Section 5.4

Fundamental Theorem of Calculus

Learning Goals

Learning Goal	*Homework Problems*
5.4.1 Compute definite integrals using the Fundamental Theorem of Calculus, Part 2 (FTC2).	1-44, 77, 78, 86.
5.4.2 Find derivatives of area functions (i.e., accumulation functions) using the Fundamental Theorem of Calculus, Part 1 (FTC1)	39-56, 79-83.
5.4.3 Find the area of a region between two graphs of functions.	57-64.
5.4.4 Use the FTC to solve an initial value problem.	65-70, 75, 76.
5.4.5 Use the FTC to solve for unknowns and analyze functions.	71, 73, 77-86.

Fundamental Theorem of Calculus (Part I)

Suppose $F(x)$ is an antiderivative of $f(x)$ on $[a,b]$, then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)} = \left([F(x)]_a^b\right)$$

So FTC I tells us how we can compute definite integrals using an antiderivative of the integrand.

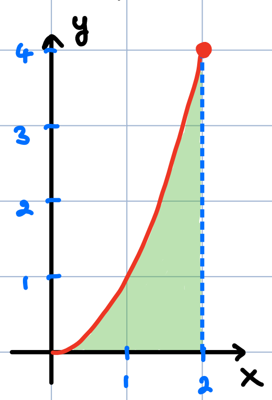
⚠ Do not get confused between indefinite / definite integrals.

$\int f(x)\,dx = F(x) + C$ the family of antiderivatives of f (a family of functions)

$\int_a^b f(x)\,dx =$ net area between $y = f(x)$ and x-axis on $[a,b]$, a number.

FTC I tells us how they are related: we can use antiderivatives to compute net areas.

Examples: 1) In the previous lecture, we used limits of Riemann sums to show that



$$\int_0^2 x^2\,dx = \frac{8}{3}.$$

We can also calculate this with FTC I:

$$\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}.$$

↳ could use any antiderivative of x^2 ($\frac{x^3}{3}+1$, etc.) and get the same result.

2) Evaluate the following integrals.

a) $\int_1^2 \frac{1}{x}\,dx = \Big[\ln|x|\Big]_1^2 = \ln|2| - \ln|1| = \boxed{\ln(2)}$.

b) $\int_0^4 \left(\sqrt{x} - e^{3x}\right)dx = \int_0^4 \left(x^{1/2} - e^{3x}\right)dx$

$$= \left[\frac{x^{3/2}}{3/2} - \frac{e^{3x}}{3}\right]_0^4$$

$$= \left(\frac{4^{3/2}}{3/2} - \frac{e^{12}}{3}\right) - \left(\frac{0^{3/2}}{3/2} - \frac{e^0}{3}\right)$$

$$= \frac{16}{3} - \frac{e^{12}}{3} + \frac{1}{3} = \boxed{\frac{17 - e^{12}}{3}}.$$

c) $\int_0^1 \frac{1}{x^2+1}\,dx = \Big[\arctan(x)\Big]_0^1$

$$= \arctan(1) - \arctan(0) = \boxed{\frac{\pi}{4}}.$$

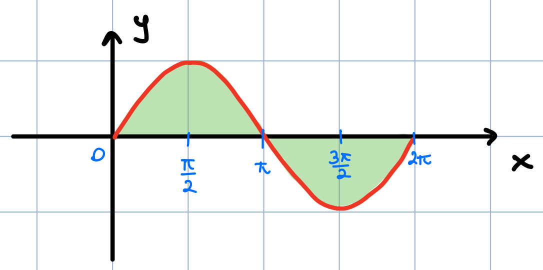
d) $\int_{\frac{\pi}{8}}^{\frac{\pi}{6}} \sec(2\theta)\tan(2\theta)\,d\theta = \left[\frac{1}{2}\sec(2\theta)\right]_{\frac{\pi}{8}}^{\frac{\pi}{6}}$

$$= \frac{1}{2}\left(\sec\left(\frac{\pi}{3}\right) - \sec\left(\frac{\pi}{4}\right)\right)$$

$$= \boxed{\frac{1}{2}\left(2 - \sqrt{2}\right)}.$$

e) $\int_0^2 \frac{1}{\sqrt{16-x^2}}\,dx = \left[\sin^{-1}\left(\frac{x}{4}\right)\right]_0^2 = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0) = \boxed{\frac{\pi}{6}}.$

3) Find the total area between $y = \sin(x)$ and the x_axis on $[0, 2\pi]$.



