Rutgers University
Math 151

**Math 151 All Worksheets**
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# Contents

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**Chapter 1: Review of Algebra & Precalculus - Worksheet**

1. **Composite functions:** recall that given two functions $f$ and $g$, the function $f \circ g$ (called $f$ *composed with* $g$) is
$$(f \circ g)(x) = f(g(x)).$$
   (a) Given $f(x) = \sqrt{x}$ and $g(x) = (x-3)^2$, find and simplify the following.

   i. $(f \circ g)(x)$ ii. $(g \circ f)(x)$ iii. $(f \circ f \circ f)(x)$

   (b) Let $H(x) = \cos(3x^2) + 1$. Complete the table below to find pairs of functions $f(x)$ and $g(x)$ such that $H(x) = f(g(x))$.

| | $f(x) =$ | $g(x) =$ |
|---|---|---|
| i. | $\cos(x) + 1$ | |
| ii. | | $x^2$ |
| iii. | | $\cos(3x^2)$ |
| iv. | $x$ | |
| v. | | $\cos(3x^2) + 7$ |

2. **Trigonometry:**

   (a) Suppose that $\cos(a) = \frac{2}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$. Evaluate the following.

   i. $\tan(a)$ ii. $\sin(2a)$ iii. $\cos(2a)$

   (b) Evaluate the following.

   i. $\sec\left(\frac{4\pi}{3}\right)$
   ii. $\tan^{-1}(1)$
   iii. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
   iv. $\csc^{-1}(2)$
   v. $\sin\left(\sin^{-1}(0.8)\right)$
   vi. $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$

   (c) Simplify the following. Your answers should be algebraic expressions of $x$ (not involving any trigonometric or inverse trigonometric functions).

   i. $\cos\left(\cos^{-1}(x)\right)$
   ii. $\cos\left(\sin^{-1}(x)\right)$
   iii. $\sin\left(\cos^{-1}(x)\right)$
   iv. $\sec\left(\tan^{-1}(4x)\right)$
   v. $\tan\left(\cos^{-1}\left(\frac{x}{2}\right)\right)$
   vi. $\csc\left(\cot^{-1}\left(\frac{3x}{5}\right)\right)$

3. **Exponential and Logarithmic Functions:**

   (a) Evaluate the following.

i. $e^{\ln(75)-2\ln(5)}$

ii. $\log_{\frac{1}{2}}(32)$

iii. $\ln(9e^2)+\ln(\sqrt{9e})-\ln(27e^{1/3})$

(b) Solve the following equations.

i. $2^{5x-1}=4^{-3x}$

ii. $\log_4(x+5)-\log_4(x)=2$

iii. $e^{2x}-3e^x-10=0$

4. **Inverse Functions:** each function below is one-to-one. Find the inverse function.

(a) $f(x)=(x+8)^{7/4}$

(b) $f(x)=\dfrac{3-2x}{4x+7}$

(c) $f(x)=5+2e^{3x+1}$

(d) $f(x)=1-\arcsin(x^3)$

(e) $f(x)=\ln(x)-\ln(x-3)$

(f) $f(x)=\dfrac{2^x}{2^x+3}$

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**Section 2.1: Introduction to Limits - Worksheet**

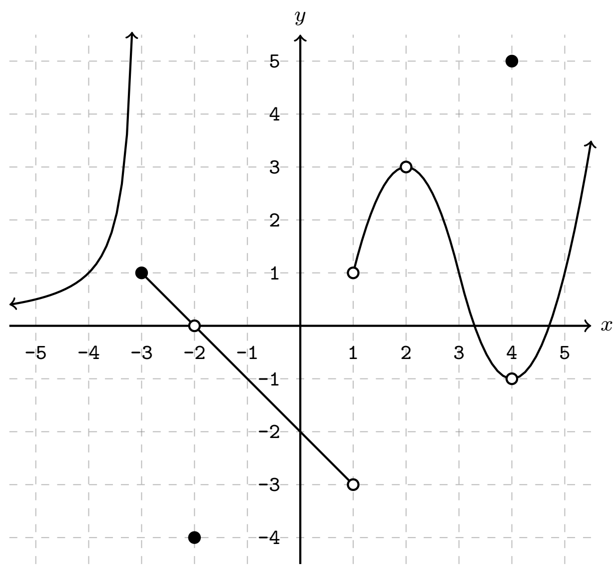
1. Calculate the average rate of change the following functions on the given intervals.

   (a) $f(x) = 2\ln(5x+1)$ on the interval $[0, 3]$.

   (b) $f(x) = \sin(4x)$ on the interval $\left[\frac{\pi}{24}, \frac{\pi}{12}\right]$.

   (c) $f(x) = \arctan(3x)$ on the interval $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

2. The graph of the function $y = f(x)$ is given below.




   Evaluate $f(a)$ and $\lim_{x \to a} f(x)$ for the following values of $a$, or say if the quantity does not exist.

   (a) $a = -3$ (b) $a = -2$ (c) $a = 1$ (d) $a = 2$ (e) $a = 4$

3. The following table of values are given for the functions $f(x)$ and $g(x)$. Use these to estimate $\lim_{x \to 3} f(x)$ and $\lim_{x \to 3} g(x)$ or say if a limit does not exist.

| $x$ | 2.9 | 3.01 | 2.999 | 3.0001 | 2.99999 |
|---|---|---|---|---|---|
| $f(x)$ | 4.15 | 3.95 | 4.05 | 3.9993 | 4.0005 |
| $g(x)$ | 7.98 | 1.001 | 7.997 | 1.0002 | 7.99992 |

4. Using a limit of average rates of change, find the instantaneous rate of change of the following functions at the given value of $x$.

   (a) $f(x) = x^2 - 3x + 7$ at $x = 0$.

   (b) $f(x) = \dfrac{x}{5 - x}$ at $x = -1$.

   (c) **[Advanced]** $f(x) = \dfrac{1}{\sqrt{2x+1}}$ at $x = 4$.

5. The position of an object moving along an axis is given by the function $s(t) = 6\sqrt{x+1}$.

   (a) Find the average velocity of the object between $t = 0$ and $t = 15$.

   (b) Find the position and instantaneous velocity of the object at $t = 3$.

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## Section 2.2: Calculating Limits - Worksheet

1. Evaluate the following limits. If a limit does not exist, explain why.

(a) $\lim_{x\to 3} \dfrac{x^2 - x - 6}{9x^{-2} - 1}$.

(b) $\lim_{t\to 2} \dfrac{\sqrt{t^2+12} - 2t}{2 - t}$.

(c) $\lim_{y\to 0} y\cot(5y)$.

(d) $\lim_{x\to 0} \dfrac{(x-2)^3 + 8 - 12x}{x^2}$.

(e) $\lim_{u\to 4} \dfrac{u - 4}{\sqrt{2u+1} - \sqrt{u+5}}$.

