Rutgers University
Math 151

Math 151 All Worksheets Solutions
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(Last updated: April 12, 2024)

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Chapter 1: Review of Algebra & Precalculus - Worksheet Solutions

1. **Composite functions:** recall that given two functions f and g, the function $f \circ g$ (called f *composed with* g) is

$$(f \circ g)(x) = f(g(x)).$$

(a) Given $f(x) = \sqrt{x}$ and $g(x) = (x-3)^2$, find and simplify the following.

i. $(f \circ g)(x)$

Solution.

$$(f \circ g)(x) = f(g(x)) = f((x-3)^2) = \sqrt{(x-3)^2} = \boxed{|x-3|}.$$

ii. $(g \circ f)(x)$

Solution.

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \boxed{(\sqrt{x}-3)^2}.$$

iii. $(f \circ f \circ f)(x)$

Solution.

$$(f \circ f \circ f)(x) = f(f(f(x))) = \sqrt{\sqrt{\sqrt{x}}} = \left(\left(x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = x^{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}} = \boxed{x^{\frac{1}{8}}}.$$

(b) Let $H(x) = \cos(3x^2) + 1$. Complete the table below to find pairs of functions $f(x)$ and $g(x)$ such that $H(x) = f(g(x))$.

	$f(x) =$	$g(x) =$
i.	$\cos(x) + 1$	
ii.		x^2
iii.		$\cos(3x^2)$
iv.	x	
v.		$\cos(3x^2) + 7$

Solution.

	$f(x) =$	$g(x) =$
i.	$\cos(x) + 1$	$3x^2$
ii.	$\cos(3x) + 1$	x^2
iii.	$x + 1$	$\cos(3x^2)$
iv.	x	$\cos(3x^2) + 1$
v.	$x - 6$	$\cos(3x^2) + 7$

2. **Trigonometry:**

(a) Suppose that $\cos(a) = \frac{2}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$. Evaluate the following.

i. $\tan(a)$

Solution. We start by using the Pythagorean identity $\cos(a)^2 + \sin(a)^2$ to find $\sin(a)$. We get

$$\begin{aligned}
\left(\frac{2}{5}\right)^2 + \sin(a)^2 &= 1 \\
\sin(a)^2 &= 1 - \frac{4}{25} = \frac{21}{25} \\
\sqrt{\sin(a)^2} &= \sqrt{\frac{21}{25}} \\
|\sin(a)| &= \frac{\sqrt{21}}{5} \\
\sin(a) &= \pm\frac{\sqrt{21}}{5}.
\end{aligned}$$

To determine which sign is appropriate, we use the fact that $\frac{3\pi}{2} < \theta < 2\pi$ (quadrant IV), which implies $\sin(a) < 0$. Therefore, $\sin(a) = -\frac{\sqrt{21}}{5}$. Now

$$\tan(a) = \frac{\sin(a)}{\cos(a)} = \frac{-\frac{\sqrt{21}}{5}}{\frac{2}{5}} = \boxed{-\frac{\sqrt{21}}{2}}.$$

ii. $\sin(2a)$

Solution. We use a double-angle identity to get

$$\sin(2a) = 2\sin(a)\cos(a) = 2\left(-\frac{\sqrt{21}}{5}\right)\frac{2}{5} = \boxed{-\frac{4\sqrt{21}}{25}}.$$

iii. $\cos(2a)$

Solution. We use a double-angle identity to get

$$\cos(2a) = 2\cos(a)^2 - 1 = 2\left(\frac{2}{5}\right)^2 - 1 = \frac{8}{25} - 1 = \boxed{-\frac{17}{25}}.$$

(b) Evaluate the following.

i. $\sec\left(\frac{4\pi}{3}\right)$

ii. $\tan^{-1}(1)$

iii. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

iv. $\csc^{-1}(2)$

v. $\sin\left(\sin^{-1}(0.8)\right)$

vi. $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$

Solution.

i. $\sec\left(\frac{4\pi}{3}\right) = \boxed{-2}$

ii. $\tan^{-1}(1) = \boxed{\dfrac{\pi}{4}}$

iii. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{\dfrac{5\pi}{6}}$

iv. $\csc^{-1}(2) = \boxed{\dfrac{\pi}{6}}$

v. $\sin\left(\sin^{-1}(0.8)\right) = \boxed{0.8}$

vi. $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right) = \boxed{-\dfrac{\pi}{4}}$

(c) Simplify the following. Your answers should be algebraic expressions of x (not involving any trigonometric or inverse trigonometric functions).

i. $\cos\left(\cos^{-1}(x)\right)$

Solution. By definition of the inverse function, $\boxed{\cos\left(\cos^{-1}(x)\right) = x}$.

ii. $\cos\left(\sin^{-1}(x)\right)$

Solution. We use the Pythagorean identity $\cos(\theta)^2+\sin(\theta)^2 = 1$ with $\theta = \sin^{-1}(x)$. By definition of $\sin^{-1}$, we know that $\sin(\theta) = x$ and θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We get

$$\begin{aligned}
\cos(\theta)^2 + x^2 &= 1 \\
\cos(\theta)^2 &= 1 - x^2 \\
\sqrt{\cos(\theta)^2} &= \sqrt{1 - x^2} \\
|\cos(\theta)| &= \sqrt{1 - x^2} \\
\cos(\theta) &= \pm\sqrt{1 - x^2}
\end{aligned}$$

To determine which sign is appropriate, recall that $\theta = \sin^{-1}(x)$ is an angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\cos(\theta) \geqslant 0$. Hence

$$\boxed{\cos(\sin^{-1}(x)) = \sqrt{1 - x^2}}.$$

iii. $\sin\left(\cos^{-1}(x)\right)$

Solution. We use the Pythagorean identity $\cos(\theta)^2+\sin(\theta)^2 = 1$ with $\theta = \cos^{-1}(x)$. By definition of $\cos^{-1}$, we know that $\cos(\theta) = x$ and θ is in $[0, \pi]$. We get

$$\begin{aligned}
x^2 + \sin(\theta)^2 &= 1 \\
\sin(\theta)^2 &= 1 - x^2 \\
\sqrt{\sin(\theta)^2} &= \sqrt{1 - x^2} \\
|\sin(\theta)| &= \sqrt{1 - x^2} \\
\sin(\theta) &= \pm\sqrt{1 - x^2}
\end{aligned}$$

To determine which sign is appropriate, recall that $\theta = \cos^{-1}(x)$ is an angle in $[0, \pi]$, so $\sin(\theta) \geqslant 0$. Hence

$$\boxed{\sin(\cos^{-1}(x)) = \sqrt{1 - x^2}}.$$

iv. $\sec\left(\tan^{-1}(4x)\right)$

Solution. We use the Pythagorean identity $1+\tan(\theta)^2 = \sec(\theta)^2$ with $\theta = \tan^{-1}(4x)$. By definition of $\tan^{-1}$, we know that $\tan(\theta) = 4x$ and θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We get

$$\begin{aligned}
\sec(\theta)^2 &= 1+(4x)^2 = 1+16x^2 \\
\sqrt{\sec(\theta)^2} &= \sqrt{1+16x^2} \\
|\sec(\theta)| &= \sqrt{1+16x^2} \\
\sec(\theta) &= \pm\sqrt{1+16x^2}
\end{aligned}$$

To determine which sign is appropriate, recall that $\theta = \tan^{-1}(x)$ is an angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\sec(\theta) > 0$. Hence

$$\boxed{\sec(\tan^{-1}(4x)) = \sqrt{1+16x^2}}.$$

v. $\tan\left(\cos^{-1}\left(\frac{x}{2}\right)\right)$

Solution. We start by using the Pythagorean identity $\cos(\theta)^2 + \sin(\theta)^2 = 1$ with $\theta = \cos^{-1}\left(\frac{x}{2}\right)$ to find $\sin(\theta)$. By definition of $\cos^{-1}$, we know that $\cos(\theta) = \frac{x}{2}$ and θ is in $[0,\pi]$. We get

$$\begin{aligned}
\left(\frac{x}{2}\right)^2 + \sin(\theta)^2 &= 1 \\
\sin(\theta)^2 &= 1 - \frac{x^2}{4} = \frac{4-x^2}{4} \\
\sqrt{\sin(\theta)^2} &= \sqrt{\frac{4-x^2}{4}} \\
|\sin(\theta)| &= \frac{\sqrt{4-x^2}}{2} \\
\sin(\theta) &= \pm\frac{\sqrt{4-x^2}}{2}
\end{aligned}$$

To determine which sign is appropriate, recall that $\theta = \cos^{-1}\left(\frac{x}{2}\right)$ is an angle in $[0,\pi]$, so $\sin(\theta) \geqslant 0$. Hence $\sin(\theta) = \frac{\sqrt{4-x^2}}{2}$ and

$$\tan\left(\cos^{-1}\left(\frac{x}{2}\right)\right) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{\sqrt{4-x^2}}{2}}{\frac{x}{2}} \boxed{\frac{\sqrt{4-x^2}}{x}}.$$

vi. $\csc\left(\cot^{-1}\left(\frac{3x}{5}\right)\right)$

Solution. We use the Pythagorean identity $1+\cot(\theta)^2 = \csc(\theta)^2$ with $\theta = \cot^{-1}\left(\frac{3x}{5}\right)$. By definition of $\cot^{-1}$, we know that $\cot(\theta) = \frac{3x}{5}$ and θ is in $(0,\pi)$. We get

$$\begin{aligned}
\csc(\theta)^2 &= 1 + \left(\frac{3x}{5}\right)^2 = 1 + \frac{9x^2}{25} = \frac{25+9x^2}{25} \\
\sqrt{\csc(\theta)^2} &= \sqrt{\frac{25+9x^2}{25}} \\
|\csc(\theta)| &= \frac{\sqrt{25+9x^2}}{5}
\end{aligned}$$

$$\csc(\theta) = \pm\frac{\sqrt{25+9x^2}}{5}$$

To determine which sign is appropriate, recall that $\theta = \cot^{-1}\left(\frac{3x}{5}\right)$ is an angle in $(0,\pi)$, so $\csc(\theta) > 0$. Hence

$$\boxed{\csc\left(\cot^{-1}\left(\frac{3x}{5}\right)\right) = \frac{\sqrt{25+9x^2}}{5}}.$$

3. **Exponential and Logarithmic Functions:**

(a) Evaluate the following.

i. $e^{\ln(75)-2\ln(5)}$

Solution.

