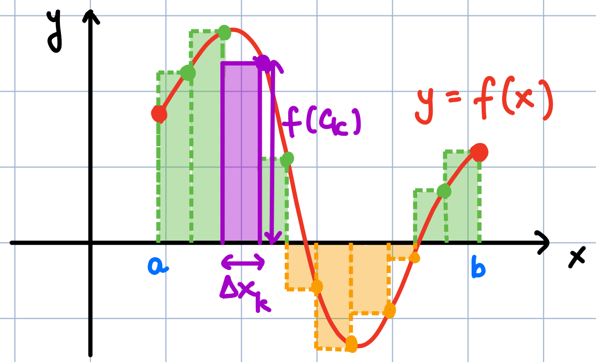
Section 5.3

# Definite Integrals

## Learning Goals

| *Learning Goal* | *Homework Problems* |
|---|---|
| 5.3.1 Express limits of Riemann sums as definite integrals. | 1-8. |
| 5.3.2 Evaluate definite integrals using either a limit of a Riemann sum, properties of definite integrals, or known area formulas. | 9-54, 63-70. |
| 5.3.3 Find average values of functions. | 55-62. |
| 5.3.4 Use the properties of the definite integral to analyze integrals and prove relationships relating them. | 71-85. |

**Recall** : Net area between graph and x - axis on $[a,b] \simeq R_n$.

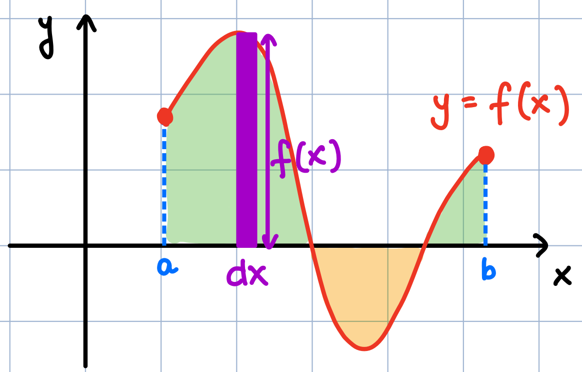



$$R_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

↳ Riemann sum with $n$ rectangles

To find the exact net area, we take the limit as the number of rectangles $n$ goes to $\infty$.

**Definition** : the definite integral of $f$ on $[a,b]$ is




$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x_k}$$

**Remarks** :

- This limit may or may not exist. We say that $f$ is integrable on $[a,b]$ when it does. All functions which are continuous (except at finitely many points) are integrable.

- The value of $\int_a^b f(x)dx$ is the net area between the graph of $f$ and the x - axis on $[a,b]$.

- We will usually work with equal length subintervals and right sums. Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(a+k\Delta x)\,\Delta x$$

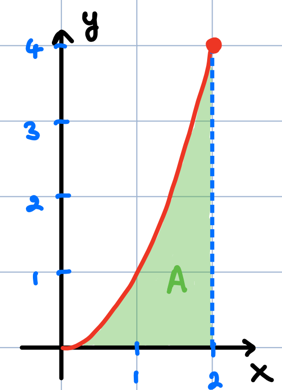
$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n}$$

- MyLab writes the definition in the equivalent form :

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k .$$

where $P$ denotes a partition of $[a,b]$ and $\|P\|$ is the length of the subintervals. ( i.e. width of rectangles) in $P$.

Examples : 1) In the previous lecture, for $f(x) = x^2$ on $[0,2]$ , we found

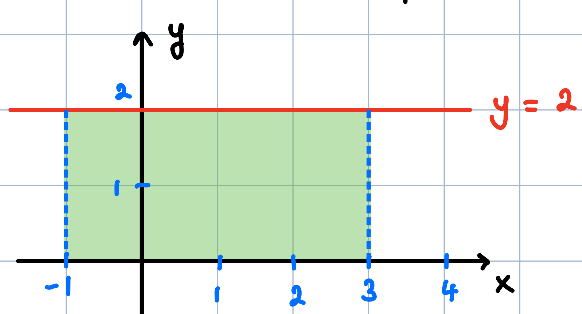



$$R_n = \sum_{k=1}^{n} \left(\frac{4k}{n}\right)^2 \frac{4k}{n} = \frac{4\,(n+1)(2n+1)}{3n^2}$$

And $\lim_{n\to\infty} R_n = \frac{8}{3}$

So $\boxed{\int_0^2 x^2\,dx = \frac{8}{3}}$, $A = \frac{8}{3}$ square units.