(f) $\lim_{x\to 0} \dfrac{\sin^2(4x)}{x\sin(3x)}$.

(g) $\lim_{x\to 1} \dfrac{\sqrt{4x^2+7} - \sqrt{x+10}}{x - 1}$

(h) $\lim_{h\to 0} \dfrac{\frac{6}{3+7h} - 2}{h}$

(i) $\lim_{x\to 0} \dfrac{x\sin(5x)}{\tan^2(3x)}$

[**Advanced**]

(j) $\lim_{\theta\to 0} \dfrac{\sin^2(3\theta)}{\cos(5\theta) - 1}$.

(k) $\lim_{x\to 0} x\sin(\ln|x|)$.

(l) $\lim_{h\to 1} \dfrac{\sqrt[3]{h} - 1}{h - 1}$.

2. Suppose that $f$ is a function such that for any number $x$, we have

$$x - 8 \leqslant f(x) \leqslant x^2 - 3x - 4.$$

For which values of $a$ can you determine $\lim_{x\to a} f(x)$? For these values of $a$, evaluate $\lim_{x\to a} f(x)$.

3. [**Advanced**] Suppose that $f$ is a function such that

$$\lim_{x\to 0} \frac{f(x)}{\sin(3x)} = 2.$$

Evaluate the following limits.

(a) $\lim_{x\to 0} f(x)$.

(b) $\lim_{x\to 0} \dfrac{f(x)}{x}$.

(c) $\lim_{x\to 0} \dfrac{1 - \cos(x)}{f(2x)^2}$.

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**Section 2.4: One-Sided Limits - Worksheet**

1. Evaluate the following limits. If a limit does not exist, explain why.

(a) $\lim_{x\to 3^-} \dfrac{x^2-4x+3}{|x-3|}$.

(b) $\lim_{x\to 3} \dfrac{x^2-4x+3}{|x-3|}$.

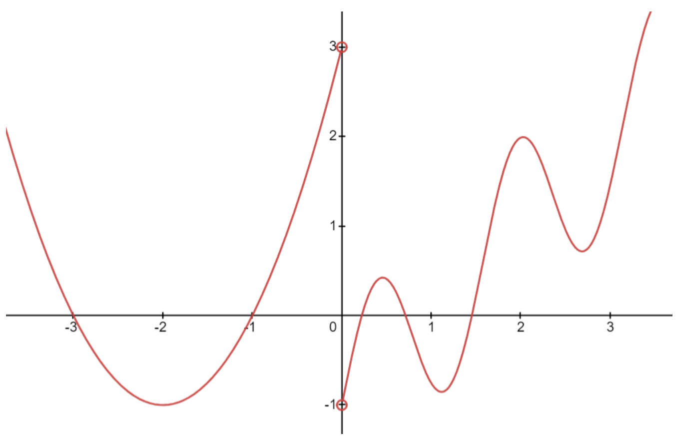
(c) $\lim_{h\to 0^-} \dfrac{1-(1-|h|)^3}{h}$.

(d) $\lim_{t\to 1} \dfrac{t^3-2t^2+t}{|t-1|}$.

(e) $\lim_{x\to -2} f(x)$ where $f(x) = \begin{cases} 3x+8 & \text{if } x<-2 \\ 8 & \text{if } x=-2 \\ \dfrac{x+2}{\sqrt{x+3}-1} & \text{if } x>-2 \end{cases}$.

(f) $\lim_{x\to 0} f(x)$ where $f(x) = \begin{cases} \dfrac{\sin(3x)}{x} & \text{if } x<0 \\ 2e^{\cos(x)-1} & \text{if } x\geqslant 0 \end{cases}$.

2. Consider the function $f(x) = \dfrac{\tan(8x)}{|2x|}$ and suppose that the graph of another function $g$ is given below.




(a) Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ or explain why it does not exist.

(b) Find $\lim_{x\to 0} |f(x)|$ and $\lim_{x\to 0} |g(x)|$ or explain why it does not exist.

(c) Find $\lim_{x\to 0} f(x)+2g(x)$ or explain why it does not exist.

(d) [**Advanced**] Find the value of the constant $a$ for which $\lim_{x\to 0} \dfrac{g(x)}{f(x)+a}$ exists. For this value of $a$, find the value of the limit.

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**Section 2.5: Continuity - Worksheet**

1. For each function, find the values of the constants $a, b$ that make it continuous.

(a) $f(x) = \begin{cases} 3x - b & \text{if } x \leqslant 1 \\ ax + 4 & \text{if } 1 < x \leqslant 3 \\ bx - 2a & \text{if } x > 3 \end{cases}$.

(b) $f(x) = \begin{cases} bx + 4 & \text{if } x < 1 \\ a & \text{if } x = 1 \\ \dfrac{x^{-1} - 1}{x^2 - 1} & \text{if } x > 1 \end{cases}$.

(c) [**Advanced**] $f(x) = \begin{cases} \dfrac{\sin(ax)}{3x} & \text{if } x < 0 \\ b & \text{if } x = 0 \\ \dfrac{x^2 + 5x}{\sqrt{x+4} - 2} & \text{if } x > 0 \end{cases}$.

2. Consider the function $f(x) = \begin{cases} x^2 + 4x + 5 & \text{if } x < -2 \\ 3 & \text{if } x = -2 \\ \cos(\pi x) & \text{if } -2 < x < 3 \\ x + 2 & \text{if } 3 \leqslant x \leqslant 4 \\ 6 - \ln(x - 3) & \text{if } x > 4 \end{cases}$.

(a) Find the values of $a$ for which $\lim_{x \to a} f(x)$ does not exist.

(b) Find the values of $x$ where $f$ is discontinuous.

3. Show that each equation has a solution in the given interval.

(a) $x^3 = 14 + 2\sqrt{x}$ in $[0, 4]$.

(b) $\ln(x) = 2 - x$ in $[1, e]$.

(c) [**Advanced**] $\cos(x) = \arcsin(x)$ in $[0, 1]$.

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## Section 2.6: Limits Involving Infinity - Worksheet

1. Evaluate the following limits. If a limit does not exist, explain why. If a limit is infinite, specify it and determine if it is $\infty$ or $-\infty$.

(a) $\displaystyle\lim_{x\to -1^-} \frac{x^2+3x+2}{(x+1)^2}$.

(b) $\displaystyle\lim_{x\to \infty} \frac{3x\sqrt{x}+2}{\sqrt{4x^3+1}}$.

(c) $\displaystyle\lim_{x\to 2\pi} \frac{x}{\cos(x)-1}$.

(d) $\displaystyle\lim_{x\to 2} \frac{x-5}{x^2-2x}$.

(e) $\displaystyle\lim_{x\to -\infty} \frac{x^3+2}{\sqrt{16x^6+1}}$.