$$e^{\ln(75)-2\ln(5)} = e^{\ln\left(\frac{75}{5^2}\right)} = \frac{75}{5^2} = \boxed{3}.$$

ii. $\log_{\frac{1}{2}}(32)$

Solution. Recall that by definition of logarithms, $\log_{\frac{1}{2}}(32)$ is the exponent to which the base $\frac{1}{2}$ must be raised to obtain 32. Therefore $\boxed{\log_{\frac{1}{2}}(32) = -5}$.

iii. $\ln(9e^2) + \ln(\sqrt{9e}) - \ln(27e^{1/3})$

Solution.

$$\ln(9e^2)+\ln(\sqrt{9e})-\ln(27e^{1/3}) = \ln\left(\frac{9e^2\sqrt{9e}}{27e^{1/3}}\right) = \ln\left(\frac{e^2e^{1/2}}{e^{1/3}}\right) = \ln\left(e^{2+1/2-1/3}\right) = \ln\left(e^{13/6}\right) = \boxed{\frac{13}{6}}.$$

(b) Solve the following equations.

i. $2^{5x-1} = 4^{-3x}$

Solution.

$$\begin{aligned}
2^{5x-1} &= 4^{-3x} \\
2^{5x-1} &= \left(2^2\right)^{-3x} \\
2^{5x-1} &= 2^{-6x} \\
5x-1 &= -6x \\
11x &= 1 \\
&\boxed{x = \frac{1}{11}}.
\end{aligned}$$

ii. $\log_4(x+5) - \log_4(x) = 2$

Solution.

$$\log_4(x+5) - \log_4(x) = 2$$

$$\begin{aligned}
&\log_4\left(\frac{x+5}{x}\right) = 2\\
&\frac{x+5}{x} = 4^2\\
&x+5 = 16x\\
&15x = 5\\
&\boxed{x = \frac{1}{3}}
\end{aligned}$$

iii. $e^{2x} - 3e^x - 10 = 0$

Solution.

$$\begin{aligned}
&e^{2x} - 3e^x - 10 = 0\\
&(e^x)^2 - 3e^x - 10 = 0\\
&(e^x - 5)(e^x + 2) = 0\\
&e^x = 5 \text{ or } e^x = -2\\
&\boxed{x = \ln(5)}
\end{aligned}$$

4. **Inverse Functions:** each function below is one-to-one. Find the inverse function.

(a) $f(x) = (x+8)^{7/4}$

Solution.

$$\begin{aligned}
&f(y) = x\\
&(y+8)^{7/4} = x\\
&\left((y+8)^{7/4}\right)^{4/7} = x^{4/7}\\
&y+8 = x^{4/7}\\
&y = x^{4/7} - 8\\
&\boxed{f^{-1}(x) = x^{4/7} - 8}
\end{aligned}$$

(b) $f(x) = \dfrac{3-2x}{4x+7}$

Solution.

$$\begin{aligned}
&f(y) = x\\
&\frac{3-2y}{4y+7} = x\\
&3-2y = x(4y+7)\\
&3-2y = 4xy + 7x
\end{aligned}$$

$$
\begin{aligned}
4xy + 2y &= 3 - 7x \\
y(4x + 2) &= 3 - 7x \\
y &= \frac{3 - 7x}{4x + 2} \\
\Aboxed{f^{-1}(x) &= \frac{3 - 7x}{4x + 2}}
\end{aligned}
$$

(c) $f(x) = 5 + 2e^{3x+1}$

Solution.

$$
\begin{aligned}
f(y) &= x \\
5 + 2e^{3y+1} &= x \\
e^{3y+1} &= \frac{x - 5}{2} \\
3y + 1 &= \ln\left(\frac{x - 5}{2}\right) \\
y &= \frac{1}{3}\left(\ln\left(\frac{x - 5}{2}\right) - 1\right) \\
\Aboxed{f^{-1}(x) &= \frac{1}{3}\left(\ln\left(\frac{x - 5}{2}\right) - 1\right)}
\end{aligned}
$$

(d) $f(x) = 1 - \arcsin(x^3)$

Solution.

$$
\begin{aligned}
f(y) &= x \\
1 - \arcsin(y^3) &= x \\
\arcsin(y^3) &= 1 - x \\
y^3 &= \sin(1 - x) \\
y &= \sqrt[3]{\sin(1 - x)} \\
\Aboxed{f^{-1}(x) &= \sqrt[3]{\sin(1 - x)}}
\end{aligned}
$$

(e) $f(x) = \ln(x) - \ln(x - 3)$

Solution.

$$
\begin{aligned}
f(y) &= x \\
\ln(y) - \ln(y - 3) &= x \\
\ln\left(\frac{y}{y - 3}\right) &= x \\
\frac{y}{y - 3} &= e^x
\end{aligned}
$$

$$y = e^x(y-3)$$
$$y = e^x y - 3e^x$$
$$e^x y - y = 3e^x$$
$$y(e^x - 1) = 3e^x$$
$$y = \frac{3e^x}{e^x - 1}$$
$$\boxed{f^{-1}(x) = \frac{3e^x}{e^x - 1}}$$

(f) $f(x) = \dfrac{2^x}{2^x + 3}$

Solution.

$$f(y) = x$$
$$\frac{2^y}{2^y + 3} = x$$
$$2^y = x(2^y + 3)$$
$$2^y = x2^y + 3x$$
$$2^y - x2^y = 3x$$
$$2^y(1 - x) = 3x$$
$$2^y = \frac{3x}{1 - x}$$
$$y = \log_2\left(\frac{3x}{1 - x}\right)$$
$$\boxed{f^{-1}(x) = \log_2\left(\frac{3x}{1 - x}\right)}$$

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Section 2.1: Introduction to Limits - Worksheet Solutions

1. Calculate the average rate of change of the following functions on the given intervals.

(a) $f(x) = 2\ln(5x+1)$ on the interval $[0,3]$.

Solution.

$$\begin{aligned}\frac{\Delta f}{\Delta x} &= \frac{f(3)-f(0)}{3-0}\\ &= \frac{2\ln(16)-2\ln(1)}{3}\\ &= \boxed{\frac{8\ln(2)}{3}}.\end{aligned}$$

(b) $f(x) = \sin(4x)$ on the interval $\left[\frac{\pi}{24}, \frac{\pi}{12}\right]$.

Solution.

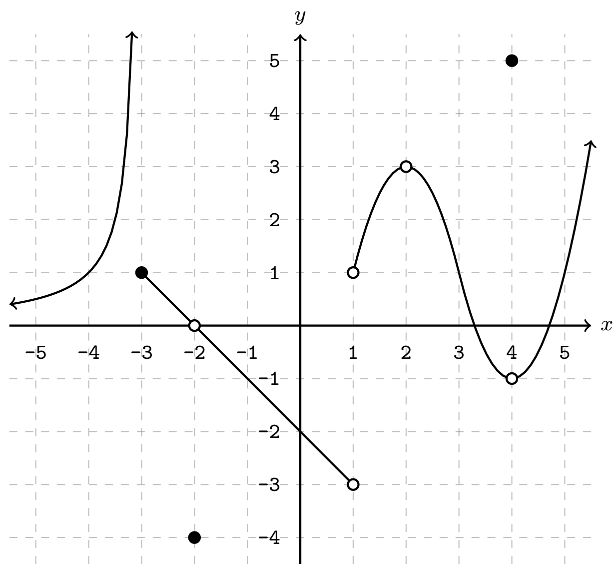
$$\begin{aligned}\frac{\Delta f}{\Delta x} &= \frac{f\left(\frac{\pi}{12}\right)-f\left(\frac{\pi}{24}\right)}{\frac{\pi}{12}-\frac{\pi}{24}}\\ &= \frac{\sin\left(\frac{\pi}{3}\right)-\sin\left(\frac{\pi}{6}\right)}{\frac{\pi}{24}}\\ &= \frac{\frac{\sqrt{3}}{2}-\frac{1}{2}}{\frac{\pi}{24}}\\ &= \boxed{\frac{12\left(\sqrt{3}-1\right)}{\pi}}.\end{aligned}$$

(c) $f(x) = \arctan(3x)$ on the interval $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

Solution.

$$\begin{aligned}\frac{\Delta f}{\Delta x} &= \frac{f\left(\frac{1}{3}\right)-f\left(-\frac{1}{3}\right)}{\frac{1}{3}-\left(-\frac{1}{3}\right)}\\ &= \frac{\arctan(1)-\arctan(-1)}{\frac{2}{3}}\\ &= \frac{\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)}{\frac{2}{3}}\\ &= \boxed{\frac{3\pi}{4}}.\end{aligned}$$

2. The graph of the function $y = f(x)$ is given below.



Evaluate $f(a)$ and $\lim_{x \to a} f(x)$ for the following values of a, or say if the quantity does not exist.

(a) $a = -3$

Solution. $\boxed{f(-3) = 1}$ and $\boxed{\lim_{x \to -3} f(x) \text{ does not exist}}$.

(b) $a = -2$

Solution. $\boxed{f(-2) = -4}$ and $\boxed{\lim_{x \to -2} f(x) = 0}$.

(c) $a = 1$

Solution. $\boxed{f(1) \text{ is undefined}}$ and $\boxed{\lim_{x \to 1} f(x) \text{ does not exist}}$.

(d) $a = 2$

Solution. $\boxed{f(2) \text{ is undefined}}$ and $\boxed{\lim_{x \to 2} f(x) = 3}$.

(e) $a = 4$

Solution. $\boxed{f(4) = 5}$ and $\boxed{\lim_{x \to 5} f(x) = -1}$.

3. The following table of values are given for the functions $f(x)$ and $g(x)$. Use these to estimate $\lim_{x \to 3} f(x)$ and $\lim_{x \to 3} g(x)$ or say if a limit does not exist.

x	2.9	3.01	2.999	3.0001	2.99999
$f(x)$	4.15	3.95	4.05	3.9993	4.0005
$g(x)$	7.98	1.001	7.997	1.0002	7.99992

Solution. $\boxed{\lim_{x\to 3} f(x) = 4}$ and $\boxed{\lim_{x\to 3} g(x) \text{ does not exist}}$.

4. Using a limit of average rates of change, find the instantaneous rate of change of the following functions at the given value of x.

   (a) $f(x) = x^2 - 3x + 7$ at $x = 0$.

   Solution.

$$\begin{aligned}\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0} \frac{h^2 - 3h + 7 - 7}{h} \\ &= \lim_{h\to 0} \frac{h^2 - 3h}{h} \\ &= \lim_{h\to 0} h - 3 \\ &= \boxed{-3}.\end{aligned}$$

   (b) $f(x) = \dfrac{x}{5-x}$ at $x = -1$.

