Rutgers University
Math 151

Sections 5.5-6: Substitution Method - Worksheet Solutions

1. Evaluate the following integrals.

(a) $\displaystyle\int (3x^4+6)\sec(x^5+10x)dx$

Solution. We substitute $u = x^5+10x$, so $du = (5x^4+10)dx = 5(x^4+2)dx$. We obtain

$$\begin{aligned}\int (3x^4+6)\sec(x^5+10x)dx &= \int 3\sec(x^5+10x)(x^4+2)dx \\ &= \int 3\sec(u)\frac{1}{5}du \\ &= \frac{3}{5}\int \sec(u)du \\ &= \frac{3}{5}\ln|\sec(u)+\tan(u)|+C \\ &= \boxed{\frac{3}{5}\ln|\sec(x^5+10x)+\tan(x^5+10x)|+C}.\end{aligned}$$

(b) $\displaystyle\int \frac{dx}{x\sqrt{3\ln(x)+5}}$

Solution 1. We substitute $u = 3\ln(x)+5$, so $du = \frac{3}{x}dx$, which gives $\frac{dx}{x} = \frac{du}{3}$. Therefore,

$$\begin{aligned}\int \frac{dx}{x\sqrt{3\ln(x)+5}} &= \int \frac{du}{3\sqrt{u}} \\ &= \frac{2}{3}\sqrt{u}+C \\ &= \boxed{\frac{2}{3}\sqrt{3\ln(x)+5}+C}.\end{aligned}$$

Solution 2. We substituted $u = \sqrt{3\ln(x)+5}$, so $du = \frac{1}{2\sqrt{3\ln(x)+5}}\cdot\frac{3}{x}dx$. This gives $\frac{dx}{x\sqrt{3\ln(x)+5}} = \frac{2}{3}du$, so

$$\begin{aligned}\int \frac{dx}{x\sqrt{3\ln(x)+5}} &= \int \frac{2}{3}du \\ &= \frac{2}{3}u+C \\ &= \boxed{\frac{2}{3}\sqrt{3\ln(x)+5}+C}.\end{aligned}$$

(c) $\displaystyle\int x^2\sqrt{x-1}dx$

Solution. We use the substitution $u = x - 1$, so $du = dx$. Then we have $x = u + 1$, so we get

$$\begin{aligned}
\int x^2\sqrt{x-1}dx &= \int (u+1)^2\sqrt{u}du \\
&= \int (u^2+2u+1)u^{1/2}du \\
&= \int (u^{5/2}+2u^{3/2}+u^{1/2})du \\
&= \frac{u^{7/2}}{7/2} + \frac{2u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2} + C \\
&= \boxed{\frac{2(x-1)^{7/2}}{7} + \frac{4(x-1)^{5/2}}{5} + \frac{2(x-1)^{1/2}}{3} + C}.
\end{aligned}$$

(d) $\displaystyle\int x^3\sin(x^4+2)dx$

Solution. We use the substitution $u = x^4 + 2$, $du = 4x^3dx$. Therefore, $x^3dx = \frac{du}{4}$, and we obtain

$$\begin{aligned}
\int x^3\sin(x^4+2)dx &= \int \frac{1}{4}\sin(u)du \\
&= -\frac{1}{4}\cos(u) + C \\
&= \boxed{-\frac{1}{4}\cos(x^4+2) + C}.
\end{aligned}$$

(e) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{3+x^2}}dx$

Solution 1. We use the substitution $u = 3 + x^2$. So $du = 2xdx$, that is $xdx = \frac{du}{2}$. The extraneous factor x^2 in the numerator can be expressed in terms of u using $x^2 = u - 3$. Finally, the bounds become

$$\begin{aligned}
x = 0 &\Rightarrow u = 3 + 0^2 = 3, \\
x = 1 &\Rightarrow u = 3 + 1^2 = 4.
\end{aligned}$$