2) Evaluate $\int_{-1}^{3} 2\,dx$.




$$\int_{-1}^{3} 2\,dx = \text{net area between } y=2 \text{ and } x\text{-axis on } [-1,3]$$

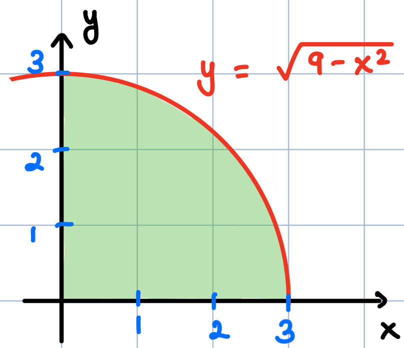
$= \text{height} \cdot \text{base}$ (rectangle)

$= 2\cdot(3-(-1)) = \boxed{8}$.

In general, if k is a constant, then

$$\boxed{\int_a^b k\,dx = k(b-a)} \quad \text{(area of a rectangle)}.$$

3) Evaluate $\int_0^3 \sqrt{9-x^2}\,dx$.




$y = \sqrt{9-x^2} \Rightarrow y^2 = 9-x^2 \Rightarrow x^2+y^2=9.$

So $y = \sqrt{9-x^2}$ is a semi-circle of radius 3 centered at $(0,0)$.

$$\int_0^3 \sqrt{9-x^2}\,dx = \text{area of quarter disk of radius } 3$$
$$= \frac{1}{4}\cdot\pi\cdot 3^2 = \boxed{\frac{9\pi}{4}}.$$

4) Find a function $f(x)$ and an interval $[a,b]$ such that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{4k}{n}\right)^2 \frac{4}{n}$$

We match with the general form of a right Riemann sum

$$\sum_{k=1}^{n} \left(3+\frac{4k}{n}\right)^2 \frac{4}{n} = \sum_{k=1}^{n} f(a+k\Delta x)\,\Delta x$$

We can take $a = 3$ and $\Delta x = \frac{4}{n} = \frac{b-3}{n}$, so $b=7$

Then $f(a+k\Delta x) = f\Big(\underbrace{3+\frac{4k}{n}}_{x}\Big) = \Big(\underbrace{3+\frac{4k}{n}}_{x}\Big)^2$

so $\boxed{f(x) = x^2 \quad \text{on} \quad [3,7]}$

Note: we can choose a different interval if we "shift" $f(x)$.

If $a=0$, then $\Delta x = \frac{4}{n} = \frac{b-0}{n}$ so $b=4$.

Then $f(a+k\Delta x) = f\left(\frac{4k}{n}\right) = \left(3+\frac{4k}{n}\right)^2$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; x \quad\quad\quad\quad\quad\quad\; x$

so $\boxed{f(x) = (3+x)^2 \quad \text{on} \quad [0,4]}$ also works

5) Find a function $f(x)$ and a value of $b$ so that

$$\int_0^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(\frac{6k}{n}+5\right)\frac{3}{n}$$

We try to match the sum with a Riemann sum:

$$\sum_{k=1}^{n} \ln\left(\frac{6k}{n}+5\right)\frac{3}{n} = \sum_{k=1}^{n} f(a+k\Delta x)\,\Delta x$$

We have $a=0$ and $\Delta x = \frac{3}{n} = \frac{b-0}{n}$, so $\boxed{b=3}$.

Then $f(a+k\Delta x) = f\left(\frac{3k}{n}\right) = \ln\left(\frac{6k}{n}+5\right) = \ln\left(2\cdot\frac{3k}{n}+5\right)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; x \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; x$

so $\boxed{f(x) = \ln(2x+5)}$

6) Find a function $f(x)$ and a value of $b$ so that

$$\int_0^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \arctan\left(\frac{50k^2}{n^2} - \frac{15k}{n} - 8\right)\frac{5}{n}$$

We try to match the sum with a Riemann sum:

$$\sum_{k=1}^{n} \arctan\left(\frac{50k^2}{n^2} - \frac{15k}{n} - 8\right)\frac{5}{n} = \sum_{k=1}^{n} f(a+k\Delta x)\,\Delta x$$

We have $a = 0$ and $\Delta x = \frac{5}{n} = \frac{5-0}{n}$ so $\boxed{b = 5}$.