   Solution.

$$\begin{aligned}\lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} &= \lim_{h\to 0} \frac{\frac{-1+h}{5-(-1+h)} - \left(-\frac{1}{6}\right)}{h} \\ &= \lim_{h\to 0} \frac{\frac{-1+h}{6-h} + \frac{1}{6}}{h} \\ &= \lim_{h\to 0} \frac{6(-1+h) + 6 - h}{6h(6-h)} \\ &= \lim_{h\to 0} \frac{5h}{6h(6-h)} \\ &= \lim_{h\to 0} \frac{5}{6(6-h)} \\ &= \boxed{\frac{5}{36}}.\end{aligned}$$

   (c) **[Advanced]** $f(x) = \dfrac{1}{\sqrt{2x+1}}$ at $x = 4$.

   Solution.

$$\begin{aligned}\lim_{h\to 0} \frac{f(4+h) - f(4)}{h} &= \lim_{h\to 0} \frac{\frac{1}{\sqrt{9+2h}} - \frac{1}{3}}{h} \\ &= \lim_{h\to 0} \frac{3 - \sqrt{9+2h}}{3h\sqrt{9+2h}} \cdot \frac{3 + \sqrt{9+2h}}{3 + \sqrt{9+2h}}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{9-(9+2h)}{3h\sqrt{9+2h}\left(3+\sqrt{9+2h}\right)} \\
&= \lim_{h\to 0} \frac{-2h}{3h\sqrt{9+2h}\left(3+\sqrt{9+2h}\right)} \\
&= \lim_{h\to 0} \frac{-2}{3\sqrt{9+2h}\left(3+\sqrt{9+2h}\right)} \\
&= \frac{-2}{3\sqrt{9+0}\left(3+\sqrt{9+0}\right)} \\
&= \boxed{-\frac{1}{27}}.
\end{aligned}
$$

5. The position of an object moving along an axis is given by the function $s(t) = 6\sqrt{x+1}$.

   (a) Find the average velocity of the object between $t = 0$ and $t = 15$.

   Solution.

$$
\begin{aligned}
\frac{\Delta s}{\Delta t} &= \frac{s(15)-s(0)}{15-0} \\
&= \frac{6\sqrt{15+1}-6\sqrt{0+1}}{15} \\
&= \frac{24-6}{15} \\
&= \frac{18}{15} \\
&= \boxed{\frac{6}{5}}.
\end{aligned}
$$

   (b) Find the position and instantaneous velocity of the object at $t = 3$.

   Solution. At $t = 3$, the position is

$$
s(3) = 6\sqrt{3+1} = \boxed{12}.
$$

   The velocity is

$$
\begin{aligned}
v(3) &= \lim_{h\to 0} \frac{s(3+h)-s(3)}{h} \\
&= \lim_{h\to 0} \frac{6\sqrt{4+h}-12}{h} \\
&= \lim_{h\to 0} 6\frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} \\
&= \lim_{h\to 0} 6\frac{4+h-4}{h\left(\sqrt{4+h}+2\right)} \\
&= \lim_{h\to 0} 6\frac{h}{h\left(\sqrt{4+h}+2\right)} \\
&= \lim_{h\to 0} 6\frac{1}{\sqrt{4+h}+2}
\end{aligned}
$$

$$= 6\frac{1}{\sqrt{4+0}+2}$$

$$= \boxed{\frac{3}{2}}.$$

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Section 2.2: Calculating Limits - Worksheet Solutions

1. Evaluate the following limits. If a limit does not exist, explain why.

(a) $\lim_{x\to 3} \dfrac{x^2 - x - 6}{9x^{-2} - 1}$.

Solution.

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2 - x - 6}{9x^{-2} - 1} &= \lim_{x\to 3} \frac{(x-3)(x+2)}{\frac{9}{x^2} - 1} \\
&= \lim_{x\to 3} \frac{(x-3)(x+2)}{\frac{9-x^2}{x^2}} \\
&= \lim_{x\to 3} \frac{(x-3)(x+2)x^2}{(3-x)(3+x)} \\
&= \lim_{x\to 3} -\frac{(x+2)x^2}{3+x} \\
&= -\frac{(3+2)3^2}{3+3} \\
&= \boxed{-\frac{15}{2}}.
\end{aligned}$$

(b) $\lim_{t\to 2} \dfrac{\sqrt{t^2+12} - 2t}{2-t}$.

Solution.

$$\begin{aligned}
\lim_{t\to 2} \frac{\sqrt{t^2+12} - 2t}{2-t} &= \lim_{t\to 2} \frac{\sqrt{t^2+12} - 2t}{2-t} \cdot \frac{\sqrt{t^2+12} + 2t}{\sqrt{t^2+12} + 2t} \\
&= \lim_{t\to 2} \frac{\left(\sqrt{t^2+12}\right)^2 - (2t)^2}{(2-t)\left(\sqrt{t^2+12} + 2t\right)} \\
&= \lim_{t\to 2} \frac{t^2 + 12 - 4t^2}{(2-t)\left(\sqrt{t^2+12} + 2t\right)} \\
&= \lim_{t\to 2} \frac{12 - 3t^2}{(2-t)\left(\sqrt{t^2+12} + 2t\right)} \\
&= \lim_{t\to 2} \frac{3(4 - t^2)}{(2-t)\left(\sqrt{t^2+12} + 2t\right)} \\
&= \lim_{t\to 2} \frac{3(2-t)(2+t)}{(2-t)\left(\sqrt{t^2+12} + 2t\right)} \\
&= \lim_{t\to 2} \frac{3(2+t)}{\sqrt{t^2+12} + 2t}
\end{aligned}$$

$$= \frac{3(2+2)}{\sqrt{2^2+12}+4}$$

$$= \boxed{\frac{3}{2}}.$$

(c) $\lim_{y \to 0} y \cot(5y)$.

Solution

$$\begin{aligned}
\lim_{y \to 0} y \cot(5y) &= \lim_{y \to 0} \frac{y \cos(5y)}{\sin(5y)} \cdot \frac{5y}{5y} \\
&= \lim_{y \to 0} \frac{y \cos(5y)}{5y} \cdot \frac{5y}{\sin(5y)} \\
&= \left(\lim_{y \to 0} \frac{\cos(5y)}{5} \right) \left(\lim_{y \to 0} \frac{5y}{\sin(5y)} \right) \\
&= \frac{\cos(0)}{5} \cdot 1 \\
&= \boxed{\frac{1}{5}}.
\end{aligned}$$

(d) $\lim_{x \to 0} \frac{(x-2)^3 + 8 - 12x}{x^2}$.

Solution. Recall that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Using this, we get

$$\begin{aligned}
\lim_{x \to 0} \frac{(x-2)^3 + 8 - 12x}{x^2} &= \lim_{x \to 0} \frac{x^3 - 6x^2 + 12x - 8 + 8 - 12x}{x^2} \\
&= \lim_{x \to 0} \frac{x^3 - 6x^2}{x^2} \\
&= \lim_{x \to 0} x - 6 \\
&= \boxed{-6}.
\end{aligned}$$

(e) $\lim_{u \to 4} \frac{u-4}{\sqrt{2u+1} - \sqrt{u+5}}$.

Solution.

$$\begin{aligned}
\lim_{u \to 4} \frac{u-4}{\sqrt{2u+1} - \sqrt{u+5}} &= \lim_{u \to 4} \frac{u-4}{\sqrt{2u+1} - \sqrt{u+5}} \cdot \frac{\sqrt{2u+1} + \sqrt{u+5}}{\sqrt{2u+1} + \sqrt{u+5}} \\
&= \lim_{u \to 4} \frac{(u-4)\left(\sqrt{2u+1} + \sqrt{u+5}\right)}{(\sqrt{2u+1})^2 - (\sqrt{u+5})^2} \\
&= \lim_{u \to 4} \frac{(u-4)\left(\sqrt{2u+1} + \sqrt{u+5}\right)}{2u+1-(u+5)} \\
&= \lim_{u \to 4} \frac{(u-4)\left(\sqrt{2u+1} + \sqrt{u+5}\right)}{u-4}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{u\to 4} \sqrt{2u+1} + \sqrt{u+5} \\
&= \sqrt{2\cdot 4+1} + \sqrt{4+5} \\
&= \boxed{6}.
\end{aligned}
$$

(f) $\displaystyle\lim_{x\to 0} \frac{\sin^2(4x)}{x\sin(3x)}$.

Solution.

$$
\begin{aligned}
\lim_{x\to 0} \frac{\sin^2(4x)}{x\sin(3x)} &= \lim_{x\to 0} \frac{\sin^2(4x)}{x\sin(3x)} \cdot \frac{(4x)^2}{(4x)^2} \cdot \frac{3x}{3x} \\
&= \lim_{x\to 0} \left(\frac{\sin(4x)}{4x}\right)^2 \cdot \frac{3x}{\sin(3x)} \cdot \frac{(4x)^2}{x(3x)} \\
&= \left(\lim_{x\to 0} \frac{\sin(4x)}{4x}\right)^2 \left(\lim_{x\to 0} \frac{3x}{\sin(3x)}\right) \left(\lim_{x\to 0} \frac{16}{3}\right) \\
&= 1^2 \cdot 1 \cdot \frac{16}{3} \\
&= \boxed{\frac{16}{3}}.
\end{aligned}
$$

(g) $\displaystyle\lim_{x\to 1} \frac{\sqrt{4x^2+7} - \sqrt{x+10}}{x-1}$

Solution. Direct substitution gives $\frac{0}{0}$, so we rewrite the expression by rationalizing the numerator and canceling out common factors.

$$
\begin{aligned}
\lim_{x\to 1} \frac{\sqrt{4x^2+7} - \sqrt{x+10}}{x-1} &= \lim_{x\to 1} \frac{\sqrt{4x^2+7} - \sqrt{x+10}}{x-1} \cdot \frac{\sqrt{4x^2+7} + \sqrt{x+10}}{\sqrt{4x^2+7} + \sqrt{x+10}} \\
&= \lim_{x\to 1} \frac{4x^2+7-(x+10)}{(x-1)(\sqrt{4x^2+7} + \sqrt{x+10})} \\
&= \lim_{x\to 1} \frac{4x^2-x-3}{(x-1)(\sqrt{4x^2+7} + \sqrt{x+10})} \\
&= \lim_{x\to 1} \frac{(x-1)(4x+3)}{(x-1)(\sqrt{4x^2+7} + \sqrt{x+10})} \\
&= \lim_{x\to 1} \frac{4x+3}{\sqrt{4x^2+7} + \sqrt{x+10}} \\
&= \boxed{\frac{7}{2\sqrt{11}}}.
\end{aligned}
$$

