, $y(3) = 1$.

General antiderivative: $y(x) = \int \frac{1}{9+x^2}\,dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

Find C using initial condition $y(3) = 1$

$$\frac{1}{3}\tan^{-1}(1) + C = 1$$

$$\frac{\pi}{12} + C = 1$$

$$\Rightarrow \quad C = 1 - \frac{\pi}{12}.$$

So the solution is $\boxed{y(x) = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + 1 - \frac{\pi}{12}}$

3) The acceleration of a particle moving on an axis is $a(t) = -16$. We know that $v(0) = 3$ and $s(0) = 30$. Find the position $s(t)$.

Since $a(t) = v'(t)$, $v(t) = \int a(t)\,dt = \int -16\,dt = -16t + C$.
To find C, we use the initial condition $v(0) = 3$, which gives $-16 \cdot 0 + C = 3 \Rightarrow C = 3$.
So $v(t) = -16t + 3$.

Since $v(t) = s'(t)$,

$$s(t) = \int v(t)\,dt = \int (-16t + 3)\,dt = -8t^2 + 3t + D.$$

To find D, we use the intial condition $s(0) = 30$, which gives $-8 \cdot 0^2 + 3 \cdot 0 + D = 30 \Rightarrow D = 30$

So $\boxed{s(t) = -8t^2 + 3t + 30}$.