Then $f(a+k\Delta x) = f\Big(\underbrace{\frac{5k}{n}}_{x}\Big) = \arctan\left(\frac{50k^2}{n^2} - \frac{15k}{n} - 8\right)$

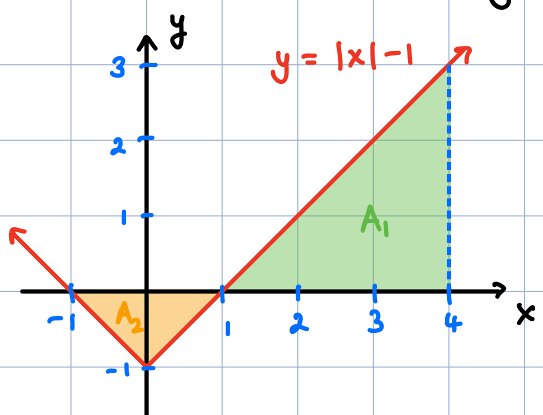
$$= \arctan\left(2\Big(\underbrace{\frac{5k}{n}}_{x}\Big)^2 - 3\underbrace{\frac{5k}{n}}_{x} - 8\right)$$

so $\boxed{f(x) = \arctan(2x^2 - 3x - 8)}$.

7) The <u>average value</u> of a function $f$ on an interval $[a,b]$ is

$$\boxed{av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

Calculate the average value of $f(x) = |x|-1$ on $[-1, 4]$.




$$\int_{-1}^{4} (|x|-1)\,dx = A_1 - A_2 = \frac{9}{2} - 1 = \frac{7}{2}$$

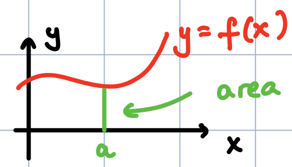
$$A_1 = \frac{1}{2}.\text{base}.\text{height} = \frac{1}{2}.3.3 = \frac{9}{2}$$

$$A_2 = \frac{1}{2}.\text{base}.\text{height} = \frac{1}{2}.2.1 = 1$$

So $av(f) = \frac{1}{4-(-1)}\int_{-1}^{4} f(x)\,dx = \frac{1}{5}\cdot\frac{7}{2} = \boxed{\frac{7}{10}}$.

# Properties of definite integrals :

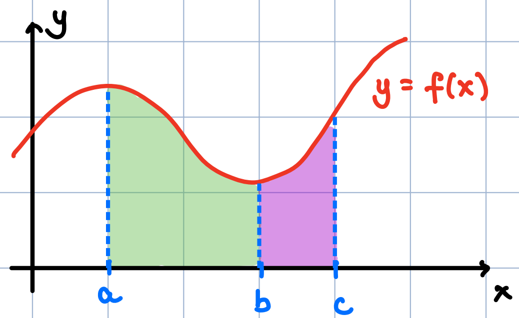
i) $\boxed{\int_a^a f(x)\,dx = 0}$




area of a line segment is 0.

ii) Reverse bounds : $\boxed{\int_a^b f(x)dx = -\int_b^a f(x)dx}$

iii) Additivity / subdivision

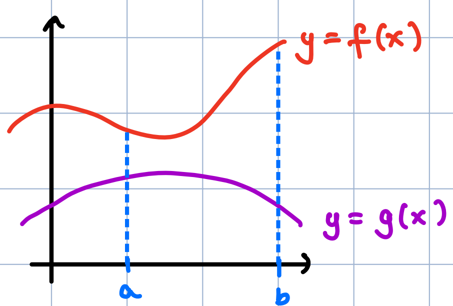



$$\boxed{\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx}.$$

iv) Linearity : $\boxed{\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx}$

$\boxed{\int_a^b kf(x)\,dx = k\int_a^b f(x)dx}$ ($k$ constant)

