Rutgers University
Math 151

Sections 5.1-2: Areas Estimations and Riemann Sums - Worksheet Solutions

1. (a) Approximate the net area between the graph of $f(x) = 9 - x^2$ and the x-axis on $[-1, 3]$ using 4 rectangles of equal width and (i) left endpoints, (ii) right endpoints.

Solution. Partitioning the interval $[-1, 3]$ into 4 subintervals of equal length will give us subintervals of length $\dfrac{3-(-1)}{4} = 1$. Therefore, we get the 4 subintervals $[-1, 0]$, $[0, 1]$, $[1, 2]$ and $[2, 3]$.

(i) Picking the value at the left endpoint for the height of the rectangles gives the sum

$$\begin{aligned} A &\simeq f(-1)\cdot 1 + f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 \\ &= (9-(-1)^2) + (9-0^2) + (9-1^2) + (9-2^2) \\ &= \boxed{30 \text{ square units}}. \end{aligned}$$

(ii) Picking the value at the right endpoint for the height of the rectangles gives the sum

$$\begin{aligned} A &\simeq f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 \\ &= (9-0^2) + (9-1^2) + (9-2^2) + (9-3^2) \\ &= \boxed{22 \text{ square units}}. \end{aligned}$$

(b) Approximate the net area between the graph of $f(x) = 2\cos(x)$ and the x-axis on $\left[0, \frac{\pi}{2}\right]$ using 3 rectangles of equal width and (i) left endpoints, (ii) right endpoints.

Solution. Partitioning the interval $\left[0, \frac{\pi}{2}\right]$ into 3 subintervals of equal length will give us subintervals of length $\frac{\frac{\pi}{2}-0}{3} = \frac{\pi}{6}$. Therefore, we get the 4 subintervals $\left[0, \frac{\pi}{6}\right]$, $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ and $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$.

(i) Picking the value at the left endpoint for the height of the rectangles gives the sum

$$\begin{aligned} A &\simeq f(0)\cdot\frac{\pi}{6} + f\left(\frac{\pi}{6}\right)\cdot\frac{\pi}{6} + f\left(\frac{\pi}{3}\right)\cdot\frac{\pi}{6} \\ &= \frac{\pi}{6}\left(2\cos(0) + 2\cos\left(\frac{\pi}{6}\right) + 2\cos\left(\frac{\pi}{3}\right)\right) \\ &= \frac{\pi}{6}\left(2 + 2\frac{\sqrt{3}}{2} + 2\frac{1}{2}\right) \\ &= \boxed{\frac{\pi(3+\sqrt{3})}{6} \text{ square units}}. \end{aligned}$$

(ii) Picking the value at the right endpoint for the height of the rectangles gives the sum

$$A \simeq f\left(\frac{\pi}{6}\right)\cdot\frac{\pi}{6} + f\left(\frac{\pi}{3}\right)\cdot\frac{\pi}{6} + f\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{6}$$

$$= \frac{\pi}{6}\left(2\cos\left(\frac{\pi}{6}\right) + 2\cos\left(\frac{\pi}{3}\right) + 2\cos\left(\frac{\pi}{2}\right)\right)$$
$$= \frac{\pi}{6}\left(2\frac{\sqrt{3}}{2} + 2\frac{1}{2} + 2\cdot 0\right)$$
$$= \boxed{\frac{\pi(1+\sqrt{3})}{6} \text{ square units}}.$$

2. Suppose that the function f has the following values.

$$f(0) = 3,\ f(1) = 7,\ f(2) = 5,\ f(3) = 1,\ f(4) = 2,\ f(5) = 8,$$
$$f(6) = 0,\ f(7) = 1,\ f(8) = 5,\ f(9) = 3,\ f(10) = 1.$$

Approximate the net area between the graph of $g(x) = f(8x+2)$ and the x-axis on the interval $[0,1]$ using a midpoint sum with 4 rectangles of equal width.