(h) $\displaystyle\lim_{h\to 0} \frac{\frac{6}{3+7h} - 2}{h}$

Solution. Direct substitution gives $\frac{0}{0}$, so we rewrite the expression as a simple fraction and cancel the common factors.

$$\begin{aligned}
\lim_{h\to 0} \frac{\frac{6}{3+7h} - 2}{h} &= \lim_{h\to 0} \frac{\frac{6-2(3+7h)}{3+7h}}{h} \\
&= \lim_{h\to 0} \frac{-14h}{h(3+7h)} \\
&= \lim_{h\to 0} \frac{-14}{3+7h} \\
&= \boxed{-\frac{14}{3}}.
\end{aligned}$$

(i) $\displaystyle\lim_{x\to 0} \frac{x\sin(5x)}{\tan^2(3x)}$

Solution.

$$\begin{aligned}
\lim_{x\to 0} \frac{x\sin(5x)}{\tan^2(3x)} &= \lim_{x\to 0} \frac{x\sin(5x)\cos^2(3x)}{\sin^2(3x)} \cdot \frac{5x}{5x} \cdot \frac{(3x)^2}{(3x)^2} \\
&= \lim_{x\to 0} \frac{\sin(5x)}{5x} \cdot \left(\frac{3x}{\sin(3x)}\right)^2 \cdot \frac{x(5x)\cos^2(3x)}{(3x)^2} \\
&= \left(\lim_{x\to 0} \frac{\sin(5x)}{5x}\right) \cdot \left(\lim_{x\to 0} \frac{3x}{\sin(3x)}\right)^2 \cdot \left(\lim_{x\to 0} \frac{5\cos^2(3x)}{9}\right) \\
&= 1 \cdot 1^2 \cdot \frac{5\cos^2(0)}{9} \\
&= \boxed{\frac{5}{9}}.
\end{aligned}$$

[Advanced]

(j) $\displaystyle\lim_{\theta\to 0} \frac{\sin(3\theta)^2}{\cos(5\theta) - 1}$.

Solution.

$$\begin{aligned}
\lim_{\theta\to 0} \frac{\sin(3\theta)^2}{\cos(5\theta) - 1} &= \lim_{\theta\to 0} \frac{\sin(3\theta)^2}{\cos(5\theta) - 1} \cdot \frac{(3\theta)^2}{(3\theta)^2} \cdot \frac{(5\theta)^2}{(5\theta)^2} \\
&= \lim_{\theta\to 0} \left(\frac{\sin(3\theta)}{3\theta}\right)^2 \cdot \frac{(5\theta)^2}{\cos(5\theta) - 1} \cdot \frac{(3\theta)^2}{(5\theta)^2} \\
&= \left(\lim_{\theta\to 0} \frac{\sin(3\theta)}{3\theta}\right)^2 \left(\lim_{\theta\to 0} -\frac{(5\theta)^2}{1-\cos(5\theta)}\right) \left(\lim_{\theta\to 0} \frac{9}{25}\right) \\
&= 1^2\,(-2)\,\frac{9}{25} \\
&= \boxed{-\frac{18}{25}}.
\end{aligned}$$

(k) $\lim_{x\to 0} x\sin(\ln|x|)$.

Solution. We use the Squeeze Theorem. For any $x \neq 0$, we have

$$-1 \leqslant \sin(\ln|x|) \leqslant 1,$$

so

$$-|x| \leqslant x\sin(\ln|x|) \leqslant |x|.$$

Since $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$, we conclude that

$$\boxed{\lim_{x\to 0} x\sin(\ln|x|) = 0}.$$

(l) $\lim_{h\to 1} \dfrac{\sqrt[3]{h}-1}{h-1}$.

Solution. We would like to rationalize the numerator to be able to simplify the h in the denominator. To this end, we will want to use the identity

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

with $a = \sqrt[3]{h}$ and $b = 1$. Therefore, we will multiply the numerator and denominator by $a^2+ab+b^2 = \sqrt[3]{h^2} + \sqrt[3]{h} + 1$. This gives

$$\begin{aligned}
\lim_{h\to 1} \frac{\sqrt[3]{h}-1}{h-1} &= \lim_{h\to 1} \frac{\sqrt[3]{h}-1}{h-1} \cdot \frac{\sqrt[3]{h^2}+\sqrt[3]{h}+1}{\sqrt[3]{h^2}+\sqrt[3]{h}+1} \\
&= \lim_{h\to 1} \frac{(\sqrt[3]{h})^3 - 1^3}{(h-1)\left(\sqrt[3]{h^2}+\sqrt[3]{h}+1\right)} \\
&= \lim_{h\to 1} \frac{h-1}{(h-1)\left(\sqrt[3]{h^2}+\sqrt[3]{h}+1\right)} \\
&= \lim_{h\to 1} \frac{h-1}{(h-1)\left(\sqrt[3]{h^2}+\sqrt[3]{h}+1\right)} \\
&= \lim_{h\to 1} \frac{1}{\sqrt[3]{h^2}+\sqrt[3]{h}+1} \\
&= \frac{1}{\sqrt[3]{1^2}+\sqrt[3]{1}+1} \\
&= \boxed{\frac{1}{3}}.
\end{aligned}$$

2. Suppose that f is a function such that for any number x, we have

$$x - 8 \leqslant f(x) \leqslant x^2 - 3x - 4.$$

For which values of a can you determine $\lim_{x\to a} f(x)$? For these values of a, evaluate $\lim_{x\to a} f(x)$.

Solution. We have $\lim_{x\to a} x - 8 = a - 8$ and $\lim_{x\to a} x^2 - 3x - 4 = a^2 - 3a - 4$. The Squeeze Theorem will guarantee that $\lim_{x\to a} f(x)$ exists when $a - 8 = a^2 - 3a - 4$, that is $a^2 - 4a + 4 = 0$. This equation gives $(a-2)^2$, or $\boxed{a = 2}$. For $a = 2$ we will have

$$\lim_{x\to 2} f(x) = 2 - 8 = \boxed{-6}.$$

3. **[Advanced]** Suppose that f is a function such that

$$\lim_{x\to 0} \frac{f(x)}{\sin(3x)} = 2.$$

Evaluate the following limits.

(a) $\lim_{x\to 0} f(x)$.

Solution.

$$\begin{aligned}\lim_{x\to 0} f(x) &= \lim_{x\to 0} f(x)\cdot\frac{\sin(3x)}{\sin(3x)}\\ &= \left(\lim_{x\to 0}\frac{f(x)}{\sin(3x)}\right)\left(\lim_{x\to 0}\sin(3x)\right)\\ &= 2\cdot\sin(0)\\ &= 0.\end{aligned}$$

(b) $\lim_{x\to 0} \frac{f(x)}{x}$.

Solution.

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)}{x} &= \lim_{x\to 0}\frac{f(x)}{x}\cdot\frac{\sin(3x)}{\sin(3x)}\cdot\frac{3x}{3x}\\ &= \left(\lim_{x\to 0}\frac{f(x)}{\sin(3x)}\right)\left(\lim_{x\to 0}\frac{\sin(3x)}{3x}\right)\left(\lim_{x\to 0}\frac{3x}{x}\right)\\ &= 2\cdot 1\cdot 3\\ &= \boxed{6}.\end{aligned}$$

(c) $\lim_{x\to 0}\frac{1-\cos(x)}{f(2x)^2}$.

Solution.

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos(x)}{f(2x)^2} &= \lim_{x\to 0}\frac{1-\cos(x)}{f(2x)^2}\cdot\frac{x^2}{x^2}\cdot\frac{(2x)^2}{(2x)^2}\\ &= \left(\lim_{x\to 0}\frac{1-\cos(x)}{x^2}\right)\left(\lim_{x\to 0}\frac{2x}{f(2x)}\right)^2\left(\lim_{x\to 0}\frac{x^2}{(2x)^2}\right)\\ &= \frac{1}{2}\cdot\left(\frac{1}{6}\right)^2\cdot\frac{1}{4}\\ &= \boxed{\frac{1}{288}}.\end{aligned}$$

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Section 2.4: One-Sided Limits - Worksheet Solutions

1. Evaluate the following limits. If a limit does not exist, explain why.

(a) $\displaystyle\lim_{x\to 3^-} \frac{x^2-4x+3}{|x-3|}$.

Solution. When $x \to 3^-$, we have $x < 3$, so $x - 3 < 0$ and $|x-3| = -(x-3)$. So

$$\begin{aligned}\lim_{x\to 3^-} \frac{x^2-4x+3}{|x-3|} &= \lim_{x\to 3^-} \frac{(x-3)(x-1)}{-(x-3)}\\ &= \lim_{x\to 3^-} -(x-1)\\ &= -(3-1)\\ &= \boxed{-2}.\end{aligned}$$

(b) $\displaystyle\lim_{x\to 3} \frac{x^2-4x+3}{|x-3|}$.

Solution. We have already computed the left limit in the previous question. Let us compute the right limit. This time, when $x \to 3^+$, we have $x > 3$, so $x-3 > 0$ and $|x-3| = x-3$. It follows that

$$\begin{aligned}\lim_{x\to 3^+} \frac{x^2-4x+3}{|x-3|} &= \lim_{x\to 3^+} \frac{(x-3)(x-1)}{x-3}\\ &= \lim_{x\to 3^+} (x-1)\\ &= (3-1)\\ &= 2.\end{aligned}$$

Since $\displaystyle\lim_{x\to 3^-} \frac{x^2-4x+3}{|x-3|} \neq \lim_{x\to 3^+} \frac{x^2-4x+3}{|x-3|}$, we conclude that $\boxed{\displaystyle\lim_{x\to 3} \frac{x^2-4x+3}{|x-3|} \text{ does not exist}}$.

(c) $\displaystyle\lim_{h\to 0^-} \frac{1-(1-|h|)^3}{h}$.

Solution. When $h \to 0^-$, we have $h < 0$, so $|h| = -h$. Therefore

$$\begin{aligned}\lim_{h\to 0^-} \frac{1-(1-|h|)^3}{h} &= \lim_{h\to 0^-} \frac{1-(1+h)^3}{h}\\ &= \lim_{h\to 0^-} \frac{1-(1+3h+3h^2+h^3)}{h}\\ &= \lim_{h\to 0^-} \frac{-3h-3h^2-h^3}{h}\end{aligned}$$

$$= \lim_{h\to 0^-} -3-3h-h^2$$
$$= \boxed{-3}.$$

(d) $\lim_{t\to 1} \dfrac{t^3-2t^2+t}{|t-1|}$.

