Sections 5.5, 5.6

Substitution Method

Learning Goals

Learning Goal	*Homework Problems*
5.5.1 Compute indefinite integrals using substitution.	1-44, 49 -72.
5.5.2 Solve initial value problems.	73-80.
5.5.3 Solve applications involving indefinite integrals and the substitution method.	79, 80.
5.5.4 Answer conceptual questions involving the substitution method.	67-72.
Learning Goal	*Homework Problems*
5.6.1 Compute definite integrals using substitution.	1-54, 72, 81, 87, 89, 90, 93, 100-104, 113, 114, 117-120.

Which of these integrals can we evaluate ?

A	B	C
$\int \cos(2x)\,dx$	$\int \cos(x^2)\,dx$	$\int 2x\cos(x^2)\,dx$

A. and C. : reverse chain rule.

$$\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + C \quad \text{because} \quad \frac{d}{dx}(\sin(2x)) = \cos(2x)(2)$$

$$\int 2x\cos(x^2)\,dx = \sin(x^2) + C \quad \text{because} \quad \frac{d}{dx}(\sin(x^2)) = \cos(x^2)(2x)$$

B cannot be evaluated explicitly.

⚠ $$\int \cos(x^2)\,dx \neq \frac{\sin(x^2)}{2x} + C$$

because $$\frac{d}{dx}\left(\frac{\sin(x^2)}{2x}\right) = \frac{2x\cos(x^2)2x - 2\sin(x^2)}{(2x)^2} \neq \cos(x^2).$$

If we do not see the answer for A. and C. directly, we can use the <u>substitution method</u>, which is the reverse chain rule for integrals.

A. $\int \cos(\underbrace{2x}_{u})\,\underbrace{dx}_{\frac{1}{2}du}$ — choose inside function : $u = 2x$; differential : $du = 2dx \Rightarrow dx = \frac{1}{2}du$

$= \int \cos(u)\,\frac{1}{2}du$ — rewrite integral in terms of u and du

$= \frac{1}{2}\sin(u) + C$ — evaluate u- integral

$= \boxed{\frac{1}{2}\sin(2x) + C.}$ — back substitute to write in terms of x.

c. $\int \underbrace{2x}_{} \cos(\underbrace{x^2}_{u}) \, dx$ choose inside function : $u = x^2$
differential : $du = 2x\,dx$

$= \int \cos(u)\, du$ rewrite integral in terms of u and du

$= \sin(u) + C$ evaluate u-integral

$= \boxed{\sin(x^2) + C}$ back substitute to write in terms of x.

Substitution Method for Antiderivatives :

$$\boxed{\int f(g(x))\,g'(x)\,dx = \int f(u)\,du \quad \text{with} \quad \begin{array}{l} u = g(x) \\ du = g'(x)\,dx \end{array}}$$

Tip to choose u : look for an "inside function" whose derivative appears as a factor (up to a constant multiple).

Examples : 1) $\int x^2 e^{x^3} dx$

$$\boxed{\begin{array}{l} u = x^3 \\ du = 3x^2 dx \\ \Rightarrow \ x^2 dx = \frac{1}{3} du \end{array}}$$

$= \int e^{\overset{u}{x^3}} \, \overset{\frac{1}{3}du}{x^2 dx}$

$= \int \frac{1}{3} e^u du = \frac{1}{3} e^u + C = \boxed{\frac{1}{3} e^{x^3} + C}$.

2) $\int (x^3 + x)^{14} (6x^2 + 2)\, dx$

$$\boxed{\begin{array}{l} u = x^3 + x \\ du = (3x^2 + 1)\,dx \end{array}}$$

$= \int (\underbrace{x^3 + x}_{u})^{14}\, 2\underbrace{(3x^2 + 1)\,dx}_{du}$

$$= \int 2u^{14}\,du = \frac{2u^{15}}{15} + C = \boxed{\frac{2(x^3+x)}{15} + C}.$$