(f) $\displaystyle\lim_{t\to \infty} \sqrt{9t^2+8t}-\sqrt{9t^2-5t}$.

**[Advanced]**

(g) $\displaystyle\lim_{\theta\to -\infty} \frac{2\theta+5\sin(3\theta)}{7\theta}$.

(h) $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{\sqrt[3]{x}}-\frac{1}{\sqrt{x}}\right)$.

(i) $\displaystyle\lim_{t\to \infty} \frac{t\arctan(3t)}{\sqrt{t^2+1}}$.

2. Find the vertical and horizontal asymptotes of the following functions, if any. Also, determine the limit to the left and right of any vertical asymptote.

(a) $f(x)=\dfrac{x^2-3x-4}{\sqrt{x}-2}$.

(b) $f(x)=\dfrac{x^2-1}{|x+1|^3}$.

(c) $f(x)=\dfrac{7+2e^x}{5e^x-4}$.

(d) $f(x)=\dfrac{\sqrt{x^2+25}+3x}{2x+5}$.

(e) $f(x)=\dfrac{\sin(7x)}{x^2+3x}$.

(f) $f(x)=x^2\cos\left(\dfrac{2}{x}\right)$.

**[Advanced]**

(g) $f(x)=\dfrac{3x\arctan(x)+7}{x-1}$.

(h) $f(x)=\dfrac{3e^{2x}-5e^{-x}}{2e^{-x}+e^{4x}}$.

(i) $f(x)=\dfrac{1-\cos(5x)}{x^2+x^3}$.

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**Section 3.1-2: Derivatives and Tangent Lines - Worksheet**

1. For the functions below, find the value of the derivative and an equation of the tangent line at the point indicated. (You must use the limit definition of the derivative in this problem - you cannot use derivative rules.)

   (a) $f(x) = \dfrac{3x}{1-2x}$ at $x = 1$.

   (b) $f(x) = \sqrt{5x-1}$ at $x = 2$.

   (c) $f(x) = 18x^{-2}$ at $x = -3$.

   (d) $f(x) = 2x^3 + 5x + 3$ at $x = -1$.

   **[Advanced]**

   (e) $f(x) = 3\tan(4x)$ at $x = 0$.

   (f) $f(x) = x^{2/3}$ at $x = 8$.

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**Sections 3.3, 3.5: Differentiation Rules - Worksheet**

1. Calculate the derivatives of the following functions.

   (a) $f(x) = 5x^4 - 8\sqrt[5]{x} - e^4$.

   (b) $f(x) = 7x\cos(x)e^x$.

   (c) $f(x) = ex^e + 4\dfrac{\sqrt{x}}{\sin(x)}$.

   (d) $f(x) = \dfrac{3}{5+x^4}$.

   (e) $f(x) = 3\sin(1)7^x - x^{4/3}$.

   (f) $f(x) = \dfrac{x^2}{xe^x - 1}$.

   (g) $f(x) = 2^x x^2$.

   (h) $f(x) = \dfrac{\cos(x)}{\sin(x)+1}$.

   (i) $f(x) = \dfrac{x\cos(x)\sin(x)}{5^x}$.

2. (a) Find the points on the graph of $f(x) = 2\sec(x) + \tan(x)$, $-\dfrac{\pi}{2} < x < \frac{\pi}{2}$, where the tangent line is horizontal.

   (b) Find the points on the graph of $f(x) = \dfrac{1}{1-2x}$ where the tangent line passes through the origin.

   (c) [**Advanced**] Find the values of the constant $a$ for which the tangent lines to the graph of $f(x) = x^3 + 3x^2 + 5x$ at $x = a$ and $x = a+1$ are parallel.

3. Find the second derivative of the functions below.

   (a) $f(x) = x^3e^x$.

   (b) $f(x) = \dfrac{3x+5}{2x+7}$.

   (c) $f(x) = \dfrac{7\cos(x)}{x}$.

4. Suppose that $f$ is a differentiable function such that $y = -2x + 1$ is tangent to the graph of $f$ at $x = 3$. Evaluate the following

   (a) $f(3)$.

   (b) $f'(3)$.

   (c) $\dfrac{d}{dx}\left(2f(x) - x^3\right)_{|x=3}$.

   (d) $\dfrac{d}{dx}\left(\dfrac{f(x)}{x}\right)_{|x=3}$.

   (e) [**Advanced**] $\dfrac{d}{dx}\left(e^x f(x)^2\right)_{|x=3}$.

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## Section 3.4: Rates of Change - Worksheet

1. The position of a body moving an axis is given by $s(t) = \dfrac{t^4}{4} - 2t^3 + 8$.

   (a) Find the body's displacement and average velocity on the time interval $[0, 2]$.

   (b) Find the velocity and acceleration of the body.

   (c) When does the body change direction?

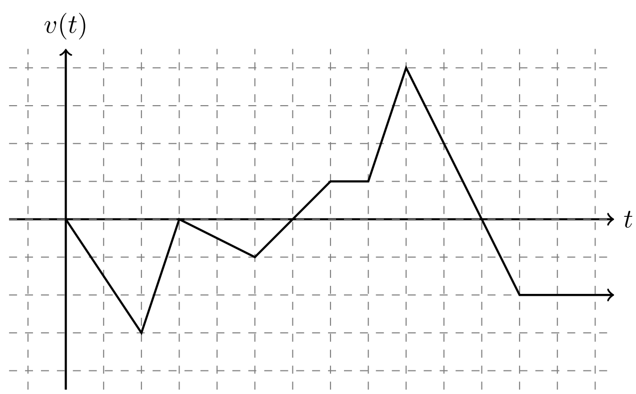
2. A projectile is thrown at $t = 0$ straight up in the air from an altitude of 99 m at a speed of 24 m/sec. The projectile being subject to gravity only, physicists tell us that the elevation of the projectile is subject to a law of the form $h(t) = at^2 + bt + c$, where $a, b, c$ are unspecified constants.

   (a) Find $b$ and $c$ using the information given.

   (b) Suppose that the projectile reaches its maximum elevation 4 seconds after being thrown. Find the value of the constant $a$.

   (c) When will the projectile hit the ground?

3. The graph below shows the velocity $v$ of an object moving along an axis.




   (a) When is the object moving forward? backward?

   (b) When does the object reverse direction?

   (c) Sketch the graph of the acceleration of the object.