So the integral becomes

$$\begin{aligned}
\int_0^1 \frac{x^3}{\sqrt{3+x^2}}dx &= \int_0^1 \frac{x^2}{\sqrt{3+x^2}}xdx \\
&= \int_3^4 \frac{u-3}{2\sqrt{u}}du \\
&= \int_3^4 \left(\frac{1}{2}\sqrt{u} - \frac{3}{2\sqrt{u}}\right)du \\
&= \left[\frac{1}{2}\cdot\frac{2}{3}u^{3/2} - 3\sqrt{u}\right]_3^4
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{3}4^{3/2} - 3\sqrt{4}\right) - \left(\frac{1}{3}3^{3/2} - 3\sqrt{3}\right) \\
&= \boxed{2\sqrt{3} - \frac{10}{3}}.
\end{aligned}
$$

Solution 2. We use the substitution $u = \sqrt{3+x^2}$. So $du = \frac{xdx}{\sqrt{3+x^2}}$. The extraneous factor x^2 in the numerator can be expressed in terms of u using $x^2 = u^2 - 3$. Finally, the bounds become

$$
\begin{aligned}
x = 0 &\Rightarrow u = \sqrt{3+0^2} = \sqrt{3}, \\
x = 1 &\Rightarrow u = \sqrt{3+1^2} = 2.
\end{aligned}
$$

So the integral becomes

$$
\begin{aligned}
\int_0^1 \frac{x^3}{\sqrt{3+x^2}}dx &= \int_0^1 x^2 \frac{xdx}{\sqrt{3+x^2}} \\
&= \int_{\sqrt{3}}^2 (u^2-3)du \\
&= \left[\frac{1}{3}u^3 - 3u\right]_{\sqrt{3}}^2 \\
&= \left(\frac{1}{3}2^3 - 3\cdot 2\right) - \left(\frac{1}{3}\sqrt{3}^3 - 3\sqrt{3}\right) \\
&= \boxed{2\sqrt{3} - \frac{10}{3}}.
\end{aligned}
$$

(f) $\displaystyle\int t\sec^2\left(3t^2\right)e^{7\tan\left(3t^2\right)}dt$

Solution. We use the substitution $u = 7\tan\left(3t^2\right)$. This gives

$$du = 7\sec^2(3t^2)\cdot 6tdt = 42t\sec^2(3t^2)dt.$$

So

$$t\sec^2\left(3t^2\right)dt = \frac{du}{42}.$$

We get

$$
\begin{aligned}
\int t\sec^2\left(3t^2\right)e^{7\tan\left(3t^2\right)}dt &= \int \frac{1}{42}e^u du \\
&= \frac{1}{42}e^u + C \\
&= \boxed{\frac{1}{42}e^{7\tan\left(3t^2\right)} + C}.
\end{aligned}
$$

(g) $\displaystyle\int e^x(e^x-2)^{2/3}dx$

Solution. We use the substitution $u = e^x - 2$, so that $du = e^x dx$. This gives

$$\begin{aligned}\int e^x(e^x-2)^{2/3}dx &= \int u^{2/3}du \\ &= \frac{3}{5}u^{5/3} + C \\ &= \boxed{\frac{3}{5}(e^x-2)^{5/3} + C}.\end{aligned}$$

(h) $\displaystyle\int e^{2x}(e^x-2)^{2/3}dx$

Solution. We again use the substitution $u = e^x - 2$, $du = e^x dx$. But this time, we have an extraneous factor e^x since $e^{2x} = e^x e^x$. We can express this extraneous factor in terms of u as $e^x = u + 2$. Therefore

$$\begin{aligned}\int e^{2x}(e^x-2)^{2/3}dx &= \int e^x(e^x-2)^{2/3}e^x dx \\ &= \int (u+2)u^{2/3}du \\ &= \int \left(u^{5/3} + 2u^{2/3}\right) du \\ &= \frac{3}{8}u^{8/3} + \frac{6}{5}u^{5/3} + C \\ &= \boxed{\frac{3}{8}(e^x-2)^{8/3} + \frac{6}{5}(e^x-2)^{5/3} + C}.\end{aligned}$$

(i) $\displaystyle\int_{e^3}^{e^6} \frac{dt}{t\ln(t)}$

Solution. We use the substitution $u = \ln(t)$, so that $du = \frac{dt}{t}$. The bounds change as follows

$$\begin{aligned} t = e^3 &\Rightarrow u = \ln(e^3) = 3, \\ t = e^6 &\Rightarrow u = \ln(e^6) = 6. \end{aligned}$$

