Rutgers University
Math 151

Section 5.4: Fundamental Theorem of Calculus - Worksheet Solutions

1. Evaluate the following definite integrals.

   (a) $\displaystyle\int_1^3 \frac{3x^2 - 2x + 1}{x} dx$

   Solution.

$$\begin{aligned}
\int_1^3 \frac{3x^2 - 2x + 1}{x} dx &= \int_1^3 \left(3x - 2 + \frac{1}{x}\right) dx \\
&= \left[\frac{3x^2}{2} - 2x + \ln|x|\right]_1^3 \\
&= \left(\frac{3 \cdot 3^2}{2} - 2 \cdot 3 + \ln(3)\right) - \left(\frac{3}{2} - 2 + \ln(1)\right) \\
&= \boxed{\ln(3) - 1}
\end{aligned}$$

   (b) $\displaystyle\int_0^{1/2} \frac{dt}{\sqrt{1 - t^2}}$

   Solution.

$$\begin{aligned}
\int_0^{1/2} \frac{dt}{\sqrt{1 - t^2}} &= \left[\sin^{-1}(t)\right]_0^{1/2} \\
&= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0) \\
&= \boxed{\frac{\pi}{6}}.
\end{aligned}$$

   (c) $\displaystyle\int_0^{\ln(2)} (e^x + 1)^2 \, dx$

   Solution.

$$\begin{aligned}
\int_0^{\ln(2)} (e^x + 1)^2 \, dx &= \int_0^{\ln(2)} \left(e^{2x} + 2e^x + 1\right) dx \\
&= \left[\frac{e^2 x}{2} + 2e^x + x\right]_0^{\ln(2)} \\
&= \left(\frac{e^{2\ln(2)}}{2} + 2e^{\ln(2)} + \ln(2)\right) - \left(\frac{e^0}{2} + 2e^0 + 0\right) \\
&= \boxed{\frac{7}{2} + \ln(2)}.
\end{aligned}$$

(d) $\int_{\pi/30}^{\pi/20} \sec^2(5\theta) d\theta$

Solution.

$$\begin{aligned}
\int_{\pi/30}^{\pi/20} \sec^2(5\theta) d\theta &= \left[\frac{1}{5}\tan(5\theta)\right]_{\pi/30}^{\pi/20} \\
&= \frac{1}{5}\tan\left(\frac{\pi}{4}\right) - \frac{1}{5}\tan\left(\frac{\pi}{6}\right) \\
&= \frac{1}{5} - \frac{\sqrt{3}}{15} \\
&= \boxed{\frac{3-\sqrt{3}}{15}}.
\end{aligned}$$

(e) $\int_{-3}^{\sqrt{3}} \frac{4}{x^2+3} dx$

Solution.

$$\begin{aligned}
\int_{-3}^{\sqrt{3}} \frac{4}{x^2+3} dx &= \left[\frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_{-3}^{\sqrt{3}} \\
&= \frac{4}{\sqrt{3}}\left(\tan^{-1}(1) - \tan^{-1}\left(-\sqrt{3}\right)\right) \\
&= \frac{4\sqrt{3}}{12}\left(\frac{\pi}{4} - \left(-\frac{\pi}{3}\right)\right) \\
&= \boxed{\frac{28\sqrt{3}\pi}{144}}.
\end{aligned}$$

(f) $\int_0^5 \frac{dz}{4z+7}$

Solution.

$$\begin{aligned}
\int_0^5 \frac{dz}{4z+7} &= \left[\frac{1}{4}\ln|4z+7|\right]_0^5 \\
&= \frac{1}{4}(\ln(27) - \ln(7)) \\
&= \boxed{\frac{1}{4}\ln\left(\frac{27}{7}\right)}.
\end{aligned}$$

(g) $\int_1^4 \sqrt{x}\left(x - \frac{4}{x}\right) dx$

Solution.

$$\begin{aligned}
\int_1^4 \sqrt{x}\left(x - \frac{4}{x}\right) dx &= \int_1^4 \left(x^{3/2} - 4x^{-1/2}\right) dx \\
&= \left[\frac{x^{5/2}}{5/2} - 4\frac{x^{1/2}}{1/2}\right]_1^4 \\
&= \left(\frac{4^{5/2}}{5/2} - 4\frac{4^{1/2}}{1/2}\right) - \left(\frac{1}{5/2} - \frac{4}{1/2}\right) \\
&= \left(\frac{64}{5} - 16\right) - \left(\frac{2}{5} - 8\right) \\
&= \boxed{\frac{22}{5}}.
\end{aligned}$$

(h) $\displaystyle\int_0^{2\pi} \left(\sin\left(\frac{x}{3}\right) + 1\right) d\theta$

Solution.