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**Section 3.6: Chain Rule - Worksheet**

1. Calculate the derivatives of the following functions.

   (a) $f(x) = 2\sec(4x^3 + 7)$

   (b) $f(x) = 14\sqrt[7]{4x - \sin(5x)}$

   (c) $f(x) = \cos(x^2) - \cos(x)^2$

   (d) $f(x) = 3\left(\tan\left(\frac{x}{7}\right) + 1\right)^{21}$

   (e) $f(x) = \sqrt{25 - 4x^2}$

   (f) $f(x) = e^{5\cos(3x)}$

   (g) $f(x) = x5^{3x^2}$

   (h) $f(x) = 6\cos(x^3\sin(1 - 2x))$

   (i) $f(x) = \frac{2x}{\sqrt{\cos(3x)}}$

2. Find the $x$-values of the points on the graph of $f(x) = (2x + 1)e^{-x^2}$ where the tangent line is horizontal.

3. **[Advanced]** Suppose that $f$ is a differentiable function such that

$$f(0) = -1, \quad f(1) = 3, \quad f(2) = -5, \quad f(4) = 7,$$
$$f'(0) = -2, \quad f'(1) = 4, \quad f(2) = 3, \quad f'(4) = -1.$$

   Find an equation of the tangent lines to each of the following functions at the given point.

   (a) $g(x) = f(-2x)$ at $x = -1$.

   (b) $g(x) = f(x^2)$ at $x = 2$.

   (c) $g(x) = \sec\left(\frac{\pi f(x)}{12}\right)$ at $x = 1$.

   (d) $g(x) = f(4x)e^{3x}$ at $x = 0$.

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**Section 3.7: Implicit Differentiation - Worksheet**

1. Calculate $\dfrac{dy}{dx}$ for the following curves.

   (a) $e^{5xy} + 11\tan(x) = y^2$

   (b) $x^3 - x\sin(y) = 3xy$

   (c) $\sqrt{x^2 + y^2} = 3^y$

   (d) $x^4 + 6xy^2 + 5y^3 = 0$

2. Consider the curve of equation $x^2 + 6xy - y^2 = 40$. Find the points on the curve, if any, where the tangent line is (a) horizontal, (b) vertical, (c) [**Advanced**] perpendicular to $y = 2x + 9$.

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**Sections 3.8-9: Derivatives of Inverse Functions - Worksheet**

1. Calculate the derivatives of the following functions.

   (a) $f(x) = \sin^{-1}(4x)$
   (b) $f(x) = \ln(2\arctan(5x) + 1)$
   (c) $f(x) = x\sec^{-1}(7x)$
   (d) $f(x) = \ln(x)^2 + 8\arccos(-x)$
   (e) $f(x) = \cot^{-1}(e^{3x})$
   (f) $f(x) = \cos(x)\log_7(\sec(x))$
   (g) $f(x) = x^{3\tan^{-1}(2x)}$
   (h) $f(x) = \cos(x)^{\ln(x)}$
   (i) $f(x) = (1-5x)^{x^2}$

2. Simplify each of the following. Your answer should not contain any trigonometric or inverse trigonometric functions.

   (a) $\cos\left(\sin^{-1}(x+1)\right)$
   (b) $\sin(2\cos^{-1}(3x))$
   (c) $\csc\left(\tan^{-1}\left(\frac{2x}{3}\right)\right)$
   (d) $\sec(\theta)$ given that $\cot(\theta) = 5$ and $\sin(\theta) < 0$

3. Suppose that $f$ is a one-to-one function and that the tangent line to the graph of $y = f(x)$ at $x = 3$ is $y = -4x + 5$. Find an equation of the tangent line to the graph of $y = f^{-1}(x)$ at $x = f(3)$.

4. Consider the one-to-one function $f(x) = 3xe^{x^2-4}$. Calculate $f(2)$ and find an equation of the tangent line to the graph of $y = f^{-1}(x)$ at $x = f(2)$.

5. Suppose that $f$ and $g$ are differentiable functions such that

$$\begin{array}{lll} f(-1) = 4, & f(0) = 2, & f(1) = 4, \\ f'(-1) = 3, & f'(0) = -5, & f'(1) = 8, \\ g(-1) = 2, & g(0) = 3, & g(1) = -2, \\ g'(-1) = 7, & g'(0) = -4, & g'(1) = 6. \end{array}$$

   (a) For $F(x) = \ln\left(f(x^2) + g(x)\right)$, evaluate $F'(-1)$.
   (b) For $G(x) = \arctan\left(3\sqrt{f(x)}\right)$, evaluate $G'(1)$.
   (c) For $H(x) = 2^{f(x)}g(3x+1)$, evaluate $H'(0)$.
   (d) **[Advanced]** For $K(x) = f(2x)^{g(x)}$, evaluate $K'(0)$.

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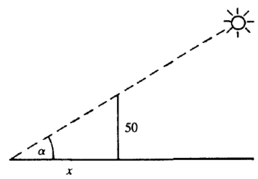
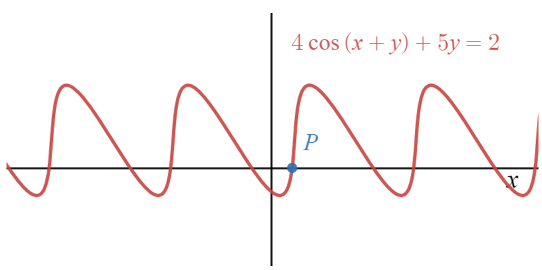
**Section 3.10: Related Rates - Worksheet**

1. How fast is the shadow cast on level ground by a pole 50 feet tall lengthening when the angle a of elevation of the sun is 45° and is decreasing by $\frac{1}{4}$ radian per hour? (See figure below.)




2. A sphere of radius 5 in fills with water at a rate of 4 $\text{in}^3$/min. When the water level inside the sphere is 6 in, how fast is it increasing? *(Hint: the volume of a spherical cap of height $h$ in a sphere of radius $r$ is $V = \frac{\pi}{3}(3rh^2 - h^3)$.)*

3. A particle travels toward the right on the graph of the implicit function $4\cos(x + y) + 5y = 2$, see the figure below.




When the particle first crosses the positive $x$-axis (at the point $P$ on the figure), its $x$-coordinate increases at 6 units/sec. At what rate is the $y$-coordinate of the particle changing at that time?

4. A 5-foot person is walking toward a 20-foot lamppost at the rate of 6 feet per second. How fast is the length of their shadow (cast by the lamp) changing?

5. The legs of an isosceles triangle of base 6 cm are increasing at a rate of 14 cm/hour, causing the vertex angle to decrease. When the legs are 4 cm, how fast is the vertex angle decreasing?

6. **[Advanced]** An object moves along the graph of a function $y = f(x)$. At a certain point, the slope of the graph is $-4$ and the $y$-coordinate of the object is increasing at the rate of 3 units per second. At that point, how fast is the $x$-coordinate of the object changing?

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## Section 3.11: Linear Approximations - Worksheet

1. Use a well-chosen linear approximation to estimate the following quantities.

(a) $\sqrt[3]{62}$

(b) $e^{-0.8}$

(c) $\sqrt{49.6}$

(d) $\ln(1+5\sin(0.06))$

(e) $\cot\left(\frac{\pi}{6}+0.02\right)-\sqrt{3}$

(f) $\sqrt[4]{17}-\sqrt[4]{16}$

2. Suppose that $f$ is a function such that $f(3)=-7$ and $f'(3)=2$. Use a linear approximation to estimate the following quantities.