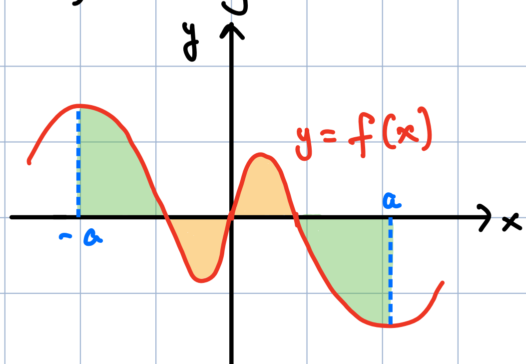
v) Domination :



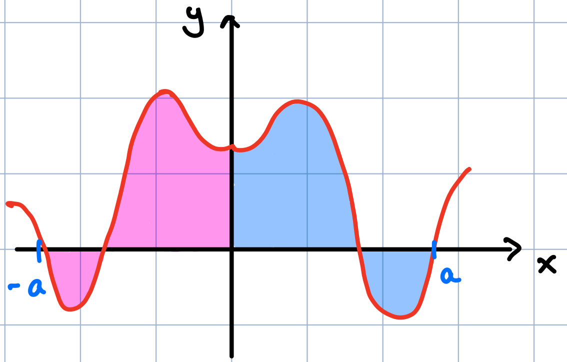

If $f(x) \geqslant g(x)$ on $[a, b]$, then

$$\boxed{\int_a^b f(x)dx \geqslant \int_a^b g(x)dx}.$$

vi) Parity :




$$\boxed{\text{If } f \text{ is odd, } \int_{-a}^{a} f(x)dx = 0}$$




$$\boxed{\text{If } f \text{ is even, } \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx}$$

Examples : 1) Suppose that $\int_{-1}^{3} f(x)\,dx = 6$ and $\int_{2}^{3} f(x)\,dx = 2$.

Evaluate $\int_{-1}^{2} (2f(x) - 5)\,dx$.

$$\int_{-1}^{2} (2f(x) - 5)\,dx = 2\int_{-1}^{2} f(x)\,dx - \int_{-1}^{2} 5\,dx$$

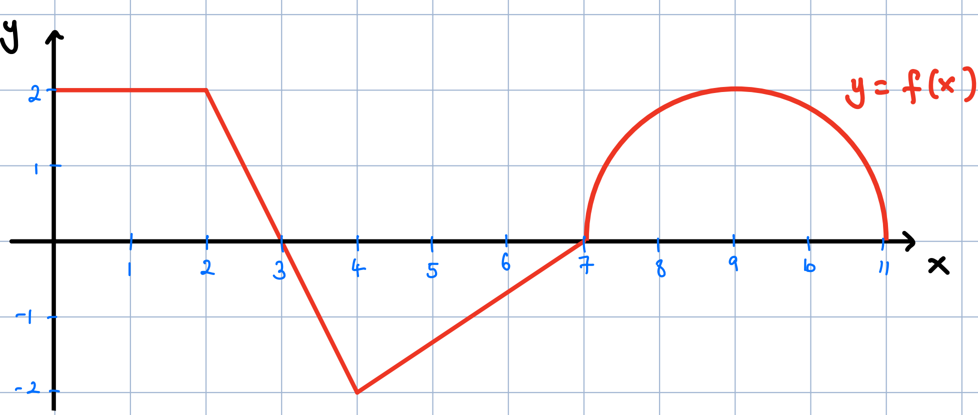
We know that $\int_{-1}^{3} f(x)\,dx = \int_{-1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx$

$$6 = \int_{-1}^{2} f(x)\,dx + 2 \Rightarrow \int_{-1}^{2} f(x)\,dx = 4.$$

So $\int_{-1}^{2} (2f(x) - 5)\,dx = 2\int_{-1}^{2} f(x)\,dx - \int_{-1}^{2} 5\,dx$

