Rutgers University
Math 151

Section 4.8: Antiderivatives - Worksheet Solutions

1. Evaluate the following antiderivatives.

(a) $\displaystyle\int \frac{7}{1+x^2}dx$

Solution. $\displaystyle\int \frac{7}{1+x^2}dx = 7\int \frac{1}{1+x^2}dx = \boxed{7\tan^{-1}(x)+C}$.

(b) $\displaystyle\int \frac{3}{\sqrt{16-x^2}}dx$

Solution. $\displaystyle\int \frac{3}{\sqrt{16-x^2}}dx = 3\int \frac{1}{\sqrt{4^2-x^2}} = \boxed{3\sin^{-1}\left(\frac{x}{4}\right)+C}$.

(c) $\displaystyle\int (3x+1)\left(x^2-\frac{5}{x}\right)dx$

Solution. We fully distribute the integrand, then use the power rule. This gives

$$\begin{aligned}\int (3x+1)\left(x^2-\frac{5}{x}\right)dx &= \int \left(3x^3+x^2-15-\frac{5}{x}\right)dx\\ &= \boxed{\frac{3}{4}x^4+\frac{1}{3}x^3-15x-5\ln|x|+C}.\end{aligned}$$

(d) $\displaystyle\int \left(e^{5x}+\cos(1)\right)dx$

Solution. Warning: an antiderivative of $\cos(1)$ is **not** $\sin(1)$, because $\cos(1)$ is a constant. The correct way to integrate $\cos(1)$ with respect to x is $\cos(1)x$. With this in mind, we have

$$\int \left(e^{5x}+\cos(1)\right)dx = \boxed{\frac{1}{5}e^{5x}+\cos(1)x+C}.$$

(e) $\displaystyle\int \left(5\sqrt[7]{x^3}+\frac{4}{81+x^2}\right)dx$

Solution.

$$\begin{aligned}\int \left(5\sqrt[7]{x^3}+\frac{4}{81+x^2}\right)dx &= 5\int x^{3/7}dx + 4\int \frac{1}{9^2+x^2}dx\\ &= 5\frac{x^{10/7}}{10/7}+\frac{4}{9}\tan^{-1}\left(\frac{x}{9}\right)+C\\ &= \boxed{\frac{35}{10}x^{10/7}+\frac{4}{9}\tan^{-1}\left(\frac{x}{9}\right)+C}.\end{aligned}$$

(f) $\displaystyle\int \csc(5\theta)\left(\sin(5\theta) - \cot(5\theta)\right) d\theta$

Solution.

$$\begin{aligned}\int \csc(5\theta)\left(\sin(5\theta) - \cot(5\theta)\right) d\theta &= \int \left(\csc(5\theta)\sin(5\theta) - \csc(5\theta)\cot(5\theta)\right) d\theta \\ &= \int \left(1 - \csc(5\theta)\cot(5\theta)\right) d\theta \\ &= \boxed{\theta + \frac{1}{5}\csc(5\theta) + C}.\end{aligned}$$

(g) $\displaystyle\int \frac{7t - 11}{\sqrt{t}} dt$

Solution.

$$\begin{aligned}\int \frac{7t - 11}{\sqrt{t}} dt &= \int \left(\frac{7t}{\sqrt{t}} - \frac{11}{\sqrt{t}}\right) dt \\ &= \int \left(7t^{1/2} - 11t^{-1/2}\right) dt \\ &= 7\frac{t^{3/2}}{3/2} - 11\frac{t^{1/2}}{1/2} + C \\ &= \boxed{\frac{14}{3}t^{3/2} - 22t^{1/2} + C}.\end{aligned}$$

(h) $\displaystyle\int \left(2^x - \frac{1}{7x}\right) dx$

Solution. $\displaystyle\int \left(2^x - \frac{1}{7x}\right) dx = \boxed{\frac{2^x}{\ln(2)} - \frac{\ln|x|}{7} + C}.$

(i) $\displaystyle\int \frac{\tan(3x) + 5\sec(3x)}{\cos(3x)} dx$

Solution.

$$\begin{aligned}\int \frac{\tan(3x) + 5\sec(3x)}{\cos(3x)} dx &= \int \left(\frac{\tan(3x)}{\cos(3x)} + 5\frac{\sec(3x)}{\cos(3x)}\right) dx \\ &= \int \left(\tan(3x)\sec(3x) + 5\sec^2(3x)\right) dx \\ &= \boxed{\frac{1}{3}\sec(3x) + \frac{5}{3}\tan(3x) + C}.\end{aligned}$$

(j) $\displaystyle\int \left(\frac{1}{z^{7/4}} - \frac{3}{36 + z^2}\right) dz$

Solution.

$$\begin{aligned}\int\left(\frac{1}{z^{7/4}}-\frac{3}{36+z^2}\right)dz &= \int\left(z^{-7/4}-3\frac{1}{6^2+z^2}\right)dz\\ &= \frac{z^{-3/4}}{-3/4}-\frac{3}{6}\tan^{-1}\left(\frac{z}{6}\right)+C\\ &= \boxed{-\frac{4}{3z^{3/4}}-\frac{1}{2}\tan^{-1}\left(\frac{z}{6}\right)+C}.\end{aligned}$$

2. Solve the following initial value problems.

(a) $\dfrac{dy}{dx}=2-7x$ and $y(2)=0$.