The integral becomes

$$\begin{aligned}\int_{e^3}^{e^6} \frac{dt}{t\ln(t)} &= \int_3^6 \frac{du}{u} \\ &= [\ln|u|]_3^6 \\ &= \ln(6) - \ln(3) \\ &= \ln\left(\frac{6}{3}\right) \\ &= \boxed{\ln(2)}.\end{aligned}$$

(j) $\displaystyle\int \frac{dx}{5x + 4\sqrt{x}}$

Solution. We can first factor out a $\sqrt{x}$ from the denominator, which gives

$$\int \frac{dx}{5x + 4\sqrt{x}} = \int \frac{dx}{\sqrt{x}\,(5\sqrt{x} + 4)}.$$

We can then use the substitution $u = 5\sqrt{x} + 4$, which gives $du = \frac{5dx}{2\sqrt{x}}$. This gives $\frac{dx}{\sqrt{x}} = \frac{2du}{5}$, and the integral becomes

$$\begin{aligned}\int \frac{dx}{\sqrt{x}\,(5\sqrt{x} + 4)} &= \int \frac{2du}{5u} \\ &= \frac{2}{5}\ln|u| + C \\ &= \boxed{\frac{2}{5}\ln|5\sqrt{x} + 4| + C}.\end{aligned}$$

(k) $\displaystyle\int \frac{dx}{\sqrt{2 - x^2}}$

Solution. Recall the reference antiderivative

$$\int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1}(u) + C.$$

We can use this antiderivative after factoring out a 2 from the square root and letting $u = \frac{x}{\sqrt{2}}$. This gives

$$\begin{aligned}\int \frac{dx}{\sqrt{2 - x^2}} &= \int \frac{dx}{\sqrt{2\left(1 - \frac{x^2}{2}\right)}} \\ &= \int \frac{dx}{\sqrt{2}\sqrt{1 - \left(\frac{x}{\sqrt{2}}\right)^2}} \\ &= \int \frac{du}{\sqrt{1 - u^2}} \\ &= \sin^{-1}(u) + C \\ &= \boxed{\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + C}.\end{aligned}$$

(l) $\displaystyle\int_0^1 \frac{xdx}{\sqrt{2 - x^2}}$

Solution 1. This time, the numerator is (up to a constant factor) the derivative of the inside of the square root. Therefore, we can compute this integral with the substitution $u = 2 - x^2$, $du = -2xdx$. Thus we have $xdx = -\frac{du}{2}$, and the bounds change to

$$x = 0 \;\Rightarrow\; u = 2 - 0^2 = 2,$$

$$x = 1 \;\Rightarrow\; u = 2 - 1^2 = 1.$$

We obtain

$$\begin{aligned}
\int_0^1 \frac{x\,dx}{\sqrt{2-x^2}} &= \int_2^1 -\frac{du}{2\sqrt{u}} \\
&= \left[-\sqrt{u}\right]_2^1 \\
&= \boxed{1-\sqrt{2}}.
\end{aligned}$$

Solution 2. We can be more ambitious with the substitution and let $u = \sqrt{2-x^2}$. The bounds change to

$$\begin{aligned}
x = 0 &\;\Rightarrow\; u = \sqrt{2-0^2} = \sqrt{2}, \\
x = 1 &\;\Rightarrow\; u = \sqrt{2-1^2} = 1.
\end{aligned}$$

Differentiating gives $du = -\frac{x\,dx}{\sqrt{2-x^2}}$, which is the entire integrand up to a negative sign. So the integral becomes

$$\int_0^1 \frac{x\,dx}{\sqrt{2-x^2}} = \int_{\sqrt{2}}^1 -\,du = \boxed{\sqrt{2}-1}.$$