3) $$\int e^{3x}\sec^2(e^{3x})\,dx$$

$u = e^{3x}$
$du = 3e^{3x}dx$
$\Rightarrow e^{3x}dx = \frac{1}{3}du$

$$= \int \sec^2(\underbrace{e^{3x}}_{u})\,\underbrace{e^{3x}dx}_{\frac{1}{3}du}$$

$$= \int \sec^2(u)\,\frac{1}{3}du = \frac{1}{3}\tan(u) + C = \boxed{\frac{1}{3}\tan(e^{3x}) + C}$$

4) $$\int \frac{5x+1}{5x^2+2x+1}\,dx$$

$u = 5x^2+2x+1$
$du = (10x+2)dx$
$\Rightarrow (5x+1)dx = \frac{1}{2}du$

$$= \int \frac{1}{\underbrace{5x^2+2x+1}_{u}}\,\underbrace{(5x+1)dx}_{\frac{1}{2}du}$$

$$= \int \frac{1}{2}\cdot\frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln|5x^2+2x+1| + C}.$$

5) $$\int \tan(x)\,dx = \int \frac{\overbrace{\sin(x)\,dx}^{-du}}{\underbrace{\cos(x)}_{u}}$$

$u = \cos(x)$
$du = -\sin(x)dx$

$$= \int -\frac{du}{u} = -\ln|u| + C = -\ln|\cos(x)| + C$$

$$= \ln\left|\frac{1}{\cos(x)}\right| + C$$

So $$\boxed{\int \tan(x)\,dx = \ln|\sec(x)| + C}.$$

* memorize

$$\int \sec(x)\,dx = \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx$$

$$= \int \frac{\overbrace{\sec^2(x) + \sec(x)\tan(x) \, dx}^{du}}{\underbrace{\sec(x) + \tan(x)}_{u}}$$

$$\boxed{\begin{aligned} u &= \sec(x) + \tan(x) \\ du &= (\sec^2(x) + \sec(x)\tan(x))dx \end{aligned}}$$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec(x) + \tan(x)| + C.$$

So $$\boxed{\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C}.$$

* memorize

Other formulas : $$\boxed{\int \cot(x)dx = \ln|\sin(x)| + C}$$

$$\boxed{\int \csc(x)dx = -\ln|\csc(x) + \cot(x)| + C}$$

6) $$\int \frac{\tan^{-1}(x)}{1+x^2}dx$$

$$\boxed{\begin{aligned} u &= \tan^{-1}(x) \\ du &= \frac{1}{1+x^2}dx \end{aligned}}$$

$$= \int \underbrace{\tan^{-1}(x)}_{u} \underbrace{\frac{1}{1+x^2}dx}_{du}$$

$$= \int u\,du = \frac{u^2}{2} + C = \boxed{\frac{(\tan^{-1}(x))^2}{2} + C}.$$

7) $$\int \frac{e^x}{\sqrt{1-e^{2x}}}dx$$

$$\boxed{\begin{aligned} u &= e^x \\ du &= e^x dx \end{aligned}}$$

$$= \int \frac{1}{\sqrt{1-(\underbrace{e^x}_{u})^2}} \underbrace{e^x dx}_{du}$$

$$= \int \frac{1}{\sqrt{1-u^2}}du = \sin^{-1}(u) + C = \boxed{\sin^{-1}(e^x) + C}.$$

Sometimes, after substituting $g(x) = u$ and $g'(x)\,dx = du$, there are still expressions involving x in the integral. When this happens, we need to solve for $x = g^{-1}(u)$ and substitute in the integral.

Examples : 1) $\int x\sqrt{2x+1}\,dx$

$$\boxed{\begin{array}{l} u = 2x+1 \\ du = 2dx \\ \Rightarrow dx = \frac{1}{2}du \end{array}}$$

$$= \int x\sqrt{u}\,\frac{1}{2}du$$

↳ need to express x in terms of u : $u = 2x+1$

$\Rightarrow x = \frac{u-1}{2}$

$$= \int \frac{u-1}{2}\sqrt{u}\,\frac{1}{2}du$$

$$= \frac{1}{4}\int (u-1)u^{1/2}\,du = \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right)du$$

$$= \frac{1}{4}\left(\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3}\right) + C$$

$$= \boxed{\frac{1}{4}\left(\frac{2(2x+1)^{5/2}}{5} - \frac{2(2x+1)^{3/2}}{3}\right) + C}.$$