⚠ $A \neq \int_0^{2\pi} \sin(x)\,dx = \left[-\cos(x)\right]_0^{2\pi} = 0.$

because the integral is the net area (below x-axis counts as negative)

$$A = \int_0^{\pi} \sin(x)\,dx - \int_{\pi}^{2\pi} \sin(x)\,dx$$

$$= \left[-\cos(x)\right]_0^{\pi} - \left[-\cos(x)\right]_{\pi}^{2\pi}$$

$$= -\cos(\pi) - (-\cos(0)) - \left(-\cos(2\pi) - (-\cos(\pi))\right) = \boxed{4 \text{ square units}}.$$

4) Suppose that the velocity of a particle moving on an axis is $v(t) = \frac{16}{t^3} + 1$ m/s. Find the displacement from $t=1$ to $t=2$.

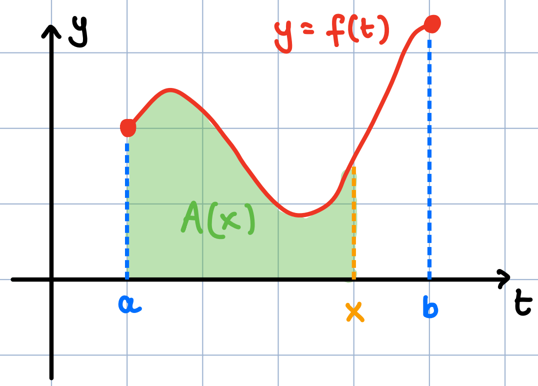
Displacement $= s(2) - s(1) = \int_1^2 v(t)\,dt$ since $s(t)$ is an antiderivative of $v(t)$

$$= \int_1^2 \left(\frac{16}{t^3} + 1\right) dt = \int_1^2 (16t^{-3} + 1)\,dt = \left[\frac{16t^{-2}}{-2} + t\right]_1^2 = \left[-\frac{8}{t^2} + t\right]_1^2$$

$$= \left(-\frac{8}{4} + 2\right) - \left(-\frac{8}{1} + 1\right)$$

$$= \boxed{7 \text{ m}}.$$

<u>Area functions</u>: if f is continuous on $[a,b]$, the area function is



$$A(x) = \int_a^x f(t)\,dt$$

The input x of $A(x)$ is the right boundary of the region.

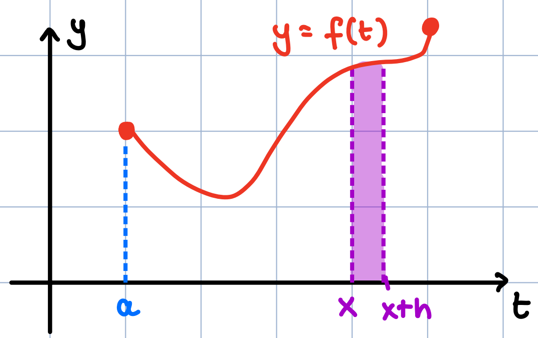
The output is the net area.

<u>FTC Part II</u>:

$$\boxed{\begin{array}{l} A'(x) = f(x) \\ \text{or} \quad \dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x) \end{array}}$$

So $A(x)$ is an antiderivative of $f(x)$.

Why does FTC II work?



$$A(x+h) - A(x) = \int_x^{x+h} f(t)\,dt$$

When h is very small, the purple region is approximately a rectangle of height $f(x)$ and width h.