(m) $\displaystyle\int_0^{2/3} \frac{dz}{4+9z^2}$

Solution. This integral will make use of the reference antiderivative

$$\int \frac{du}{1+u^2} = \tan^{-1}(u) + C.$$

To get to this form, we factor out a 4 from the denominator to obtain

$$\int_0^{2/3} \frac{dz}{4+9z^2} = \int_0^{2/3} \frac{dz}{4\left(1+\frac{9z^2}{4}\right)} = \int_0^{2/3} \frac{dz}{4\left(1+\left(\frac{3z}{2}\right)^2\right)}$$

We can then use the substitution $u = \frac{3z}{2}$, which gives $du = \frac{3dz}{2}$, so $dz = \frac{2du}{3}$. The bounds change to

$$\begin{aligned}
x = 0 &\;\Rightarrow\; u = 0, \\
x = \frac{2}{3} &\;\Rightarrow\; u = 1.
\end{aligned}$$

We obtain

$$\begin{aligned}
\int_0^{2/3} \frac{dz}{4\left(1+\left(\frac{3z}{2}\right)^2\right)} &= \int_0^1 \frac{1}{4}\cdot\frac{2}{3}\cdot\frac{du}{1+u^2} \\
&= \left[\frac{1}{6}\tan^{-1}(u)\right]_0^1 \\
&= \frac{1}{6}\left(\tan^{-1}(1) - \tan^{-1}(0)\right)
\end{aligned}$$

$$= \frac{1}{6}\left(\frac{\pi}{4} - 0\right)$$
$$= \boxed{\frac{\pi}{24}}.$$

(n) $\displaystyle\int \frac{e^{4\arcsin(5x)}}{\sqrt{1-25x^2}}dx.$

Solution. We use the substitution $u = 4\arcsin(5x)$, so that $du = \frac{20dx}{\sqrt{1-25x^2}}$. Therefore, $\frac{dx}{\sqrt{1-25x^2}} = \frac{du}{20}$ and we obtain

$$\begin{aligned}\int \frac{e^{4\arcsin(5x)}}{\sqrt{1-25x^2}}dx &= \int \frac{e^u}{20}du \\ &= \frac{e^u}{20} + C \\ &= \boxed{\frac{e^{4\arcsin(5x)}}{20} + C}.\end{aligned}$$

(o) $\displaystyle\int_0^{\pi/10} \frac{\sin^3(5x)}{\cos(5x)+3}dx.$

Solution. We can write $\sin^3(5x) = \sin^2(5x)\sin(5x) = (1-\cos^2(5x))\sin(5x)$, which allows us to use the substitution $u = \cos(5x)$, $du = -5\sin(5x)dx$. The bounds of the integral change as follows:

$$\begin{aligned}x = 0 &\Rightarrow u = \cos(0) = 1, \\ x = \frac{\pi}{10} &\Rightarrow u = \cos\left(\frac{\pi}{2}\right) = 0.\end{aligned}$$

Therefore we get

$$\begin{aligned}\int_0^{\pi/10} \frac{\sin^3(5x)}{\cos(5x)+3}dx &= \int_0^{\pi/10} \frac{1-\cos^2(5x)}{\cos(5x)+3}\sin(5x)dx \\ &= \int_1^0 -\frac{1-u^2}{5(u+3)}du \\ &= \frac{1}{5}\int_0^1 \frac{1-u^2}{u+3}du.\end{aligned}$$

To evaluate this last integral, we can substitute $w = u+3$, so that $dw = du$. The u in the numerator can be expressed in terms of w as $u = w-3$. We get

$$\begin{aligned}\frac{1}{5}\int_0^1 \frac{1-u^2}{u+3}du &= \frac{1}{5}\int_3^4 \frac{1-(w-3)^2}{w}dw \\ &= \frac{1}{5}\int_3^4 \frac{1-w^2+6w-9}{w}dw \\ &= \frac{1}{5}\int_3^4 \frac{6w-w^2-8}{w} \\ &= \frac{1}{5}\int_3^4 \left(6 - w - \frac{8}{w}\right)dw\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{5}\left[6w - \frac{w^2}{2} - 8\ln|w|\right]_3^4 \\
&= \frac{1}{5}\left(24 - 8 - 8\ln(4) - 18 + \frac{9}{2} + 8\ln(3)\right) \\
&= \boxed{\frac{5 + 16\ln(3/4)}{10}}.
\end{aligned}
$$