Solution. Let us compute the left limit and the right limit. For the left limit, we have $t<1$, so $|t-1| = -(t-1)$. So

$$\begin{aligned}\lim_{t\to 1^-} \frac{t^3-2t^2+t}{|t-1|} &= \lim_{t\to 1^-} \frac{t(t-1)^2}{-(t-1)} \\ &= \lim_{t\to 1^-} -t(t-1) \\ &= -1(1-1) \\ &= 0.\end{aligned}$$

For the right limit, we have $t>1$ so $|t-1| = t-1$. Therefore

$$\begin{aligned}\lim_{t\to 1^+} \frac{t^3-2t^2+t}{|t-1|} &= \lim_{t\to 1^+} \frac{t(t-1)^2}{t-1} \\ &= \lim_{t\to 1^-} t(t-1) \\ &= 1(1-1) \\ &= 0.\end{aligned}$$

Since $\lim_{t\to 1^-} \dfrac{t^3-2t^2+t}{|t-1|} = \lim_{t\to 1^+} \dfrac{t^3-2t^2+t}{|t-1|} = 0$, it follows that $\boxed{\lim_{t\to 1^-} \dfrac{t^3-2t^2+t}{|t-1|} = 0}$.

(e) $\lim_{x\to -2} f(x)$ where $f(x) = \begin{cases} 3x+8 & \text{if } x<-2 \\ 8 & \text{if } x=-2 \\ \dfrac{x+2}{\sqrt{x+3}-1} & \text{if } x>-2 \end{cases}$.

Solution. We have

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} 3x+8 = 3(-2)+8 = 2$$

and

$$\begin{aligned}\lim_{x\to -2^+} f(x) &= \lim_{x\to -2^+} \frac{x+2}{\sqrt{x+3}-1}\cdot\frac{\sqrt{x+3}+1}{\sqrt{x+3}+1} \\ &= \lim_{x\to -2^+} \frac{(x+2)(\sqrt{x+3}+1)}{(\sqrt{x+3})^2-1^2} \\ &= \lim_{x\to -2^+} \frac{(x+2)(\sqrt{x+3}+1)}{x-2} \\ &= \lim_{x\to -2^+} \sqrt{x+3}+1 \\ &= 2.\end{aligned}$$

Since $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x) = 2$, it follows that $\boxed{\lim_{x\to -2} f(x) = 2}$.

(f) $\lim_{x\to 0} f(x)$ where $f(x) = \begin{cases} \dfrac{\sin(3x)}{x} & \text{if } x < 0 \\ 2e^{\cos(x)-1} & \text{if } x \geqslant 0 \end{cases}$.

Solution. We have

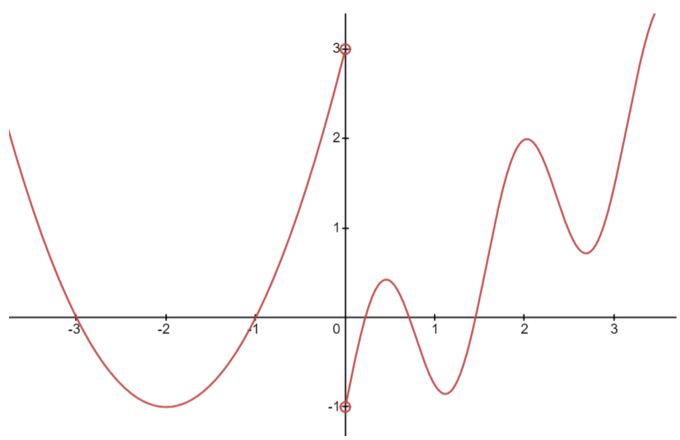
$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} \frac{\sin(3x)}{x}\cdot\frac{3x}{3x} \\ &= \lim_{x\to 0^-} \frac{\sin(3x)}{3x}\cdot\frac{3x}{x} \\ &= \left(\lim_{x\to 0^-} \frac{\sin(3x)}{3x}\right)\left(\lim_{x\to 0^-} 3\right) \\ &= 1\cdot 3 \\ &= 3 \end{aligned}$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2e^{\cos(x)-1} = 2e^{\cos(0)-1} = 2e^{1-1} = 2e^0 = 2.$$

Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, we conclude that $\boxed{\lim_{x\to 0} f(x) \text{ does not exist}}$.

2. Consider the function $f(x) = \dfrac{\tan(8x)}{|2x|}$ and suppose that the graph of another function g is given below.



(a) Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ or explain why it does not exist.

Solution. We have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin(8x)}{-2x\cos(8x)}\cdot\frac{8x}{8x} = \left(\lim_{x\to 0^-} \frac{\sin(8x)}{8x}\right)\left(\lim_{x\to 0^-} \frac{8x}{-2x\cos(8x)}\right) = -\frac{4}{\cos(0)} = -4$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin(8x)}{2x\cos(8x)}\cdot\frac{8x}{8x} = \left(\lim_{x\to 0^+} \frac{\sin(8x)}{8x}\right)\left(\lim_{x\to 0^+} \frac{8x}{2x\cos(8x)}\right) = \frac{4}{\cos(0)} = 4.$$

Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, we conclude that $\boxed{\lim_{x\to 0} f(x) \text{ does not exist}}$.

By inspection of the graph, we see that

$$\lim_{x\to 0^-} g(x) = 3, \qquad \lim_{x\to 0^+} g(x) = -1.$$

Since $\lim\limits_{x\to 0^-} g(x) \neq \lim\limits_{x\to 0^+} g(x)$, we conclude that $\boxed{\lim\limits_{x\to 0} g(x) \text{ does not exist}}$.

(b) Find $\lim\limits_{x\to 0} |f(x)|$ and $\lim\limits_{x\to 0} |g(x)|$ or explain why it does not exist.

Solution. We have

$$\lim_{x\to 0^-} |f(x)| = \left| \lim_{x\to 0^-} f(x) \right| = |-4| = 4,$$

and

$$\lim_{x\to 0^+} |f(x)| = \left| \lim_{x\to 0^+} f(x) \right| = |4| = 4.$$

Since $\lim\limits_{x\to 0^-} |f(x)| = \lim\limits_{x\to 0^+} |f(x)| = 4$, we conclude that $\boxed{\lim\limits_{x\to 0} |f(x)| = 4}$.

We have

$$\lim_{x\to 0^-} |g(x)| = \left| \lim_{x\to 0^-} g(x) \right| = |3| = 3,$$

and

$$\lim_{x\to 0^+} |g(x)| = \left| \lim_{x\to 0^+} g(x) \right| = |-1| = 1.$$

Since $\lim\limits_{x\to 0^-} |g(x)| \neq \lim\limits_{x\to 0^+} |g(x)|$, we conclude that $\boxed{\lim\limits_{x\to 0} |g(x)| \text{ does not exist}}$.

(c) Find $\lim\limits_{x\to 0} f(x) + 2g(x)$ or explain why it does not exist.

Solution. We have

$$\lim_{x\to 0^-} f(x) + 2g(x) = \left(\lim_{x\to 0^-} f(x) \right) + 2 \left(\lim_{x\to 0^-} g(x) \right) = -4 + 2 \cdot 3 = 2$$

and

$$\lim_{x\to 0^+} f(x) + 2g(x) = \left(\lim_{x\to 0^+} f(x) \right) + 2 \left(\lim_{x\to 0^+} g(x) \right) = 4 + 2 \cdot (-1) = 2.$$

Since $\lim\limits_{x\to 0^-} f(x) + 2g(x) = \lim\limits_{x\to 0^+} f(x) + 2g(x) = 2$, we conclude that $\boxed{\lim\limits_{x\to 0} f(x) + 2g(x) = 2}$.

(d) [**Advanced**] Find the value of the constant a for which $\lim\limits_{x\to 0} \dfrac{g(x)}{f(x)+a}$ exists. For this value of a, find the value of the limit.

Solution. We have

$$\lim_{x\to 0^-} \frac{g(x)}{f(x)+a} = \frac{\lim\limits_{x\to 0^-} g(x)}{\left(\lim\limits_{x\to 0^-} f(x) \right) + \left(\lim\limits_{x\to 0^-} a \right)} = \frac{3}{-4+a},$$

and

$$\lim_{x\to 0^+} \frac{g(x)}{f(x)+a} = \frac{\lim\limits_{x\to 0^+} g(x)}{\left(\lim\limits_{x\to 0^+} f(x)\right) + \left(\lim\limits_{x\to 0^+} a\right)} = \frac{-1}{4+a}.$$

The two-sided limit exists when both one-sided limits are equal. This gives the condition $\frac{3}{-4+a} = \frac{-1}{4+a}$, that is $3(4+a) = -(-4+a)$, or $12+3a = 4-a$. Solving this for a gives $\boxed{a=-2}$. Then we have

$$\boxed{\lim_{x\to 0} \frac{g(x)}{f(x)-2} = \frac{-1}{4-2} = -\frac{1}{2}}.$$

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Section 2.5: Continuity - Worksheet Solutions

1. For each function, find the values of the constants a, b that make it continuous.

(a) $f(x) = \begin{cases} 3x - b & \text{if } x \leqslant 1 \\ ax + 4 & \text{if } 1 < x \leqslant 3 \\ bx - 2a & \text{if } x > 3 \end{cases}.$

Solution. Each piece of f being continuous, it suffices to test for continuity at the transition points $x = 1$ and $x = 3$. At $x = 1$, we have

$$\begin{aligned} &\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 3x - b = 3 - b, \\ &\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} ax + 4 = a + 4, \\ &f(1) = 3(1) - b = 3 - b. \end{aligned}$$

So the continuity test gives the condition $3 - b = a + 4$, or $a + b = -1$. At $x = 3$, we have

$$\begin{aligned} &\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} ax + 4 = 3a + 4, \\ &\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} bx - 2a = 3b - 2a, \\ &f(3) = 3a + 4. \end{aligned}$$

So the continuity test gives the condition $3a + 4 = 3b - 2a$, or $5a - 3b = -4$. Therefore, for f to be continuous, the constants a, b must satisfy the equations

$$\begin{cases} a + b = -1, \\ 5a - 3b = -4. \end{cases}$$

To finish, we need to solve this system of two linear equations. Adding 3 times the first equation to the second one gives $8a = -7$, so $\boxed{a = -\frac{7}{8}}$. Then we get $\boxed{b = \frac{15}{8}}$.

