Rutgers University
Math 151

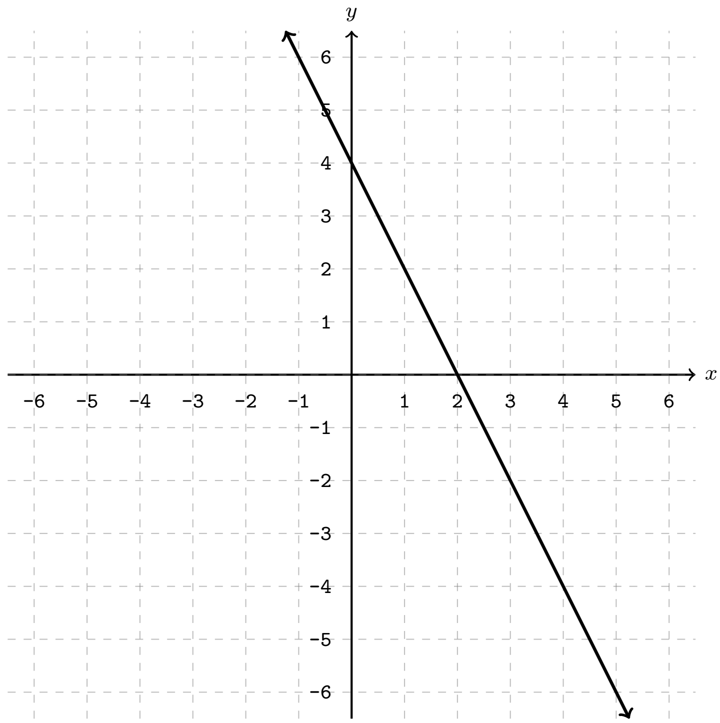
Section 5.3: Definite Integrals - Worksheet Solutions

1. Let $f(x) = 4 - 2x$. We are going to calculate $\int_0^2 f(x)dx$ using two methods.

   (a) Geometric method.

   (i) Sketch the graph of $y = f(x)$.

   Solution.



   (ii) Use your graph and a geometric formula to calculate $\int_0^2 f(x)dx$.

   Solution. $\int_0^2 f(x)dx$ is the area of a triangle with base 2 and height 4, so $\int_0^2 f(x)dx = \frac{1}{2} \cdot 2 \cdot 4 = \boxed{4}$

   (b) With Riemann sums.

   (i) Calculate R_n, the right-endpoint Riemann sum of f on $[0, 2]$ with n rectangles. Your answer should not contain the Σ or $\cdots$ symbols. *Hint: you will need to use the reference sum*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Solution. We have $\Delta x = \dfrac{b-a}{n} = \dfrac{2}{n}$ and

$$\begin{aligned}
R_n &= \sum_{k=1}^{n} f(a + k\Delta x)\Delta x \\
&= \sum_{k=1}^{n} \left(4 - 2\frac{2k}{n}\right)\frac{2}{n} \\
&= \sum_{k=1}^{n} \left(\frac{8}{n} - \frac{8k}{n^2}\right) \\
&= \left(\sum_{k=1}^{n} \frac{8}{n}\right) - \frac{8}{n^2}\sum_{k=1}^{n} k \\
&= \frac{8}{n} \cdot n - \frac{8}{n^2} \cdot \frac{n(n+1)}{2} \\
&= 8 - \frac{4(n+1)}{n} \\
&= \boxed{8 - 4\left(1 + \frac{1}{n}\right)}
\end{aligned}$$

(ii) Using your formula for R_n, calculate $\displaystyle\int_0^2 f(x)dx$.

Solution.

$$\int_0^2 f(x)dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} 8 - 4\left(1 + \frac{1}{n}\right) = 8 - 4 = \boxed{4}.$$

2. Write each limit below as the integral of a function $f(x)$ on an interval $[0, b]$.

(a) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{\frac{3k}{n} + 5}\ \frac{3}{n}.$

Solution.

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{\frac{3k}{n} + 5}\ \frac{3}{n} = \int_0^3 \sqrt{x+5}dx}.$$

(b) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} e^{12k/n}\frac{8}{n}.$

Solution.

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} e^{12k/n}\frac{8}{n} = \int_0^8 e^{3x/2}dx}.$$

(c) $\lim_{n\to\infty} \sum_{k=1}^{n} \sin\left(\frac{k^3}{n^3}\right)\frac{2}{n}$.

Solution.

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} \sin\left(\frac{k^3}{n^3}\right)\frac{2}{n} = \int_0^2 \sin\left(\frac{x^3}{8}\right) dx}.$$

(d) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2n+5k}$.

Solution. We can write the sum in the format of a Riemann sum by factoring out an n from the denominator, which gives

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2n+5k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2+5\frac{k}{n}} \cdot \frac{1}{n} = \boxed{\int_0^1 \frac{dx}{2+5x}}.$$

3. Suppose that f and g are functions such that

$$\int_{-2}^{0} f(x)dx = 4, \quad \int_{-2}^{5} f(x)dx = -1, \quad \int_{-2}^{5} g(x)dx = 10.$$

Evaluate the following integrals.