(p) $\displaystyle\int \frac{\left(\tan^{-1}(t)\right)^3}{1+t^2}\,dt.$

Solution. We use the substitution $u = \tan^{-1}(t)$, so $du = \frac{dt}{1+t^2}$. This gives

$$
\begin{aligned}
\int \frac{\left(\tan^{-1}(t)\right)^3}{1+t^2}\,dt &= \int u^3\,du \\
&= \frac{1}{4}u^4 + C \\
&= \boxed{\frac{1}{4}\left(\tan^{-1}(t)\right)^4 + C}.
\end{aligned}
$$

(q) $\displaystyle\int_e^{e^2} \frac{dx}{x\sqrt{\ln(x)}}\,dx.$

Solution. We use the substitution $u = \ln(x)$, so that $du = \frac{dx}{x}$. The bounds become

$$
\begin{aligned}
x = e &\;\Rightarrow\; u = \ln(e) = 1, \\
x = e^2 &\;\Rightarrow\; u = \ln(e^2) = 2.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\int_e^{e^2} \frac{dx}{x\sqrt{\ln(x)}}\,dx &= \int_1^2 \frac{du}{\sqrt{u}}\,du \\
&= \left[2\sqrt{u}\right]_1^2 \\
&= \boxed{2\left(\sqrt{2} - 1\right)}.
\end{aligned}
$$

(r) $\displaystyle\int \frac{\tan(3\ln(x))}{x}\,dx.$

Solution. We use the substitution $u = 3\ln(x)$, so $du = \frac{3dx}{x}$ and the integral becomes

$$
\begin{aligned}
\int \frac{\tan(3\ln(x))}{x}\,dx &= \int \frac{\tan(u)}{3}\,du \\
&= \frac{1}{3}\ln|\sec(u)| + C \\
&= \boxed{\frac{1}{3}\ln\left|\sec\left(3\ln(x)\right)\right| + C}.
\end{aligned}
$$

(s) $\displaystyle\int \frac{x^3+1}{9+x^2}dx.$

Solution. We can start by splitting up the integral into a sum of two integrals:

$$\int \frac{x^3+1}{9+x^2}dx = \int \frac{x^3}{9+x^2}dx + \int \frac{1}{9+x^2}dx.$$

The first integral can be computed using the substitution $u = 9 + x^2$, which gives $du = 2xdx$. The extraneous factor x^2 in the numerator can be replaced by $u - 9$, which gives

$$\begin{aligned}\int \frac{x^3}{9+x^2}dx &= \int \frac{x^2}{9+x^2}xdx \\ &= \int \frac{u-9}{2u}du \\ &= \frac{1}{2}\int \left(1 - \frac{9}{u}\right) du \\ &= \frac{1}{2}\left(u - 9\ln|u|\right) + C \\ &= \frac{1}{2}\left(x^2 + 9 - 9\ln(x^2+9)\right) + C \\ &= \frac{1}{2}\left(x^2 - 9\ln(x^2+9)\right) + C.\end{aligned}$$

For the second integral, we can factor out a 9 from the denominator and use the substitution $u = \frac{x}{3}$, which gives $du = \frac{dx}{3}$. We obtain

$$\begin{aligned}\int \frac{dx}{9+x^2} &= \frac{1}{9}\int \frac{dx}{1+\left(\frac{x}{3}\right)^2} \\ &= \frac{1}{3}\int \frac{du}{1+u^2} \\ &= \frac{1}{3}\tan^{-1}(u) + C \\ &= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.\end{aligned}$$

Putting the pieces together gives

$$\begin{aligned}\int \frac{x^3+1}{9+x^2}dx &= \int \frac{x^3}{9+x^2}dx + \int \frac{1}{9+x^2}dx \\ &= \boxed{\frac{1}{2}\left(x^2 - 9\ln(x^2+9)\right) + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C}.\end{aligned}$$