(b) $f(x) = \begin{cases} bx + 4 & \text{if } x < 1 \\ a & \text{if } x = 1 \\ \dfrac{x^{-1} - 1}{x^2 - 1} & \text{if } x > 1 \end{cases}.$

Solution. Each piece of f is continuous. This is obvious for the piece for $x < 1$, as it is a linear function. For the piece for $x > 1$, observe that the roots of the denominator are $x = -1, 1$. Therefore, the denominator does not cancel for $x > 1$ and the piece is a well-defined rational function (therefore continuous). So it suffices to test for continuity at $x = 1$. We have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} bx + 4 = b + 4,$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^{-1}-1}{x^2-1} = \lim_{x\to 1^+} \frac{\frac{1-x}{x}}{(x-1)(x+1)} = \lim_{x\to 1^+} -\frac{1}{x(x+1)} = -\frac{1}{2},$$
$$f(1) = a.$$

So the continuity test gives the condition $b+4 = -\frac{1}{2} = a$. This gives the values $\boxed{a = -\frac{1}{2}}$ and $\boxed{b = -\frac{9}{2}}$.

(c) [**Advanced**] $f(x) = \begin{cases} \dfrac{\sin(ax)}{3x} & \text{if } x < 0 \\ b & \text{if } x = 0 \\ \dfrac{x^2+5x}{\sqrt{x+4}-2} & \text{if } x > 0 \end{cases}.$

Solution. The pieces for $x < 0$ and $x > 0$ are both continuous (well-defined common functions). So it suffices to test for continuity at $x = 0$. We have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin(ax)}{3x} \cdot \frac{ax}{ax} = \left(\lim_{x\to 0^-} \frac{\sin(ax)}{ax}\right)\left(\lim_{x\to 0^-} \frac{ax}{3x}\right) = \frac{a}{3},$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2+5x}{\sqrt{x+4}-2} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \lim_{x\to 0^+} \frac{x(x+5)\left(\sqrt{x+4}+2\right)}{x+4-4} = \lim_{x\to 0^+} (x+5)\left(\sqrt{x+4}+2\right) = 20$$
$$f(1) = b.$$

So we get the conditions $\frac{a}{3} = 20 = b$. This gives the solutions $\boxed{a = 60}$ and $\boxed{b = 20}$.

2. Consider the function $f(x) = \begin{cases} x^2+4x+5 & \text{if } x < -2 \\ 3 & \text{if } x = -2 \\ \cos(\pi x) & \text{if } -2 < x < 3 \\ x+2 & \text{if } 3 \leqslant x \leqslant 4 \\ 6 - \ln(x-3) & \text{if } x > 4 \end{cases}.$

(a) Find the values of a for which $\lim_{x\to a} f(x)$ does not exist.

Solution. Since each piece of f is continuous (therefore has a limit at every point of its domain), it suffices to test the transition points. At $x = -2$, we have

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} x^2+4x+5 = (-2)^2 + 4(-2) + 5 = 1,$$
$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \cos(\pi x) = \cos(-2\pi) = 1.$$

Since $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x)$, $\lim_{x\to -2} f(x)$ exists.

At $x = 3$, we have

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \cos(\pi x) = \cos(3\pi) = -1,$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x+2 = 5.$$

Since $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$, $\lim_{x\to 3} f(x)$ does not exist.

At $x = 4$, we have

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} x + 2 = 6,$$
$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} 6 - \ln(x-3) = 6 - \ln(1) = 6.$$

Since $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x)$, $\lim_{x\to 4} f(x)$ exists.

In conclusion, $\lim_{x\to a} f(x)$ does not exist for $\boxed{a = 3}$.

(b) Find the values of x where f is discontinuous.

Solution. Since each piece of f is continuous, it suffices to test the transition points. We already know f is discontinuous at $x = 3$ since it does not have a limit at this point by part (a). At $x = -2$ we have $\lim_{x\to -2} f(x) = 1$ and $f(-2) = 3$, so f is discontinuous at $x = -2$. At $x = 4$, we have $\lim_{x\to 4} f(x) = 6$ and $f(4) = 6$, so f is continuous at $x = 6$.

In conclusion, f is discontinuous at $\boxed{x = -2, 3}$.

3. Show that each equation has a solution in the given interval.

(a) $x^3 = 14 + 2\sqrt{x}$ in $[0, 4]$.

Solution. We start by writing the equation as $x^3 - 2\sqrt{x} = 14$. This has the form $f(x) = y_0$ with $f(x) = x^3 - 2\sqrt{x} - 14$ and $y_0 = 14$. The function f is continuous on $[0, 4]$. We have

$$f(0) = 0^3 - 2\sqrt{0}0 < 14,$$
$$f(4) = 4^3 - 2\sqrt{4} = 60 > 14.$$

Therefore, the value $y_0 = 14$ is an intermediate value between $f(0)$ and $f(4)$. By the IVT, it follows that the equation has a solution in $[0, 4]$.

(b) $\ln(x) = 2 - x$ in $[1, e]$.

Solution. We start by writing the equation as $\ln(x) + x = 2$. This has the form $f(x) = y_0$ with $f(x) = \ln(x) + x$ and $y_0 = 2$. The function f is continuous on $[1, e]$. We have

$$f(1) = \ln(1) + 1 = 1 < 2,$$
$$f(e) = \ln(e) + e = 1 + e > 2.$$

Therefore, the value $y_0 = 2$ is an intermediate value between $f(1)$ and $f(e)$. By the IVT, it follows that the equation has a solution in $[1, e]$.

(c) **[Advanced]** $\cos(x) = \arcsin(x)$ in $[0, 1]$.

Solution. We start by writing the equation as $\cos(x) - \arcsin(x) = 0$. This has the form $f(x) = y_0$ with $f(x) = \cos(x) - \arcsin(x)$ and $y_0 = 0$. The function f is continuous on $[0, 1]$. We have

$$f(0) = \cos(0) - \arcsin(0) = 1 - 0 = 1 > 0,$$
$$f(1) = \cos(1) - \arcsin(1) = \cos(1) - \frac{\pi}{2}.$$

Observe that $\cos(1) \leqslant 1$ since the range of cos is $[-1, 1]$ and $\frac{\pi}{2} > 1$ since $\pi > 2$. Thus, $\cos(1) - \frac{\pi}{2} < 0$. It follows that the value $y_0 = 0$ is an intermediate value between $f(0)$ and $f(1)$. By the IVT, it follows that the equation has a solution in $[0, 1]$.

Rutgers University
Math 151

Section 2.6: Limits Involving Infinity - Worksheet

1. Evaluate the following limits. If a limit does not exist, explain why. If a limit is infinite, specify it and determine if it is ∞ or $-\infty$.

(a) $\displaystyle\lim_{x\to -1^-} \frac{x^2+3x+2}{(x+1)^2}$.

Solution. Substitution gives $\frac{0}{0}$, so we need more analysis. We have

$$\lim_{x\to -1^-} \frac{x^2+3x+2}{(x+1)^2} = \lim_{x\to -1^-} \frac{(x+1)(x+2)}{(x+1)^2} = \lim_{x\to -1^-} \frac{x+2}{x+1}.$$

In this simplified form, substitution gives $\frac{1}{0}$ so the one-sided limit is infinite. To determine if the limit is ∞ or $-\infty$, we use a sign analysis. As $x \to -1^-$, $x+2>0$ and $x+1<0$, so $\frac{x+2}{x+1}<0$. Therefore

$$\boxed{\lim_{x\to -1^-} \frac{x^2+3x+2}{(x+1)^2} = -\infty}.$$

(b) $\displaystyle\lim_{x\to\infty} \frac{3x\sqrt{x}+2}{\sqrt{4x^3+1}}$.

Solution. Observe that $x\sqrt{x} = \sqrt{x^3}$ when $x>0$. So

$$\begin{aligned}\lim_{x\to\infty} \frac{3x\sqrt{x}+2}{\sqrt{4x^3+1}} &= \lim_{x\to\infty} \frac{3x\sqrt{x}+2}{\sqrt{4x^3+1}} \cdot \frac{\frac{1}{\sqrt{x^3}}}{\frac{1}{\sqrt{x^3}}} \\ &= \lim_{x\to\infty} \frac{3+\frac{2}{\sqrt{x^3}}}{\sqrt{4+\frac{1}{x^3}}} \\ &= \frac{3+0}{\sqrt{4+0}} \\ &= \boxed{\frac{3}{2}}.\end{aligned}$$

(c) $\displaystyle\lim_{x\to 2\pi} \frac{x}{\cos(x)-1}$.

Solution. Substitution gives $\frac{2\pi}{0}$, so both one-sided limits are infinite. We need a sign analysis to determine if the limit is ∞ or $-\infty$ on each side. Observe that $\cos(x)-1 \leqslant 0$ for all x since the range of cos is $[-1,1]$. It follows that as $x\to 2\pi^+$ and $x \to 2\pi^-$, the values of $\frac{x}{\cos(x)-1}$ are negative (positive numerator and negative denominator). Hence,

$$\boxed{\lim_{x\to 2\pi} \frac{x}{\cos(x)-1} = -\infty}.$$

(d) $\lim_{x\to 2} \dfrac{x-5}{x^2-2x}$.

Solution. Substitution gives $\frac{-3}{0}$, so both one-sided limits are infinite. We need a sign analysis to determine if the limit is ∞ or $-\infty$ on each side. When $x \to 2^-$, we have $0 < x < 2$ so $x^2 - 2x = x(x-2) < 0$ and $\frac{x-5}{x(x-2)} > 0$. Therefore,

$$\lim_{x\to 2^-} \frac{x-5}{x^2-2x} = \infty.$$

When $x \to 2^+$, we have $0x > 2$ so $x^2 - 2x = x(x-2) > 0$ and $\frac{x-5}{x(x-2)} < 0$. Therefore,

$$\lim_{x\to 2^+} \frac{x-5}{x^2-2x} = -\infty.$$

We conclude that $\boxed{\lim_{x\to 2} \dfrac{x-5}{x^2-2x} \text{ does not exist}}$.

(e) $\lim_{x\to -\infty} \dfrac{x^3+2}{\sqrt{16x^6+1}}$.

Solution. Observe that $\sqrt{x^6} = |x^3| = -x^3$ when $x < 0$. So

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x^3+2}{\sqrt{16x^6+1}} &= \lim_{x\to-\infty} \frac{x^3+2}{\sqrt{x^6\left(16+\frac{1}{x^6}\right)}} \\
&= \lim_{x\to-\infty} \frac{x^3+2}{-x^3\sqrt{16+\frac{1}{x^6}}} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \\
&= \lim_{x\to-\infty} \frac{1+\frac{2}{x^3}}{-\sqrt{16+\frac{1}{x^6}}} \\
&= \frac{1+0}{-\sqrt{16+0}} \\
&= \boxed{-\frac{1}{4}}.
\end{aligned}$$

(f) $\lim_{t\to\infty} \sqrt{9t^2+8t} - \sqrt{9t^2-5t}$.