$$\begin{aligned}
\int_0^{2\pi} \left(\sin\left(\frac{x}{3}\right) + 1\right) d\theta &= \left[-3\cos\left(\frac{\theta}{3}\right) + \theta\right]_0^{2\pi} \\
&= \left(-3\cos\left(\frac{2\pi}{3}\right) + 2\pi\right) - (-3\cos(0) + 0) \\
&= \frac{3}{2} + 2\pi + 3 \\
&= \boxed{\frac{9}{2} + 2\pi}.
\end{aligned}$$

(i) $\displaystyle\int_{\sqrt{2}}^2 \frac{5}{3x\sqrt{x^2-1}} dx$

Solution.

$$\begin{aligned}
\int_{\sqrt{2}}^2 \frac{5}{3x\sqrt{x^2-1}} dx &= \left[\frac{5}{3}\sec^{-1}|x|\right]_{\sqrt{2}}^2 \\
&= \frac{5}{3}\left(\sec^{-1}(2) - \sec^{-1}(\sqrt{2})\right) \\
&= \frac{5}{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\
&= \boxed{\frac{5\pi}{36}}.
\end{aligned}$$

2. Evaluate the following derivatives.

(a) $\displaystyle\frac{d}{dx}\left(\int_4^x \sqrt{t^4+1}dt\right)$

Solution. $\displaystyle\frac{d}{dx}\left(\int_4^x \sqrt{t^4+1}dt\right) = \boxed{\sqrt{x^4+1}}$

(b) $\frac{d}{dx}\left(\int_x^0 \sec(5t^2)dt\right)$

Solution. $\frac{d}{dx}\left(\int_x^0 \sec(5t^2)dt\right) = -\frac{d}{dx}\left(\int_0^x \sec(5t^2)dt\right) = \boxed{-\sec(5x^2)}$.

(c) $\frac{d}{dx}\left(\int_1^{2x} \frac{dt}{t^3+t+1}\right)$

Solution. $\frac{d}{dx}\left(\int_1^{2x} \frac{dt}{t^3+t+1}\right) = \frac{1}{(2x)^3+2x+1}(2) = \boxed{\frac{2}{8x^2+2x+1}}$.

(d) $\frac{d}{dx}\left(\int_{3x^2}^7 (t^4+2)^{3/4}dt\right)$

Solution. $\frac{d}{dx}\left(\int_{3x^2}^7 (t^4+2)^{3/4}dt\right) = -\frac{d}{dx}\left(\int_7^{3x^2} (t^4+2)^{3/4}dt\right) = -((3x^2)^4+2)^{3/4}(6x) = \boxed{-6x(81x^8+2)^{3/4}}$.

(e) $\frac{d}{dx}\left(\int_{\tan(2x)}^{\sec(2x)} \cos(\sqrt{t})dt\right)$

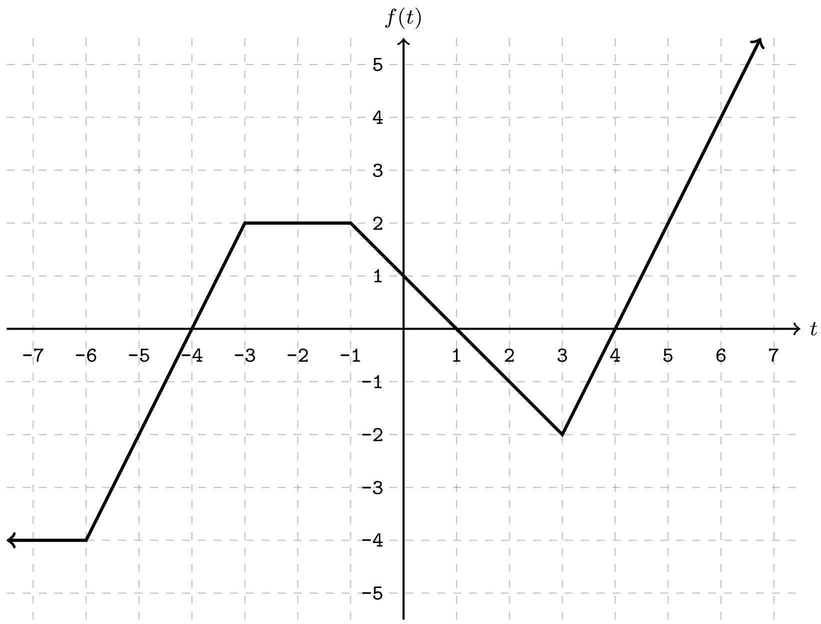
Solution.

$$\frac{d}{dx}\left(\int_{\tan(2x)}^{\sec(2x)} \cos(\sqrt{t})dt\right) = \frac{d}{dx}\left(\int_0^{\sec(2x)} \cos(\sqrt{t})dt - \int_0^{\tan(2x)} \cos(\sqrt{t})dt\right)$$
$$= \boxed{\cos(\sqrt{\sec(2x)})\sec(2x)\tan(2x)(2) - \cos(\sqrt{\tan(2x)})\sec^2(2x)(2)}.$$

(f) $\frac{d}{dx}\left(\int_0^{\sin^{-1}(3x)} t^t dt\right)$

Solution. $\frac{d}{dx}\left(\int_0^{\sin^{-1}(3x)} t^t dt\right) = \boxed{\sin^{-1}(3x)^{\sin^{-1}(3x)}\frac{3}{\sqrt{1-(3x)^2}}}$.