(t) $\displaystyle\int_0^{\pi/12} \tan^2(3\theta)\sec^2(3\theta)d\theta.$

Solution. We substitute $u = \tan(3\theta)$, so $du = 3\sec^2(3\theta)d\theta$. The bounds change as follows:

$$\begin{aligned}x = 0 &\Rightarrow u = \tan(0) = 0, \\ x = \frac{\pi}{12} &\Rightarrow u = \tan\left(\frac{\pi}{4}\right) = 1.\end{aligned}$$

Then we get

$$\begin{aligned}\int_0^{\pi/12} \tan^2(3\theta)\sec^2(3\theta)d\theta &= \int_0^1 \frac{u^2}{3}du \\ &= \left[\frac{u^2}{9}\right]_0^1 \\ &= \boxed{\frac{1}{9}}.\end{aligned}$$

(u) $\displaystyle\int \frac{e^{3x}}{\sqrt{49-e^{6x}}}dx.$

Solution. We use the substitution $u = e^3x$, so $du = 3e^{3x}dx$. Then we get

$$\begin{aligned}\int \frac{e^{3x}}{\sqrt{49-e^{6x}}}dx &= \int \frac{du}{3\sqrt{7^2-u^2}} \\ &= \frac{1}{3}\sin^{-1}\left(\frac{u}{7}\right) + C \\ &= \boxed{\frac{1}{3}\sin^{-1}\left(\frac{e^{3x}}{7}\right) + C}.\end{aligned}$$

(v) $\displaystyle\int_{-5/2}^{5/2} \frac{1+\sin(x)}{4x^2+25}dx.$

Solution. We can split up the integral into a sum of two integrals:

$$\int_{-5/2}^{5/2} \frac{1+\sin(x)}{4x^2+25}dx = \int_{-5/2}^{5/2} \frac{dx}{4x^2+25} + \int_{-5/2}^{5/2} \frac{\sin(x)}{4x^2+25}dx.$$

The second integral is the integral of an odd function on an interval symmetric about the origin since

$$\frac{\sin(-x)}{4(-x)^2+25} = \frac{-\sin(x)}{4x^2+25} = -\frac{\sin(x)}{4x^2+25}.$$

Therefore,

$$\int_{-5/2}^{5/2} \frac{\sin(x)}{4x^2+25}dx = 0.$$

For the other integral, we can use an arc tangent to get

$$\begin{aligned}\int_{-5/2}^{5/2} \frac{dx}{4x^2+25} &= \frac{1}{25}\int_{-5/2}^{5/2} \frac{dx}{\left(\frac{2x}{5}\right)^2+1} \\ &= \frac{1}{25}\left[\frac{5}{2}\tan^{-1}\left(\frac{2x}{5}\right)\right]_{-5/2}^{5/2} \\ &= \frac{1}{10}\left(\tan^{-1}(1) - \tan^{-1}(-1)\right) \\ &= \frac{\pi}{20}.\end{aligned}$$

Therefore

$$\boxed{\int_{-5/2}^{5/2} \frac{1+\sin(x)}{4x^2+25}dx = \frac{\pi}{20}}.$$

2. Suppose that f is an **even** function such that

$$\int_{-9}^{5} f(x)dx = -13 \quad \text{and} \quad \int_{0}^{9} f(x)dx = 4.$$

Evaluate the definite integrals below.

(a) $\displaystyle\int_{-9}^{9} f(x)dx$

Solution. Since f is even, by symmetry we have

$$\int_{-9}^{9} f(x)dx = 2\int_{0}^{9} f(x)dx = \boxed{8}.$$

(b) $\displaystyle\int_{0}^{5} (4x - 3f(x))dx$

Solution. Let us start by calculating $\displaystyle\int_{0}^{5} f(x)dx$. By additivity of the integral, we have

$$\int_{-9}^{5} f(x)dx = \int_{-9}^{0} f(x)dx + \int_{0}^{5} f(x)dx.$$

Since f is even, we have

$$\int_{-9}^{0} f(x)dx = \int_{0}^{9} f(x)dx = 4.$$

So we get

$$-13 = 4 + \int_{0}^{5} f(x)dx \;\Rightarrow\; \int_{0}^{5} f(x)dx = -17.$$

Now using the linearity of the integral, we obtain

$$\begin{aligned}\int_{0}^{5} (4x - 3f(x))dx &= 4\int_{0}^{5} xdx - 3\int_{0}^{5} f(x)dx \\ &= 4\left[\frac{1}{2}x^2\right]_{0}^{5} - 3(-17) \\ &= 4\frac{1}{2}(25) + 51 \\ &= \boxed{101}.\end{aligned}$$