Solution.

$$\begin{aligned}
\lim_{t\to\infty} \sqrt{9t^2+8t} - \sqrt{9t^2-5t} &= \lim_{t\to\infty} \left(\sqrt{9t^2+8t} - \sqrt{9t^2-5t}\right) \frac{\sqrt{9t^2+8t} + \sqrt{9t^2-5t}}{\sqrt{9t^2+8t} + \sqrt{9t^2-5t}} \\
&= \lim_{t\to\infty} \frac{\left(\sqrt{9t^2+8t}\right)^2 - \left(\sqrt{9t^2-5t}\right)^2}{\sqrt{9t^2+8t} + \sqrt{9t^2-5t}} \\
&= \lim_{t\to\infty} \frac{9t^2+8t - \left(9t^2-5t\right)}{\sqrt{9t^2+8t} + \sqrt{9t^2-5t}} \\
&= \lim_{t\to\infty} \frac{13t}{\sqrt{9t^2+8t} + \sqrt{9t^2-5t}} \cdot \frac{\frac{1}{t}}{\frac{1}{t}}
\end{aligned}$$

$$= \lim_{t\to\infty} \frac{13}{\sqrt{9+\frac{8}{t}}+\sqrt{9-\frac{5}{t}}}$$

$$= \frac{13}{\sqrt{9+0}+\sqrt{9+0}}$$

$$= \boxed{\frac{13}{6}}.$$

[Advanced]

(a) $\displaystyle\lim_{\theta\to-\infty} \frac{2\theta+5\sin(3\theta)}{7\theta}$.

Solution. Observe that

$$\frac{2\theta+5\sin(3\theta)}{7\theta} = \frac{2}{7}+\frac{5\sin(3\theta)}{7\theta}.$$

Since $-1 \leqslant \sin(3\theta) \leqslant 1$, we have $-5 \leqslant 5\sin(3\theta) \leqslant 5$ and $-\frac{5}{7\theta} \leqslant \frac{5\sin(3\theta)}{7\theta} \leqslant \frac{5}{7\theta}$. Additionally, we have

$$\lim_{\theta\to-\infty} -\frac{5}{7\theta} = \lim_{\theta\to-\infty} \frac{5}{7\theta} = 0.$$

So by the Squeeze Theorem, $\displaystyle\lim_{\theta\to-\infty} \frac{5\sin(3\theta)}{7\theta} = 0$. Therefore

$$\lim_{\theta\to-\infty} \frac{2\theta+5\sin(3\theta)}{7\theta} = \lim_{\theta\to-\infty} \frac{2}{7}+\frac{5\sin(3\theta)}{7\theta} = \frac{2}{7}+0 = \boxed{\frac{2}{7}}.$$

(b) $\displaystyle\lim_{x\to0^+} \left(\frac{1}{\sqrt[3]{x}}-\frac{1}{\sqrt{x}}\right)$.

Solution. For $x > 0$, we have

$$\frac{1}{\sqrt[3]{x}}-\frac{1}{\sqrt{x}} = \frac{1}{x^{1/3}}-\frac{1}{x^{1/2}} = \frac{x^{1/6}}{x^{1/3+1/6}}-\frac{1}{x^{1/2}} = \frac{x^{1/6}}{x^{1/2}}-\frac{1}{x^{1/2}} = \frac{x^{1/6}-1}{x^{1/2}}.$$

Substituting $x = 0$ in this expression would give $\frac{-1}{0}$, so we know that the one-sided limit is infinite. To determine if the limit is ∞ or $-\infty$, we look at the sign of the expression. For $x > 0$, $\sqrt{x} > 0$. As $x \to 0^+$, x is close to 0 so $x^{1/6}-1 < 0$. It follows that $\dfrac{x^{1/6}-1}{x^{1/2}} < 0$ and

$$\boxed{\lim_{x\to0^+} \left(\frac{1}{\sqrt[3]{x}}-\frac{1}{\sqrt{x}}\right) = -\infty}.$$

(c) $\displaystyle\lim_{t\to\infty} \frac{t\arctan(3t)}{\sqrt{t^2+1}}$.

Solution.

$$\lim_{t\to\infty} \frac{t\arctan(3t)}{\sqrt{t^2+1}} = \lim_{t\to\infty} \frac{t\arctan(3t)}{\sqrt{t^2+1}} \cdot \frac{\frac{1}{t}}{\frac{1}{t}}$$

$$
\begin{aligned}
&= \lim_{t\to\infty} \frac{\arctan(3t)}{\sqrt{1+\frac{1}{t^2}}} \\
&= \frac{\frac{\pi}{2}}{\sqrt{1+0}} \\
&= \boxed{\frac{\pi}{2}}.
\end{aligned}
$$

2. Find the vertical and horizontal asymptotes of the following functions, if any. Also, determine the limit to the left and right of any vertical asymptote.

   (a) $f(x) = \dfrac{x^2-3x-4}{\sqrt{x}-2}$.

   Solution. To find potential vertical asymptotes, we set the denominator equal to 0. This gives $\sqrt{x}-2=0$, that is $x=4$. Substituting 4 in $f(x)$ gives $\frac{0}{0}$, so we need to do more analysis to determine if $x=4$ is indeed a vertical asymptote. The limit at 4 is

   $$\lim_{x\to 4} \frac{x^2-3x-4}{\sqrt{x}-2}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4}\frac{(x-4)(x+1)(\sqrt{x}+2)}{x-4} = \lim_{x\to 4}(x+1)(\sqrt{x}+2) = (4+1)(\sqrt{4}+2)20.$$

   Since the limit as $x\to 4$ is finite, $x=4$ is not a vertical asymptote. So $\boxed{f \text{ has no vertical asymptote}}$.

   To find the horizontal asymptotes, we compute the limits at ∞ and $-\infty$. Note that f is undefined for $x<0$, so only the limit at ∞ makes sense. We have

   $$
   \begin{aligned}
   \lim_{x\to\infty}\frac{x^2-3x-4}{\sqrt{x}-2} &= \lim_{x\to\infty}\frac{x^2-3x-4}{\sqrt{x}-2}\cdot\frac{\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x}}} \\
   &= \lim_{x\to\infty}\frac{x^{3/2}-3x^{1/2}-\frac{4}{\sqrt{x}}}{1-\frac{2}{\sqrt{x}}} \\
   &= \frac{+\infty}{1} \\
   &= \infty.
   \end{aligned}
   $$

   Therefore, $\boxed{f \text{ has no horizontal asymptote}}$.

   (b) $f(x) = \dfrac{x^2-1}{|x+1|^3}$.

   Solution. To find potential vertical asymptotes, we set the denominator equal to 0. This gives $|x+1|^3=0$, that is $x=-1$. Substituting -1 in $f(x)$ gives $\frac{0}{0}$, so we need to do more analysis to determine if $x=-1$ is indeed a vertical asymptote. The left and right limit at -1 are

   $$
   \begin{aligned}
   \lim_{x\to -1^+}\frac{x^2-1}{|x+1|^3} &= \lim_{x\to -1^+}\frac{(x-1)(x+1)}{(x+1)^3} = \lim_{x\to -1^+}\frac{(x-1)}{(x+1)^2} = -\infty, \\
   \lim_{x\to -1^-}\frac{x^2-1}{|x+1|^3} &= \lim_{x\to -1^-}\frac{(x-1)(x+1)}{-(x+1)^3} = \lim_{x\to -1^-} -\frac{(x-1)}{(x+1)^2} = \infty.
   \end{aligned}
   $$

   So $\boxed{x=-1 \text{ is the one vertical asymptote of } f}$.

To find the horizontal asymptotes of f, we compute the limits at ∞ and $-\infty$. We have

$$\lim_{x\to\infty} \frac{x^2-1}{|x+1|^3} = \lim_{x\to\infty} \frac{x^2-1}{(x+1)^3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{1}{x^3}}{\left(1+\frac{1}{x}\right)^3} = \frac{0-0}{(1+0)^3} = 0,$$

$$\lim_{x\to-\infty} \frac{x^2-1}{|x+1|^3} = \lim_{x\to-\infty} \frac{x^2-1}{-(x+1)^3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to-\infty} -\frac{\frac{1}{x}-\frac{1}{x^3}}{\left(1+\frac{1}{x}\right)^3} = -\frac{0-0}{(1+0)^3} = 0.$$

So $\boxed{y=0 \text{ is the one horizontal asymptote of } f}$.

(c) $f(x) = \dfrac{7+2e^x}{5e^x-4}$.

Solution. To find potential vertical asymptotes, we set the denominator equal to 0. This gives $5e^x-4=0$, that is $e^x = \frac{4}{5}$, so $x = \ln\left(\frac{4}{5}\right)$. Substituting this value in $f(x)$ gives $\frac{7+2\cdot\frac{4}{5}}{0}$. This has the form $\frac{\text{non-zero number}}{0}$, so $\boxed{x = \ln\left(\dfrac{4}{5}\right) \text{ is the one vertical asymptote of } f}$.

To find the horizontal asymptotes of f, we compute the limits at ∞ and $-\infty$. Recall that $\lim\limits_{x\to\infty} e^x = \infty$ and $\lim\limits_{x\to-\infty} e^x = 0$. Therefore

$$\lim_{x\to-\infty} \frac{7+2e^x}{5e^x-4} = \frac{7+2\cdot 0}{5\cdot 0-4} = -\frac{7}{4},$$

$$\lim_{x\to\infty} \frac{7+2e^x}{5e^x-4} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim_{x\to\infty} \frac{\frac{7}{e^x}+2}{5-\frac{4}{e^x}} = \frac{0+2}{5-0} = \frac{2}{5}.$$

So $\boxed{y = -\dfrac{7}{4} \text{ and } y = \dfrac{2}{5} \text{ are the two horizontal asymptotes of } f}$.

(d) $f(x) = \dfrac{\sqrt{x^2+25}+3x}{2x+5}$.

Solution. To find potential vertical asymptotes, we set the denominator equal to 0. This gives $2x+5=0$, or $x=-\frac{2}{5}$. Substituting this value in $f(x)$ gives the form $\frac{\text{non-zero number}}{0}$. It follows that $\boxed{x = -\dfrac{2}{5} \text{ is the one vertical asymptote of } f}$.