Solution. Dividing the interval $[0,1]$ into 4 subintervals of equal width gives the intervals $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{3}{4}\right]$ and $\left[\frac{3}{4}, 1\right]$. We will pick the height using the value at the midpoint of each interval, that is $x = \frac{1}{8}$, $x = \frac{3}{8}$, $x = \frac{5}{8}$ and $x = \frac{7}{8}$. We get the approximation

$$A \simeq g\left(\frac{1}{8}\right)\frac{1}{4} + g\left(\frac{3}{8}\right)\frac{1}{4} + g\left(\frac{5}{8}\right)\frac{1}{4} + g\left(\frac{7}{8}\right)\frac{1}{4}$$
$$= \frac{1}{4}\left(f(3) + f(5) + f(7) + f(9)\right)$$
$$= \boxed{\frac{13}{4} \text{ square units}}.$$

3. Evaluate the following sums.

(a) $\displaystyle\sum_{k=0}^{5} \frac{k(k-1)}{2}$.

Solution.

$$\sum_{k=0}^{5} \frac{k(k-1)}{2} = \frac{0(0-1)}{2} + \frac{1(1-1)}{2} + \frac{2(2-1)}{2} + \frac{3(3-1)}{2} + \frac{4(4-1)}{2} + \frac{5(5-1)}{2}$$
$$= \boxed{20}.$$

(b) $\displaystyle\sum_{j=1}^{4} \cos(j\pi)j$.

Solution.

$$\sum_{j=1}^{4} \cos(j\pi)j = \cos(\pi) + 2\cos(2\pi) + 3\cos(3\pi) + 4\cos(4\pi)$$
$$= -1 + 2 - 3 + 4$$
$$= \boxed{2}.$$

(c) $\displaystyle\sum_{n=1}^{5} \left(\frac{1}{n} - \frac{1}{n+1}\right)$.

Solution.

$$\begin{aligned}\sum_{n=1}^{5} \left(\frac{1}{n} - \frac{1}{n+1}\right) &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) \\ &= 1 - \frac{1}{6} \\ &= \boxed{\frac{5}{6}}.\end{aligned}$$

4. Consider the sum $2 + 4 + 8 + 16 + 32 + 64$.

(a) Write the sum in sigma notation with the index starting at the value 1.

(b) Write the sum in sigma notation with the index starting at the value 0.

(c) Write the sum in sigma notation with the index starting at the value 3.

Solution.

$$\boxed{2 + 4 + 8 + 16 + 32 + 64 = \sum_{k=1}^{6} 2^k = \sum_{k=0}^{5} 2^{k+1} = \sum_{k=3}^{8} 2^{k-2}}.$$

5. Use the common sum formulas

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4},$$

to evaluate the following sums.

(a) $\displaystyle\sum_{k=1}^{136} (2k-3)$.

Solution.

$$\begin{aligned}\sum_{k=1}^{136} (2k-3) &= 2\sum_{k=1}^{136} k - \sum_{k=1}^{136} 3 \\ &= 2\frac{136(137)}{2} - 3 \cdot 136 \\ &= \boxed{18224}.\end{aligned}$$

(b) $\displaystyle\sum_{j=2}^{20} j^2(j-4)$.

Solution.

$$\sum_{j=2}^{20} j^2(j-4) = \sum_{j=2}^{20} (j^3 - 4j^2)$$

$$
\begin{aligned}
&= \sum_{j=2}^{20} j^3 - 4\sum_{j=2}^{20} j^2 \\
&= \sum_{j=1}^{20} j^3 - 1^3 - 4\left(\sum_{j=1}^{20} j^2 - 1^2\right) \\
&= \frac{20^2 \cdot 21^2}{4} - 1 - 4\frac{20 \cdot 21 \cdot 41}{6} + 4 \\
&= \boxed{55583}.
\end{aligned}
$$