(a) $f(3.07)$

(b) **[Advanced]** $f(1+\cos(0.1)+e^{0.2})$

3. Find the differential $dy$ of the following functions.

(a) $y=\arcsin(3x^2)$

(b) $y=4\sqrt[3]{x}-\dfrac{5}{x^2}+e^3$

(c) $y=\csc(5\theta)$

(d) $y=5^{3-t^2}$

(e) $y=x^{\cos(2x)}$

(f) $y=\sin(3e^{-7z})$

4. The volume of a sphere is computed by measuring its diameter.

(a) Suppose that the diameter of the sphere is measured at 5 cm with a precision of 0.2 cm. What is the percentage error propagated in the computation of the volume?

(b) **[Advanced]** Suppose that we want a measurement of the volume with an error of at most 1.5%. What is the maximum percentage error that can be made measuring the diameter?

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## Section 4.1: Extreme Values - Worksheet

1. Find the absolute extrema of the following functions on the given interval.

   (a) $f(x) = 2x^3 + 3x^2 - 12x + 1$ on $[-1, 2]$.

   (b) $f(x) = x(7 - x)^{2/5}$ on $[1, 6]$.

   (c) $f(x) = 3x^4 - 10x^3 + 6x^2 - 7$ on $[-2, 1]$.

   (d) $f(x) = (e^x - 2)^{4/7}$ on $[0, \ln(3)]$.

   (e) $f(x) = \dfrac{\ln(x)}{\sqrt{x}}$ on $[1, e^4]$.

   (f) [**Advanced**] $f(x) = 2\arctan(3x) - 3x$ on $\left[0, \frac{1}{\sqrt{3}}\right]$. *(Hint: use the approximations* $\pi \simeq 3.1$ *and* $\sqrt{3} \simeq 1.7$)*.)*

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**Sections 4.2-3: Mean Value Theorem and First Derivative Test - Worksheet**

1. Find the values of the constants $A, B$ for which the following function satisfies the assumptions of the Mean Value Theorem on the interval $[-2, 2]$.

$$f(x) = \begin{cases} e^{5x+B} & \text{if } x \geqslant 0 \\ \arctan(Ax+1) & \text{if } x < 0 \end{cases}$$

2. Suppose that $f$ is continuous on $[-2, 4]$, that $f(4) = 1$ and that $f'(x) \geqslant 3$ for $x$ in $(-2, 4)$. Find the largest possible value of $f(-2)$.

3. Find and classify the critical points of the following functions.

   (a) $f(x) = x^{4/7}(72 - x^2)$

   (b) $f(x) = x^5 \ln(x)$

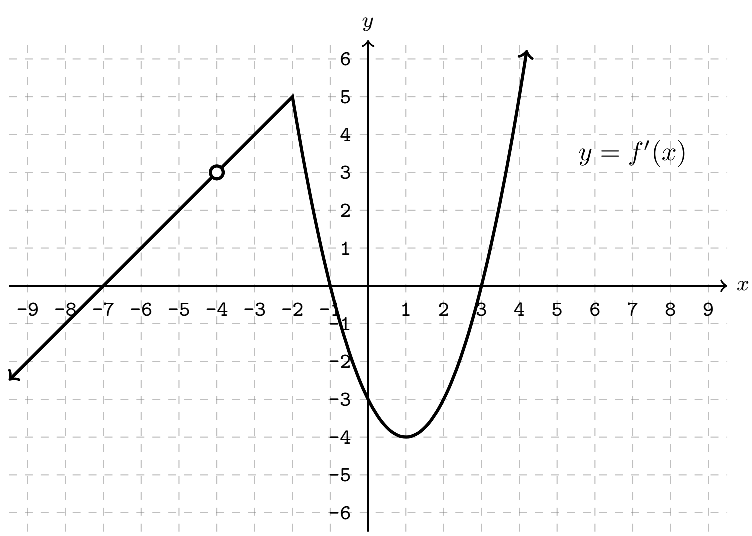
   (c) $f(x) = x + \cos(2x)$ on $\left[0, \dfrac{\pi}{2}\right]$

   (d) $f(x) = \sin^{-1}\left(e^{-x^2}\right)$

4. Suppose that $f$ is continuous on $(-\infty, \infty)$ and that $f'(x) = \dfrac{(x+3)(x-5)^2}{x^{2/3}(x-1)^{1/5}}$.

   (a) Find the critical points of $f$.
   (b) Find the intervals where $f$ is increasing and the intervals where $f$ is decreasing.
   (c) Find the location of the local extrema of $f$.

5. Suppose that $f$ is a differentiable function. The graph of the **derivative** of $f$, $y = f'(x)$, is sketched below.



(a) Find the critical points of $f$.

(b) Find the intervals where $f$ is increasing and the intervals where $f$ is decreasing.

(c) Find the location of the local extrema of $f$.

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**Sections 4.4: Concavity and Curve Sketching - Worksheet**

1. Find the intervals where the functions below are concave up, concave down and find the inflection points.

   (a) $f(x) = \dfrac{1}{x^2 + 12}$

   (b) $f(x) = x^4 e^{-3x}$

2. Sketch the graphs of the following functions. Your graph should clearly show any asymptotes, local extrema and inflection points of the functions.

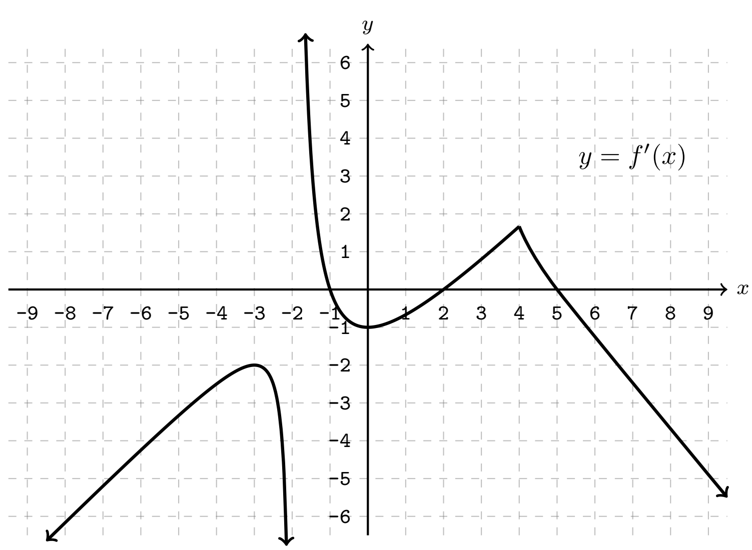
   (a) $f(x) = \frac{8}{x} - x^2$

   (b) $f(x) = \tan(2x) - 8x$ on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

3. Suppose that $f$ is continuous on $(-\infty, \infty)$, that $f'(x) = \dfrac{x}{(x+4)^{1/3}}$ and that $f''(x) = \dfrac{2x+12}{(x+4)^{4/3}}$.