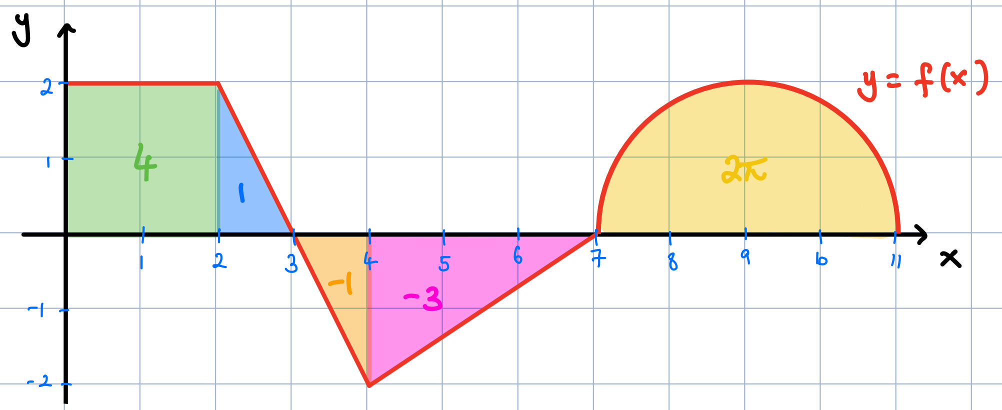
$$= 2\cdot 4 - 5\cdot 3 = \boxed{-7}.$$

2) Use the graph of $f$ to calculate each integral (Each piece of the graph is either a straight line or a circle arc.)



a) $\int_4^{11} f(x)\,dx$ b) $\int_0^9 (f(x)-3)\,dx$ c) $\int_{11}^2 f(x)\,dx$

d) $\int_0^7 |f(x)|\,dx$ e) $\int_2^4 (x - f(x))\,dx$



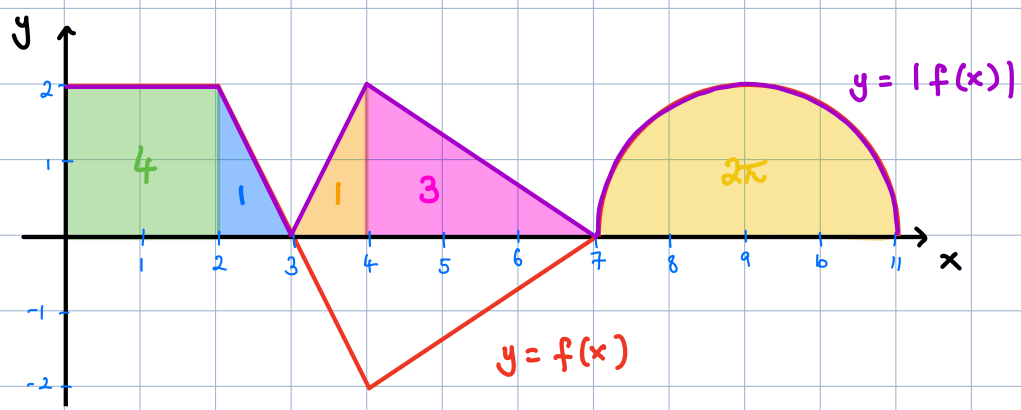

a) $\int_4^{11} f(x)\,dx = \boxed{2\pi - 3}$.

b) $\int_0^9 (f(x)-3)\,dx = \int_0^9 f(x)dx - \int_0^9 3dx$

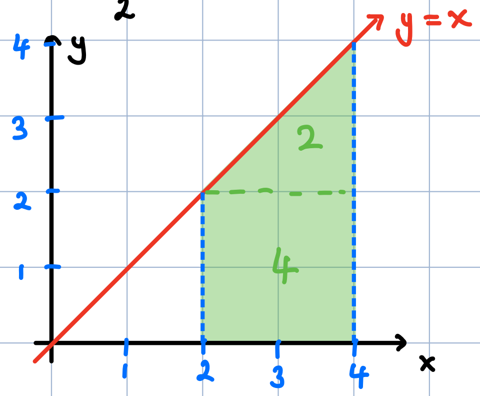
$= (4+1-1-3+\pi) - 3\cdot 9 = \boxed{\pi - 26}$.

c) $\int_{11}^2 f(x)\,dx = -\int_2^{11} f(x)\,dx = -(1-1-3+2\pi) = \boxed{3-2\pi}$.

d) $\int_0^7 |f(x)|\,dx = 4+1+1+3 = \boxed{9}$.



e) $\int_2^4 (x - f(x))\,dx = \int_2^4 x\,dx - \int_2^4 f(x)\,dx = 6 - (1-1) = \boxed{6}$.




$\int_2^4 x\,dx = 4+2 = 6$