3. For the function $f(t)$ sketched below, let $F(x) = \int_{-3}^x f(t)dt$.



(a) Evaluate the following.
Solution.

(i) $F(3) = \boxed{4}$ (ii) $F(-5) = \boxed{-3}$ (iii) $F'(-2) = \boxed{2}$ (iv) $F'(4) = \boxed{0}$

(b) Find an equation of the tangent line to the graph of $y = F(x)$ at $x = 6$.

Solution. We have $F(6) = 7$ and $F'(6) = f(6) = 4$, so an equation of the tangent line to the graph pf $y = F(x)$ at $x = 6$ is $\boxed{y - 7 = 4(x - 6)}$.

(c) Find the critical points of F.

Solution. We have $F'(x) = f(x)$. Observe that $f(x)$ is never undefined, and the solutions of $f(x) = 0$ are $\boxed{x = -4, 1, 4}$.

(d) Find the intervals on which F is increasing and the intervals on which F is decreasing.

Solution. F is increasing on $\boxed{[-4, 1], [4, \infty)}$. F is decreasing on $\boxed{(-\infty, -4], [1, 4]}$.

(e) Find the x-values at which $F(x)$ has a local maximum or a local minimum.

Solution. The location of the local maxima of F is $\boxed{x = 1}$. The location of the local minima of F are $\boxed{x = -4, 4}$.

(f) Find the intervals on which F is concave up and the intervals on which F is concave down.

Solution. F is concave up when $F' = f$ is increasing, which happens on $\boxed{[-6, -3], [3, \infty)}$. F is concave down when $F' = f$ is decreasing, which happens on $\boxed{[-1, 3]}$.

(g) Find the x-values at which $F(x)$ has an inflection point.

Solution. The inflection points of F are located where the concavity changes, which is at $\boxed{x = 3}$.

4. Let $f(x) = 7 + \int_{13}^{x} t(t-14)^{2/5} dt$.

(a) Find an equation of the tangent line to the graph of $y = f(x)$ at $x = 13$.

Solution. We have

$$f(11) = 7 + \int_{13}^{13} t(t-14)^{2/5} dt = 7 + 0 = 7$$

and

$$f'(11) = \frac{d}{dx}\left(\int_{13}^{x} t(t-14)^{2/5} dt\right)_{|x=13} = \left(x(x-14)^{2/5}\right)_{|x=13} = 13(13-12)^{2/5} = -13.$$

So an equation of the tangent line to the graph of $y = f(x)$ at $x = 13$ is $\boxed{y - 7 = -13(x - 13)}$.

(b) Find the critical points of f.

Solution. The derivative of f is $f'(x) = x(x-14)^{2/5}$ so the critical points of f are $\boxed{x = 0, 14}$.

(c) Find the intervals on which f is increasing and the intervals on which F is decreasing.

Solution. Let us test for the sign of $f'(x)$ in between the critical points.

- On $(-\infty, 0)$, the sign of $f'(x)$ is $(-)(+) = (-)$.
- On $(0, 14)$ the sign of $f'(x)$ is $(+)(+) = (+)$.
- On $(14, \infty)$, the sign of $f'(x)$ is $(+)(+) = (+)$.

Hence, f is increasing on $\boxed{[0, \infty)}$ and decreasing on $\boxed{(\infty, 0]}$.

(d) Find the x-values at which $f(x)$ has a local maximum or a local minimum.

Solution. Based on our findings in the previous question, we can deduce that f does not have a local maximum and has a local minimum at $\boxed{x = 0}$.

(e) Find the intervals on which f is concave up and the intervals on which F is concave down.

Solution. We'll need a sign chart for $f''(x)$ to determine this. We have

$$f''(x) = (x-14)^{2/5} + \frac{2x}{5(x-14)^{3/5}} = \frac{5(x-14)+2x}{5(x-14)^{3/5}} = \frac{7x-70}{(x-14)^{3/5}} = \frac{7(x-10)}{(x-14)^{3/5}}$$

Let us analyze the sign of $f''(x)$. The points where $f''(x)$ is zero or undefined are $x = 10, 14$.

- On $(-\infty, 10)$, the sign of $f''(x)$ is $\frac{(-)}{(-)} = (+)$.
- On $(10, 14)$, the sign of $f''(x)$ is $\frac{(+)}{(-)} = (-)$.

- On $(14, \infty)$, the sign of $f''(x)$ is $\frac{(+)}{(+)} = (+)$.

Hence, f is concave up on $\boxed{(-\infty, 10], [14, \infty)}$ and concave down on $\boxed{[10, 14]}$.

(f) Find the x-values at which $f(x)$ has an inflection point.

Solution. Based on our previous answer, we can deduce that f has inflection points at $\boxed{x = 10, 14}$.