So $A(x+h) - A(x) \simeq hf(x)$ as $h \to 0$

$$\Rightarrow \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(x).$$

<u>Examples</u> 1) Calculate the following derivatives.

a) $\dfrac{d}{dx}\displaystyle\int_{-1}^{x} \sqrt{t^3+1}\,dt = \sqrt{x^3+1}$

b) $\dfrac{d}{dx}\int_0^{x} \dfrac{dt}{t^4+1} = \dfrac{1}{x^4+1}$

c) $\dfrac{d}{dx}\int_{-4}^{x} \cos(u)\sin(u^2)\,du = \cos(x)\sin(x^2)$

2) FTC II only applies when the upper bound of the integral is the variable of differentiation. If this is not the case, need to use in combination with other rules.

a) $\dfrac{d}{dx}\left(\int_x^2 e^{-3t^2}\,dt\right)$ variable is not the upper bound so FTC does not apply yet.

$= \dfrac{d}{dx}\left(-\int_2^x e^{-3t^2}\,dt\right)$ reverse bounds

$= -\dfrac{d}{dx}\left(\int_2^x e^{-3t^2}\,dt\right)$ now FTC applies

$= \boxed{-e^{-3x^2}}$

b) $\dfrac{d}{dx}\left(\underbrace{\int_0^{3x^2} \sin(\theta^4)\,d\theta}_{A(3x^2)}\right)$ upper bound is not just x but an "inside function": use chain rule

$= A'(3x^2)\cdot(6x)$ chain rule

$= \sin((3x^2)^4)(6x)$ FTC for A'

$= \boxed{6x\sin(81x^8)}$

c) $\dfrac{d}{dx}\left(\int_1^{2^x} \dfrac{\cos(t)}{t}\,dt\right)$ upper bound is not just x but an "inside function": use chain rule

$= \boxed{\dfrac{\cos(2^x)}{2^x}\cdot \ln(2)2^x}$

d) $\frac{d}{dx}\left(\int_{\tan^{-1}(x)}^{3} \frac{du}{\sqrt{u}+1} \right)$

$= -\frac{d}{dx}\left(\int_{3}^{\tan^{-1}(x)} \frac{du}{\sqrt{u}+1} \right)$ first reverse bounds

$= \boxed{-\frac{1}{\sqrt{\tan^{-1}(x)}+1} \cdot \frac{1}{x^2+1}}$ chain rule

e) $\frac{d}{dx}\left(\int_{2x}^{x^2} \cos(t^3)\,dt \right)$

$= \frac{d}{dx}\left(\int_{0}^{x^2} \cos(t^3)\,dt - \int_{0}^{2x} \cos(t^3)\,dt \right)$ split into two integrals

$= \cos((x^2)^3)\cdot(2x) - \cos((2x)^3)\cdot 2$

$= \boxed{2x\cos(x^6) - 2\cos(8x^3)}$

3) We know that:

- $A(x) = \int_a^x f(t)\,dt$ is an antiderivative of f (FTC)
- $A(a) = 0$ (property of integrals)

So $A(x) = \int_a^x f(t)\,dt$ is the only antiderivative of f passing through $(a, 0)$.

We can use this to solve Initial Value Problems.

a) Solve the IVP $\frac{dy}{dx} = x^2$ and $y(1) = 6$.

$$\boxed{y(x) = 6 + \int_1^x t^2\,dt}$$

b) Solve the IVP $\frac{dy}{dx} = \sec(x^2)$ and $y(0) = -3$

$$\boxed{y(x) = -3 + \int_0^x \sec(t^2)\,dt}$$

c) Solve the IVP $\frac{dy}{dx} = \sqrt{x}\,e^x$ and $y(4) = -7$.

$$\boxed{y(x) = -7 + \int_4^x \sqrt{t}\,e^t\,dt}$$