   (a) Find the critical points of $f$.
   (b) Find the intervals where $f$ is increasing and the intervals where $f$ is decreasing.
   (c) Find the location of the local extrema of $f$.
   (d) Find the intervals where $f$ is concave up and the intervals where $f$ is concave down.
   (e) Find the $x$-coordinates of the inflection points of $f$.

4. Suppose that $f$ is a differentiable function. The graph of the **derivative** of $f$, $y = f'(x)$, is sketched below.



(a) Find the critical points of $f$.

(b) Find the intervals where $f$ is increasing and the intervals where $f$ is decreasing.

(c) Find the location of the local extrema of $f$.

(d) Find the intervals where $f$ is concave up and the intervals where $f$ is concave down.

(e) Find the $x$-coordinates of the inflection points of $f$.

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**Section 4.5: L'Hôpital's Rule - Worksheet**

1. Evaluate the following limits. **Note:** L'Hôpital's Rule is not possible/necessary for every limit.

(a) $\lim_{x\to 8} \dfrac{\sqrt[3]{x}-2}{64-x^2}$

(b) $\lim_{x\to\infty} \dfrac{\ln(x)^2}{\sqrt{x}}$

(c) $\lim_{x\to 0} \dfrac{5^x-3^x}{\sin(2x)}$

(d) $\lim_{\theta\to\frac{\pi}{2}} \dfrac{1-\csc(\theta)}{1-\sec(4\theta)}$

(e) $\lim_{x\to\infty} \ln(5x+1)-\ln(x)$

(f) $\lim_{x\to\infty} \left(1+\dfrac{2}{x}\right)^x$

(g) $\lim_{x\to 0} \dfrac{2^{\sin(x)}-1}{\sin^{-1}(5x)}$

(h) $\lim_{x\to-\infty} \dfrac{2x+3\cos(x)}{5x}$

(i) $\lim_{x\to\infty} x^{1/x}$

(j) $\lim_{x\to-\infty} x^3e^{5x+2}$

(k) $\lim_{x\to 0^+} \sqrt[3]{x}\log_2(x)$

(l) $\lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2+4}}$

(m) $\lim_{x\to 0} \cos(3x)^{1/x^2}$

(n) $\lim_{x\to\infty} \left(\dfrac{x+5}{x+3}\right)^{4x}$

(o) $\lim_{x\to\infty} x^{1/\ln(x+1)}$

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**Section 4.6: Optimization - Worksheet**

1. Farmer Brown wants to enclose rectangular pens for the animals on her farm. The three parts of this problem are independent.
   (a) Suppose that Farmer Brown wants to enclose a single pen alongside a river with 300 ft of fencing. The side of the pen alongside the river needs no fencing. What dimensions (length and width) would produce the pen with largest surface area?
   (b) Suppose that Farmer Brown has 360 ft of fencing to enclose 2 adjacent pens. Both pens have the same height, but the second one is twice as wide as the first. What is the largest total area that can be enclosed?
   (c) Suppose that Farmer Brown wants to enclose a total of 2,400 $\text{ft}^2$ in two adjacent pens having the same dimensions. What is the minimal amount of fencing needed?

2. A rectangular box has total surface area 216 $\text{in}^2$, and the length of its base is 4 times its width. Find the dimensions of such a box with largest volume.

3. A rectangular box is created by cutting equal size squares from the corners of a 10 in by 20 in cardboard rectangle and folding the sides. What size should the cut squares be for the resulting box to have the largest possible volume?

4. A rectangle has base on the $x$-axis and its two other vertices on the graph of $y = \frac{1}{25+x^2}$. Find the dimensions of such a rectangle with largest possible area.

5. A circular cone is created by cutting a circular sector from a disk of radius 9in and sealing the resulting open wedge together. What is the largest possible volume of such a cone?

6. The parts of this problem are independent.
   (a) Find the point on the line $2x + y = 5$ that is closest to the origin.
   (b) Find the point on the graph of $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

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**Section 4.8: Antiderivatives - Worksheet**

1. Evaluate the following antiderivatives.

   (a) $\displaystyle\int \frac{7}{1+x^2}dx$

   (b) $\displaystyle\int \frac{3}{\sqrt{16-x^2}}dx$

   (c) $\displaystyle\int (3x+1)\left(x^2-\frac{5}{x}\right)dx$

   (d) $\displaystyle\int \left(e^{5x}+\cos(1)\right)dx$

   (e) $\displaystyle\int \left(5\sqrt[7]{x^3}+\frac{4}{81+x^2}\right)dx$

   (f) $\displaystyle\int \csc(5\theta)\left(\sin(5\theta)-\cot(5\theta)\right)d\theta$

   (g) $\displaystyle\int \frac{7t-11}{\sqrt{t}}dt$

   (h) $\displaystyle\int \left(2^x-\frac{1}{7x}\right)dx$

   (i) $\displaystyle\int \frac{\tan(3x)+5\sec(3x)}{\cos(3x)}dx$

   (j) $\displaystyle\int \left(\frac{1}{z^{7/4}}-\frac{3}{36+z^2}\right)dz$

2. Solve the following initial value problems.

   (a) $\dfrac{dy}{dx}=2-7x$ and $y(2)=0$.

   (b) $\dfrac{dy}{dx}=x^{-6}+\dfrac{6}{x}$ and $y(1)=3$.

   (c) $\dfrac{dy}{dx}=\dfrac{5}{9+x^2}$ and $y(3)=-1$.

   (d) $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{64-x^2}}$ and $y(-4)=0$.

   (e) $\dfrac{d^2y}{dx^2}=3-e^{2x}$, $y'(0)=1$ and $y(0)=7$.

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**Sections 5.1-2: Areas Estimations and Riemann Sums - Worksheet**

1. (a) Approximate the net area between the graph of $f(x) = 9 - x^2$ and the $x$-axis on $[-1, 3]$ using 4 rectangles of equal width and (i) left endpoints, (ii) right endpoints.

   (b) Approximate the net area between the graph of $f(x) = 2\cos(x)$ and the $x$-axis on $\left[0, \frac{\pi}{2}\right]$ using 3 rectangles of equal width and (i) left endpoints, (ii) right endpoints.

2. Suppose that the function $f$ has the following values.

$$f(0) = 3,\ f(1) = 7,\ f(2) = 5,\ f(3) = 1,\ f(4) = 2,\ f(5) = 8,$$
$$f(6) = 0,\ f(7) = 1,\ f(8) = 5,\ f(9) = 3,\ f(10) = 1.$$

   Approximate the net area between the graph of $g(x) = f(8x + 2)$ and the $x$-axis on the interval $[0, 1]$ using a midpoint sum with 4 rectangles of equal width.