Solution. First, we find the general form of $y(x)$ by integrating $y'(x)$.

$$y(x)=\int(2-7x)dx=2x-\frac{7}{2}x^2+C.$$

Next, we find the value of the constant C by using the initial condition $y(2)=0$. This gives

$$2\cdot 2-\frac{7}{2}\cdot 2^2+C=0 \Rightarrow -10+C=0 \Rightarrow C=10.$$

Therefore, the solution of the initial value problem is $\boxed{y(x)=2x-\frac{7}{2}x^2+10}$.

(b) $\dfrac{dy}{dx}=x^{-6}+\dfrac{6}{x}$ and $y(1)=3$.

Solution. First, we find the general form of $y(x)$ by integrating $y'(x)$.

$$y(x)=\int\left(x^{-6}+\frac{6}{x}\right)dx=\frac{x^{-5}}{-5}+6\ln|x|+C.$$

Next, we find the value of the constant C by using the initial condition $y(1)=3$. This gives

$$\frac{1^{-5}}{-5}+6\ln|1|+C=3 \Rightarrow -\frac{1}{5}+C=3 \Rightarrow C=\frac{16}{5}.$$

Therefore, the solution of the initial value problem is $\boxed{y(x)=\frac{x^{-5}}{-5}+6\ln|x|+\frac{16}{5}}$.

(c) $\dfrac{dy}{dx}=\dfrac{5}{9+x^2}$ and $y(3)=-1$.

Solution. First, we find the general form of $y(x)$ by integrating $y'(x)$.

$$y(x)=\int\frac{5}{9+x^2}dx=\frac{5}{3}\tan^{-1}\left(\frac{x}{3}\right)+C.$$

Next, we find the value of the constant C by using the initial condition $y(3)=-1$. This gives

$$\frac{5}{3}\tan^{-1}(1)+C=-1 \Rightarrow \frac{5\pi}{12}+C=-1 \Rightarrow C=-1-\frac{5\pi}{12}.$$

Therefore, the solution of the initial value problem is $\boxed{y(x) = \frac{5}{3}\tan^{-1}\left(\frac{x}{3}\right) - 1 - \frac{5\pi}{12}}$.

(d) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{64 - x^2}}$ and $y(-4) = 0$.

Solution. First, we find the general form of $y(x)$ by integrating $y'(x)$.

$$y(x) = \int \frac{1}{\sqrt{64 - x^2}} dx = \sin^{-1}\left(\frac{x}{8}\right) + C.$$

Next, we find the value of the constant C by using the initial condition $y(-4) = 0$. This gives

$$\sin^{-1}\left(-\frac{1}{2}\right) + C = 0 \ \Rightarrow \ -\frac{\pi}{6} + C = 0 \ \Rightarrow \ C = \frac{\pi}{6}.$$

Therefore, the solution of the initial value problem is $\boxed{y(x) = \sin^{-1}\left(\frac{x}{8}\right) + \frac{\pi}{6}}$.

(e) $\dfrac{d^2y}{dx^2} = 3 - e^{2x}$, $y'(0) = 1$ and $y(0) = 7$.

Solution. We first solve the initial value problem $\dfrac{dy'}{dx} = 3 - e^{2x}$, $y'(0) = 1$ to find $y'(x)$. The general form of $y'(x)$ is

$$y'(x) = \int \left(3 - e^{2x}\right) dx = 3x - \frac{e^{2x}}{2} + C.$$

To find the value of the constant C, we use the initial condition $y'(0) = 1$. This gives

$$3 \cdot 0 - \frac{e^0}{2} + C = 1 \ \Rightarrow \ -\frac{1}{2} + C = 1 \ \Rightarrow \ C = \frac{3}{2}.$$

Therefore, $y'(x) = 3x - \dfrac{e^{2x}}{2} + \dfrac{3}{2}$. We can now find $y(x)$ by solving the initial value problem $\dfrac{dy}{dx} = 3x - \dfrac{e^{2x}}{2} + \dfrac{3}{2}$, $y(0) = 7$. We have

$$y(x) = \int \left(3x - \frac{e^{2x}}{2} + \frac{3}{2}\right) = \frac{3x^2}{2} - \frac{e^{2x}}{4} + \frac{3x}{2} + D.$$

To find the value of the constant D, we use the initial condition $y(0) = 7$. This gives

$$\frac{\cdot 0}{2} - \frac{e^0}{4} + \frac{3 \cdot 0}{2} + D = 7 \ \Rightarrow \ -\frac{1}{4} + D = 7 \ \Rightarrow \ D = \frac{29}{4}.$$

Therefore $\boxed{y(x) = \frac{3x^2}{2} - \frac{e^{2x}}{4} + \frac{3x}{2} + \frac{29}{4}}$.