(a) $\int_{-2}^{5} \frac{g(x)}{2} dx$

Solution.

$$\int_{-2}^{5} \frac{g(x)}{2} dx = \frac{1}{2}\int_{-2}^{5} g(x)dx = \boxed{5}.$$

(b) $\int_{-2}^{5} (2g(x) - 3f(x))dx$

Solution.

$$\int_{-2}^{5} (2g(x) - 3f(x))dx = 2\int_{-2}^{5} g(x)dx - 3\int_{-2}^{5} f(x)dx = 2 \cdot 10 - 3(-1) = \boxed{23}.$$

(c) $\int_{0}^{5} 7f(x)dx$

Solution.

$$\int_0^5 7f(x)dx = 7\int_0^5 f(x)dx = 7\left(\int_{-2}^{5} f(x)dx - \int_{-2}^{0} f(x)dx\right) = 7(-1-4) = \boxed{-35}.$$

(d) $\displaystyle\int_5^{-2} (f(x) + 4g(x))\, dx$

Solution.

$$\begin{aligned}\int_5^{-2} (f(x) + 4g(x))\, dx &= \int_5^{-2} f(x)dx + 4\int_5^{-2} g(x)dx \\ &= -\int_{-2}^{5} f(x)dx - 4\int_{-2}^{5} g(x)dx \\ &= -(-1) - 4 \cdot 10 \\ &= \boxed{-39}.\end{aligned}$$

(e) $\displaystyle\int_{-2}^{0} (2x + f(x) - 1)\, dx$

Solution. First, observe that

$$\int_{-2}^{0} (2x + f(x) - 1)\, dx = \int_{-2}^{0} 2xdx + \int_{-2}^{0} f(x)dx - \int_{-2}^{0} 1dx.$$

The integral $\displaystyle\int_{-2}^{0} 2xdx$ gives the area of a triangle of base 2 and height 4 located below the y-axis, therefore $\displaystyle\int_{-2}^{0} 2xdx = -4$. We know that $\displaystyle\int_{-2}^{0} f(x)dx = 4$ and $\displaystyle\int_{-2}^{0} 1dx = 2$. Hence

$$\int_{-2}^{0} (2x + f(x) - 1)\, dx = -4 + 4 - 2 = \boxed{-2}.$$

(f) $\displaystyle\int_5^{0} \left(f(x) - 4\sqrt{25 - x^2}\right) dx$

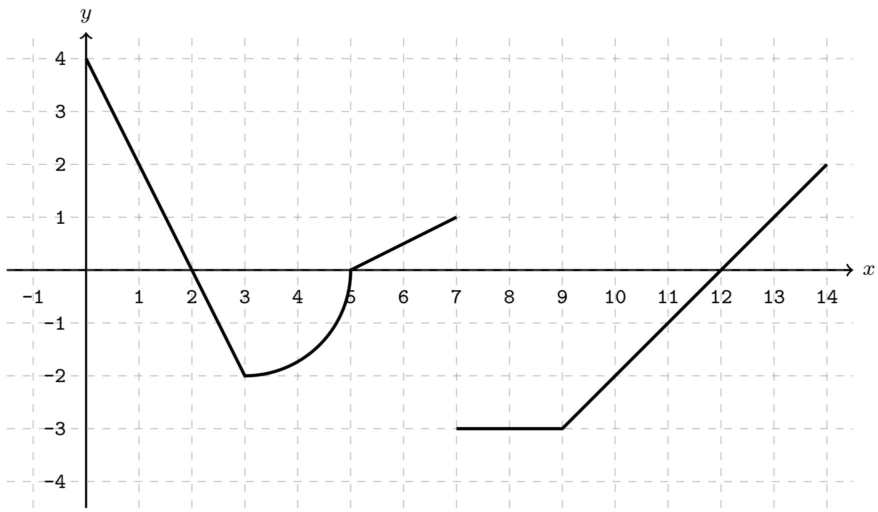
Solution. First, we have

$$\begin{aligned}\int_5^{0} \left(f(x) - 4\sqrt{25 - x^2}\right) dx &= \int_5^{0} f(x)dx - 4\int_5^{0} \sqrt{25 - x^2}dx \\ &= -\int_0^{5} f(x)dx + 4\int_0^{5} \sqrt{25 - x^2}dx.\end{aligned}$$

We know that $\displaystyle\int_0^{5} f(x)dx = -5$. The graph of $y = \sqrt{25 - x^2}$ is a semi circle of radius 5 centered at $(0, 0)$. Thus, $\displaystyle\int_0^{5} \sqrt{25 - x^2}dx$ is the area of a quarter disk of radius 5, that is $\frac{25\pi}{4}$. We obtain

$$\int_5^{0} \left(f(x) - 4\sqrt{25 - x^2}\right) dx = -(-5) + 4\frac{25\pi}{4} = \boxed{5 + 25\pi}.$$

4. Let f be the function whose graph is sketched below. You can assume that each piece of the graph of f is either a straight line or a circle arc.



Calculate the following integrals.

(a) $\int_0^5 f(x)dx$

Solution. $\int_0^5 f(x)dx = 4 - 1 - \pi = \boxed{3 - \pi}$.

(b) $\int_3^9 (3 - f(x))dx$

Solution. $\int_3^9 (3 - f(x))dx = \int_3^9 3dx - \int_3^9 f(x)dx = 3 \cdot 6 - (-\pi + 1 - 6) = \boxed{23 + \pi}$.

(c) $\int_{12}^5 f(x)dx$

Solution. $\int_{12}^5 f(x)dx = -\int_5^{12} f(x)dx = -\left(1 - 6 - \frac{9}{2}\right) = \boxed{\frac{19}{2}}$

(d) $\int_7^{14} |f(x)|dx$

Solution. $\int_7^{14} |f(x)|dx = 6 + \frac{9}{2} + 2 = \boxed{\frac{25}{2}}$