3. Evaluate the following sums.

   (a) $\displaystyle\sum_{k=0}^{5} \frac{k(k-1)}{2}$. (b) $\displaystyle\sum_{j=1}^{4} \cos(j\pi)j$. (c) $\displaystyle\sum_{n=1}^{5} \left(\frac{1}{n} - \frac{1}{n+1}\right)$.

4. Consider the sum $2 + 4 + 8 + 16 + 32 + 64$.

   (a) Write the sum in sigma notation with the index starting at the value 1.

   (b) Write the sum in sigma notation with the index starting at the value 0.

   (c) Write the sum in sigma notation with the index starting at the value 3.

5. Use the common sum formulas

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4},$$

   to evaluate the following sums.

   (a) $\displaystyle\sum_{k=1}^{136} (2k - 3)$. (b) $\displaystyle\sum_{j=2}^{20} j^2(j - 4)$.

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**Section 5.3: Definite Integrals - Worksheet**

1. Let $f(x) = 4 - 2x$. We are going to calculate $\int_0^2 f(x)dx$ using two methods.

   (a) Geometric method.

      (i) Sketch the graph of $y = f(x)$.

      (ii) Use your graph and a geometric formula to calculate $\int_0^2 f(x)dx$.

   (b) With Riemann sums.

      (i) Calculate $R_n$, the right-endpoint Riemann sum of $f$ on $[0, 2]$ with $n$ rectangles. Your answer should not contain the $\Sigma$ or $\cdots$ symbols. *Hint: you will need to use the reference sum*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

      (ii) Using your formula for $R_n$, calculate $\int_0^2 f(x)dx$.

2. Write each limit below as the integral of a function $f(x)$ on an interval $[0, b]$.

   (a) $\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{\frac{3k}{n} + 5}\, \frac{3}{n}$.

   (b) $\lim_{n\to\infty} \sum_{k=1}^{n} e^{12k/n} \frac{8}{n}$.

   (c) $\lim_{n\to\infty} \sum_{k=1}^{n} \sin\left(\frac{k^3}{n^3}\right) \frac{2}{n}$.

   (d) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2n + 5k}$.

3. Suppose that $f$ and $g$ are functions such that

$$\int_{-2}^{0} f(x)dx = 4, \quad \int_{-2}^{5} f(x)dx = -1, \quad \int_{-2}^{5} g(x)dx = 10.$$

   Evaluate the following integrals.

   (a) $\int_{-2}^{5} \frac{g(x)}{2} dx$

   (b) $\int_{-2}^{5} (2g(x) - 3f(x))dx$

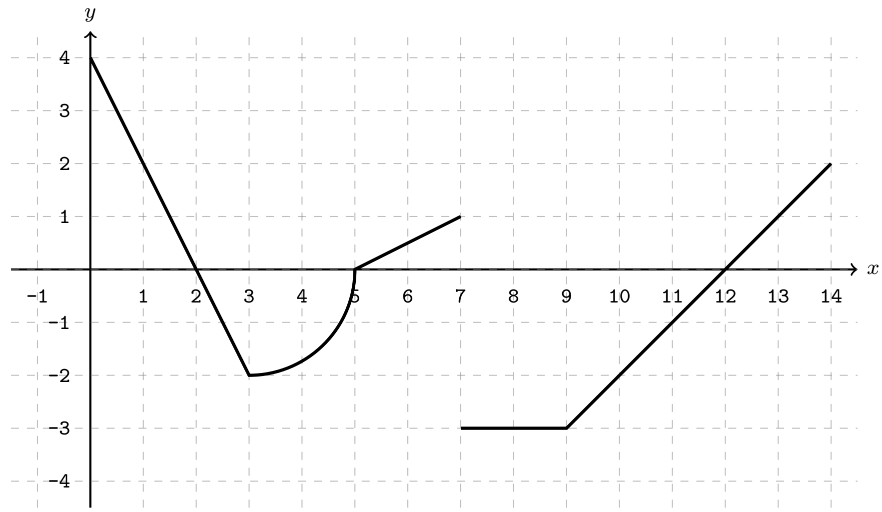
   (c) $\int_{0}^{5} 7f(x)dx$

   (d) $\int_{5}^{-2} (f(x) + 4g(x))\, dx$

   (e) $\int_{-2}^{0} (2x + f(x) - 1)\, dx$

   (f) $\int_{5}^{0} \left(f(x) - 4\sqrt{25 - x^2}\right) dx$

4. Let $f$ be the function whose graph is sketched below. You can assume that each piece of the graph of $f$ is either a straight line or a circle arc.




Calculate the following integrals.

(a) $\int_0^5 f(x)dx$ (b) $\int_3^9 (3 - f(x))dx$ (c) $\int_{12}^5 f(x)dx$ (d) $\int_7^{14} |f(x)|dx$

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**Section 5.4: Fundamental Theorem of Calculus - Worksheet**

1. Evaluate the following definite integrals.

(a) $\displaystyle\int_1^3 \frac{3x^2 - 2x + 1}{x} dx$

(b) $\displaystyle\int_0^{1/2} \frac{dt}{\sqrt{1-t^2}}$

(c) $\displaystyle\int_0^{\ln(2)} (e^x + 1)^2 \, dx$

(d) $\displaystyle\int_{\pi/30}^{\pi/20} \sec^2(5\theta) d\theta$

(e) $\displaystyle\int_{-3}^{\sqrt{3}} \frac{4}{x^2+3} dx$

(f) $\displaystyle\int_0^5 \frac{dz}{4z+7}$

(g) $\displaystyle\int_1^4 \sqrt{x}\left(x - \frac{4}{x}\right) dx$

(h) $\displaystyle\int_0^{2\pi} \left(\sin\left(\frac{x}{3}\right) + 1\right) d\theta$