(c) $\int_{-3}^{3} xf(x)dx$

Solution. Since f is even, the function $g(x) = xf(x)$ is odd, as shown below:

$$g(-x) = (-x)f(-x) = -xf(x) = -g(x).$$

Since the interval of integration $[-3, 3]$ is centered at 0, we deduce

$$\boxed{\int_{-3}^{3} xf(x) = 0}.$$

(d) $\int_{0}^{3} xf(x^2)dx$

Solution. We can evaluate this integral using the substitution $u = x^2$, which gives $du = 2xdx$, or $xdx = \frac{du}{2}$. The bounds become

$$x = 0 \Rightarrow u = 0^2 = 0,$$
$$x = 3 \Rightarrow u = 3^2 = 9.$$

Therefore

$$\begin{aligned}\int_0^3 xf(x^2)dx &= \int_0^9 \frac{1}{2} f(u)du \\ &= \frac{1}{2}\int_0^9 f(u)du \\ &= \frac{1}{2}4 \\ &= \boxed{2}.\end{aligned}$$

3. Find the average value of the following functions on the given interval.

(a) $f(x) = \frac{3}{\sqrt{100 - x^2}}$ on $[0, 5]$.

Solution. The average value is given by

$$\begin{aligned}\text{av}(f) &= \frac{1}{5-0}\int_0^5 \frac{3}{\sqrt{100-x^2}}dx \\ &= \frac{3}{5}\int_0^5 \frac{dx}{\sqrt{100\left(1 - \frac{x^2}{100}\right)}} \\ &= \frac{3}{5}\int_0^5 \frac{dx}{10\sqrt{1 - \left(\frac{x}{10}\right)^2}} \\ &= \frac{3}{5}\int_0^{1/2} \frac{du}{\sqrt{1-u^2}} \qquad \left(u = \frac{x}{10}\right) \\ &= \frac{3}{5}\left[\sin^{-1}(u)\right]_0^{1/2}\end{aligned}$$

$$= \frac{3}{5}\left(\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right)$$
$$= \frac{3}{5} \cdot \frac{\pi}{6}$$
$$= \boxed{\frac{\pi}{10}}.$$

(b) $f(x) = x\sqrt[3]{3x-7}$ on $[2, 5]$.

Solution. The average value is given by

$$\operatorname{av}(f) = \frac{1}{5-2}\int_2^5 x\sqrt[3]{3x-7}dx = \frac{1}{3}\int_2^5 x\sqrt[3]{3x-7}dx.$$

We can calculate the integral using the substitution $u = 3x - 7$. This will give $du = 3dx$, or $dx = \frac{du}{3}$. The bounds will become

$$x = 2 \;\Rightarrow\; u = 3 \cdot 2 - 7 = -1,$$
$$x = 5 \;\Rightarrow\; u = 3 \cdot 5 - 7 = 8.$$

Finally, the extraneous factor x in the integrand can be expressed in terms of u as $x = \frac{u+7}{3}$. We obtain

$$\operatorname{av}(f) = \frac{1}{3}\int_{-1}^{8} \frac{u+7}{3}\sqrt[3]{u}\frac{du}{3}$$
$$= \frac{1}{27}\int_{-1}^{8}\left(u^{4/3} + 7u^{1/3}\right) du$$
$$= \frac{1}{27}\left[\frac{3}{7}u^{7/3} + \frac{21}{4}u^{4/3}\right]_{-1}^{8}$$
$$= \frac{1}{27}\left(\left(\frac{3}{7}8^{7/3} + \frac{21}{4}8^{4/3}\right) - \left(\frac{3}{7}(-1)^{7/3} + \frac{21}{4}(-1)^{4/3}\right)\right)$$
$$= \boxed{\frac{139}{28}}.$$