2) $\int \frac{x}{(1-x)^{1/3}}\,dx$

$$\boxed{\begin{array}{l} u = 1-x \\ du = -dx \\ \Rightarrow dx = -du \end{array}}$$

$$= \int \frac{x}{u^{1/3}}(-du)$$

$u = 1-x$

$\Rightarrow x = 1-u$

$$= \int -\frac{1-u}{u^{1/3}}\,du = \int \left(u^{2/3} - u^{-1/3}\right)du = \frac{3u^{5/3}}{5} - \frac{3u^{2/3}}{2} + C$$

$$= \boxed{\frac{3(1-x)^{5/3}}{5} - \frac{3(1-x)^{2/3}}{2} + C}.$$

3) $\displaystyle\int \frac{dx}{(2+\sqrt{x})^3}$

$$\boxed{\begin{aligned} u &= 2+\sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}dx \\ \Rightarrow dx &= 2\sqrt{x}\,du \end{aligned}}$$

$$= \int \frac{2\overset{u-2}{\sqrt{x}}\,du}{u^3}$$

$$= \int \frac{2(u-2)}{u^3}du = 2\int (u^{-2} - 2u^{-3})\,du = 2\left(\frac{u^{-1}}{-1} - \frac{2u^{-2}}{-2}\right) + C$$

$$= -\frac{2}{u} + \frac{2}{u^2} + C = \boxed{-\frac{2}{2+\sqrt{x}} + \frac{2}{(2+\sqrt{x})^2} + C}.$$

Substitution Method for Definite Integrals :

If $u = g(x)$, then $\boxed{\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du}$

Bounds must be changed in terms of u.

Examples : 1) $\displaystyle\int_{-1}^{1} x^2\sqrt{x^3+1}\,dx$

$$\boxed{\begin{aligned} u &= x^3+1 \\ du &= 3x^2dx \\ \Rightarrow x^2dx &= \tfrac{1}{3}du \\ x=1 &\Rightarrow u = 1^3+1 = 2 \\ x=-1 &\Rightarrow u=(-1)^3+1 = 0 \end{aligned}}$$

$$= \int_0^2 \frac{1}{3}\sqrt{u}\,du$$

$$= \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_0^2 = \frac{1}{3}\left(\frac{2}{3}2^{3/2} - \frac{2}{3}0^{3/2}\right) = \boxed{\frac{2\cdot 2^{3/2}}{9}}.$$

DO NOT BACK SUBSTITUTE

JUST PLUG-IN u- BOUNDS

2) $\displaystyle\int_1^e \frac{\ln(x)^2}{x}\,dx$

$$\boxed{\begin{aligned} u &= \ln(x) \\ du &= \frac{dx}{x} \\ x=1 &\Rightarrow u = \ln(1) = 0 \\ x=e &\Rightarrow u = \ln(e) = 1 \end{aligned}}$$

$$= \int_0^1 u^2\,du$$

$$= \left[\frac{1}{3}u^2\right]_0^1 = \boxed{\frac{1}{3}}$$

3) $$\int_0^{\pi/6} \sec^2(2x)\tan^3(2x)\,dx$$

$u = \tan(2x)$
$du = 2\sec^2(2x)\,dx$
$x = 0 \Rightarrow u = \tan(0) = 0$
$x = \frac{\pi}{6} \Rightarrow u = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$$= \int_0^{\sqrt{3}} \frac{1}{2}u^3\,du$$

$$= \left[\frac{u^4}{8}\right]_0^{\sqrt{3}} = \boxed{\frac{9}{8}}.$$

$\tan^{-1}(5x)$

4) $$\int_0^{1/5} \frac{e^{\tan^{-1}(5x)}}{1+25x^2}\,dx$$

$u = \tan^{-1}(5x)$
$du = \frac{5}{1+25x^2}\,dx$

$x = 0 \Rightarrow u = \tan^{-1}(0) = 0$
$x = 1/5 \Rightarrow u = \tan^{-1}(1) = \pi/4$

$$= \int_0^{\pi/4} \frac{1}{5}e^u\,du$$

$$= \left[\frac{1}{5}e^u\right]_0^{\pi/4} = \boxed{\frac{e^{\pi/4}-1}{5}}.$$