(i) $\displaystyle\int_{\sqrt{2}}^2 \frac{5}{3x\sqrt{x^2-1}} dx$

2. Evaluate the following derivatives.

(a) $\displaystyle\frac{d}{dx}\left(\int_4^x \sqrt{t^4+1} dt\right)$

(b) $\displaystyle\frac{d}{dx}\left(\int_x^0 \sec(5t^2) dt\right)$

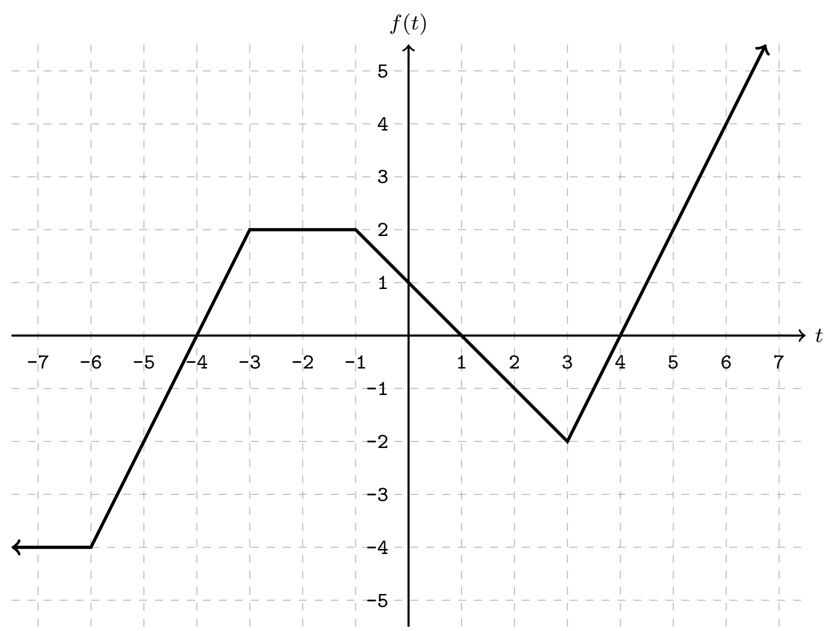
(c) $\displaystyle\frac{d}{dx}\left(\int_1^{2x} \frac{dt}{t^3+t+1}\right)$

(d) $\displaystyle\frac{d}{dx}\left(\int_{3x^2}^7 (t^4+2)^{3/4} dt\right)$

(e) $\displaystyle\frac{d}{dx}\left(\int_{\tan(2x)}^{\sec(2x)} \cos(\sqrt{t}) dt\right)$

(f) $\displaystyle\frac{d}{dx}\left(\int_0^{\sin^{-1}(3x)} t^t dt\right)$

3. For the function $f(t)$ sketched below, let $F(x) = \displaystyle\int_{-3}^x f(t) dt$.




(a) Evaluate the following.

(i) $F(3)$ (ii) $F(-6)$ (iii) $F'(-2)$ (iv) $F'(4)$

(b) Find an equation of the tangent line to the graph of $y = F(x)$ at $x = 6$.

(c) Find the critical points of $F$.

(d) Find the intervals on which $F$ is increasing and the intervals on which $F$ is decreasing.

(e) Find the $x$-values at which $F(x)$ has a local maximum or a local minimum.

(f) Find the intervals on which $F$ is concave up and the intervals on which $F$ is concave down.

(g) Find the $x$-values at which $F(x)$ has an inflection point.

4. Let $f(x) = 7 + \int_{13}^{x} t(t-14)^{2/5} dt$.

(a) Find an equation of the tangent line to the graph of $y = f(x)$ at $x = 13$.

(b) Find the critical points of $f$.

(c) Find the intervals on which $f$ is increasing and the intervals on which $F$ is decreasing.

(d) Find the $x$-values at which $f(x)$ has a local maximum or a local minimum.

(e) Find the intervals on which $f$ is concave up and the intervals on which $F$ is concave down.

(f) Find the $x$-values at which $f(x)$ has an inflection point.

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**Sections 5.5-6: Substitution Method - Worksheet**

1. Evaluate the following integrals.

(a) $\int (3x^4+6)\sec(x^5+10x)dx$

(b) $\int \frac{dx}{x\sqrt{3\ln(x)+5}}$

(c) $\int x^2\sqrt{x-1}dx$

(d) $\int x^3\sin(x^4+2)dx$

(e) $\int_0^1 \frac{x^3}{\sqrt{3+x^2}}dx$

(f) $\int t\sec^2\left(3t^2\right)e^{7\tan\left(3t^2\right)}dt$

(g) $\int e^x(e^x-2)^{2/3}dx$

(h) $\int e^{2x}(e^x-2)^{2/3}dx$

(i) $\int_{e^3}^{e^6} \frac{dt}{t\ln(t)}$

(j) $\int \frac{dx}{5x+4\sqrt{x}}$

(k) $\int \frac{dx}{\sqrt{2-x^2}}$

(l) $\int_0^1 \frac{xdx}{\sqrt{2-x^2}}$

(m) $\int_0^{2/3} \frac{dz}{4+9z^2}$

(n) $\int \frac{e^{4\arcsin(5x)}}{\sqrt{1-25x^2}}dx.$

(o) $\int_0^{\pi/10} \frac{\sin^3(5x)}{\cos(5x)+3}dx.$

(p) $\int \frac{\left(\tan^{-1}(t)\right)^3}{1+t^2}dt.$

(q) $\int_e^{e^2} \frac{dx}{x\sqrt{\ln(x)}}dx.$

(r) $\int \frac{\tan(3\ln(x))}{x}dx.$

(s) $\int \frac{x^3+1}{9+x^2}dx.$

(t) $\int_0^{\pi/12} \tan^2(3\theta)\sec^2(3\theta)d\theta.$

(u) $\int \frac{e^{3x}}{\sqrt{49-e^{6x}}}dx.$

(v) $\int_{-5/2}^{5/2} \frac{1+\sin(x)}{4x^2+25}dx.$

2. Suppose that $f$ is an **even** function such that

$$\int_{-9}^{5} f(x)dx = -13 \quad \text{and} \quad \int_0^9 f(x)dx = 4.$$

Evaluate the definite integrals below.

(a) $\int_{-9}^{9} f(x)dx$

(b) $\int_0^5 (4x-3f(x))dx$

(c) $\int_{-3}^{3} xf(x)dx$

(d) $\int_0^3 xf(x^2)dx$

3. Find the average value of the following functions on the given interval.

(a) $f(x) = \frac{3}{\sqrt{100-x^2}}$ on $[0,5]$.

(b) $f(x) = x\sqrt[3]{3x-7}$ on $[2,5]$.

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**Section 5.6: Areas Between Curves - Worksheet**

1. For each of the regions described below (i) sketch the region, clearly labeling the curves and their intersection points, (ii) calculate the area of the region using an $x$-integral and (iii) calculate the area of the region using a $y$-integral.

   (a) The region to the right of the parabola $y = 1 - (x-2)^2$, below the line $y = 1$ and to the left of the line $x - 2y = 3$.

   (b) The region bounded by the curves $y = 2x$ and $y = \sqrt[3]{32x}$.

   (c) The region bounded by the curves $y = \dfrac{4}{x+2}$ and $y = 3 - x$.

2. Calculate the area of the regions described below.

   (a) The region bounded by the parabola $x = (y+3)^2 - 4$ and the line $x = 3y + 9$.

   (b) The region bounded by $y = \dfrac{4}{3+x^2}$ and $y = 1$.

   (c) The region bounded by $y = 2\ln(x+1)$, the $x$-axis and the line $x = 4$.

   (d) The region to the right of the $y$-axis, above the graph of $y = \sec(x)^2$ and below the graph of $y = 2\sec(x)$.