To find the horizontal asymptotes of f, we calculate the limits at ∞ and $-\infty$. We have

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{x^2+25}+3x}{2x+5} &= \lim_{x\to-\infty} \frac{\sqrt{x^2\left(1+\frac{25}{x^2}\right)}+3x}{2x+5} \\
&= \lim_{x\to-\infty} \frac{|x|\sqrt{1+\frac{25}{x^2}}+3x}{2x+5} \\
&= \lim_{x\to-\infty} \frac{-x\sqrt{1+\frac{25}{x^2}}+3x}{2x+5} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \quad (x<0) \\
&= \lim_{x\to-\infty} \frac{-\sqrt{1+\frac{25}{x^2}}+3}{2+\frac{5}{x}}
\end{aligned}$$

$$= \frac{-\sqrt{1+0}+3}{2+0}$$
$$= 1,$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{x^2+25}+3x}{2x+5} &= \lim_{x\to\infty} \frac{\sqrt{x^2\left(1+\frac{25}{x^2}\right)}+3x}{2x+5} \\
&= \lim_{x\to\infty} \frac{|x|\sqrt{1+\frac{25}{x^2}}+3x}{2x+5} \\
&= \lim_{x\to\infty} \frac{x\sqrt{1+\frac{25}{x^2}}+3x}{2x+5} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \quad (x>0) \\
&= \lim_{x\to\infty} \frac{\sqrt{1+\frac{25}{x^2}}+3}{2+\frac{5}{x}} \\
&= \frac{\sqrt{1+0}+3}{2+0} \\
&= 2.
\end{aligned}$$

So $\boxed{y = 1 \text{ and } y = 2 \text{ are the two horizontal asymptotes of } f}$.

(e) $f(x) = \dfrac{\sin(7x)}{x^2+3x}$.

Solution. To find potential vertical asymptotes, we set the denominator equal to 0. This gives $x(x+3) = 0$, or $x = -3, 0$. Substituting $x = -3$ in $f(x)$ gives $\frac{\sin(-21)}{0}$, which has the form $\frac{\text{non-zero number}}{0}$, so $x = -3$ is indeed a vertical asymptote of f. Subsituting $x = 0$ gives $\frac{0}{0}$, so we need more analysis to determine whether $x = 0$ is a vertical asymptote or not. We have

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(7x)}{x^2+3x} &= \lim_{x\to 0} \frac{\sin(7x)}{x(x+3)} \cdot \frac{7x}{7x} \\
&= \lim_{x\to 0} \frac{\sin(7x)}{7x} \cdot \frac{7x}{x(x+3)} \\
&= \left(\lim_{x\to 0} \frac{\sin(7x)}{7x}\right)\left(\lim_{x\to 0} \frac{7}{x+3}\right) \\
&= 1 \cdot \frac{7}{3} \\
&= \frac{7}{3},
\end{aligned}$$

so $x = 0$ is not a vertical asymptote of f. In conclusion $\boxed{x = -3 \text{ is the one vertical asymptote of } f}$.

To find the horizontal asymptotes of f, we calculate the limits at ∞ and $-\infty$. We have $-1 \leqslant \sin(7x) \leqslant 1$ so

$$-\frac{1}{x^2+3x} \leqslant \frac{\sin(7x)}{x^2+3x} \leqslant \frac{1}{x^2+3x}.$$

Additionally, $\lim\limits_{x\to\pm\infty} -\frac{1}{x^2+3x} = \lim\limits_{x\to\pm\infty} -\frac{1}{x^2+3x} = 0$. So by the Squeeze Theorem, $\lim\limits_{x\to\pm\infty} \frac{\sin(7x)}{x^2+3x} = 0$. In conclusion, $\boxed{y = 0 \text{ is the one horizontal asymptote of } f}$.

(f) $f(x) = x^2 \cos\left(\dfrac{2}{x}\right)$.

Solution. The function f is continuous on its domain, that is $(-\infty, 0) \cup (0, \infty)$. Therefore, the only potential vertical asymptote is $x = 0$. For any $x \neq 0$, we have

$$-x^2 \leqslant x^2 \cos\left(\frac{2}{x}\right) \leqslant x^2,$$

and $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$. So by the Squeeze Theorem, $\lim_{x\to 0} x^2 \cos\left(\frac{2}{x}\right) = 0$. Hence, $\boxed{f \text{ has no vertical asymptote}}$.

To find the horizontal asymptotes, we must compute the limits at ∞ and $-\infty$. We have

$$\lim_{x\to -\infty} x^2 \cos\left(\frac{2}{x}\right) = \infty \cdot \cos(0) = \infty \cdot 1 = \infty,$$
$$\lim_{x\to \infty} x^2 \cos\left(\frac{2}{x}\right) = \infty \cdot \cos(0) = \infty \cdot 1 = \infty.$$

So $\boxed{f \text{ has no horizontal asymptote}}$.

[Advanced]

(g) $f(x) = \dfrac{3x \arctan(x) + 7}{x - 1}$.

Solution. To find potential vertical asymptotes, we set the denominator equal to 0. This gives $x - 1 = 0$, or $x = 1$. Substituting $x = 1$ in $f(x)$ gives $\frac{\frac{3\pi}{4}+7}{0}$. This has the form $\frac{\text{non-zero number}}{0}$, so $\boxed{x = 1 \text{ is the one vertical asymptote of } f}$.

To find the horizontal asymptotes, we must compute the limits at ∞ and $-\infty$. Recall that $\lim_{x\to -\infty} \arctan(x) = -\frac{\pi}{2}$ and $\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$, so we have

$$\lim_{x\to -\infty} \frac{3x \arctan(x) + 7}{x - 1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to -\infty} \frac{3\arctan(x) + \frac{7}{x}}{1 - \frac{1}{x}} = \frac{3 \cdot \left(-\frac{\pi}{2}\right) + 0}{1 - 0} = -\frac{3\pi}{2},$$
$$\lim_{x\to \infty} \frac{3x \arctan(x) + 7}{x - 1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to \infty} \frac{3\arctan(x) + \frac{7}{x}}{1 - \frac{1}{x}} = \frac{3 \cdot \frac{\pi}{2} + 0}{1 - 0} = \frac{3\pi}{2}.$$

Hence, $\boxed{y = -\dfrac{3\pi}{2} \text{ and } y = \dfrac{3\pi}{2} \text{ are the two horizontal asymptotes of } f}$.

(h) $f(x) = \dfrac{3e^{2x} - 5e^{-x}}{2e^{-x} + e^{4x}}$.

Solution. Since the denominator of $f(x)$ is positive for any value of x, the function f is continuous on $\mathbb{R}$. Hence, $\boxed{f \text{ has no vertical asymptote}}$.

To find the horizontal asymptotes, we must compute the limits at ∞ and $-\infty$. We have

$$\lim_{x\to -\infty} \frac{3e^{2x} - 5e^{-x}}{2e^{-x} + e^{4x}} \cdot \frac{\frac{1}{e^{-x}}}{\frac{1}{e^{-x}}} = \lim_{x\to -\infty} \frac{3e^{3x} - 5}{2 + e^{5x}} = \frac{3 \cdot 0 - 5}{2 + 0} = -\frac{5}{2},$$

$$\lim_{x\to\infty} \frac{3e^{2x}-5e^{-x}}{2e^{-x}+e^{4x}} \cdot \frac{\frac{1}{e^{4x}}}{\frac{1}{e^{4x}}} = \lim_{x\to-\infty} \frac{3e^{-2x}-5e^{-5x}}{2e^{-5x}+1} = \frac{3\cdot 0 - 5\cdot 0}{2\cdot 0+1} = 0.$$

Hence, $\boxed{y = -\frac{5}{2} \text{ and } y = 0 \text{ are the two horizontal asymptotes of } f}$.

(i) $f(x) = \dfrac{1-\cos(5x)}{x^2+x^3}$.

Solution. To find potential vertical asymptotes, we set the denominator equal to 0. This gives $x^2(x+1) = 0$, or $x = -1, 0$. Subsituting $x = -1$ in $f(x)$ gives the form $\frac{\text{non-zero number}}{0}$, so $x = -1$ is a vertical asymptote of f. Subsituting $x = 0$ gives $\frac{0}{0}$, so we need more analysis. We have

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos(5x)}{x^2+x^3} &= \lim_{x\to 0} \frac{1-\cos(5x)}{x^2(x+1)} \cdot \frac{(5x)^2}{(5x)^2} \\
&= \lim_{x\to 0} \frac{1-\cos(5x)}{(5x)^2} \cdot \frac{(5x)^2}{x^2(x+1)} \\
&= \left(\lim_{x\to 0} \frac{1-\cos(5x)}{(5x)^2}\right)\left(\lim_{x\to 0} \frac{25}{x+1}\right) \\
&= \frac{1}{2} \cdot \frac{25}{0+1} \\
&= \frac{25}{2}.
\end{aligned}$$

So $x = 0$ is not a vertical asymptote of f. It follows that $\boxed{x = -1 \text{ is the one vertical asymptote of } f}$.

To find the horizontal asymptotes, we must compute the limits at ∞ and $-\infty$. We have $-1 \leqslant \cos(5x) \leqslant 1$, so $0 \leqslant 1 - \cos(5x) \leqslant 2$ and

$$0 \leqslant \frac{1-\cos(5x)}{x^2+x^3} \leqslant \frac{2}{x^2+x^3}.$$

Also, $\lim_{x\to\pm\infty} \frac{2}{x^2+x^3} = \lim_{x\to\pm\infty} 0 = 0$. By the Squeeze Theorem, it follows that $\lim_{x\to\pm\infty} \frac{1-\cos(5x)}{x^2+x^3} = 0$. So $\boxed{y = 0 \text{ is the one horizontal asymptote of } f}$.

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Section 3.1-2: Derivatives and Tangent Lines - Worksheet Solutions

1. For the functions below, find the value of the derivative and an equation of the tangent line at the point indicated. (You must use the limit definition of the derivative in this problem - you cannot use derivative rules.)

   (a) $f(x) = \dfrac{3x}{1-2x}$ at $x = 1$.

   Solution. We have $f(1) = -3$ and

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0} \frac{\frac{3(1+h)}{1-2(1+h)} - (-3)}{h} \\ &= \lim_{h\to 0} \frac{\frac{3+3h}{-1-2h} + 3}{h} \\ &= \lim_{h\to 0} \frac{3+3h+3(-1-2h)}{h(-1-2h)} \\ &= \lim_{h\to 0} \frac{3+3h-3-6h}{h(-1-2h)} \\ &= \lim_{h\to 0} \frac{-3h}{h(-1-2h)} \\ &= \lim_{h\to 0} \frac{-3}{-1-2h} \\ &= \frac{-3}{-1-2\cdot 0} \\ &= \boxed{3}. \end{aligned}$$

   So the tangent line passes through $(1, -3)$ and has slope 3. Therefore, it has equation $\boxed{y = 3(x-1) - 3}$.

   (b) $f(x) = \sqrt{5x-1}$ at $x = 2$.

   Solution. We have $f(2) = 3$ and

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{5(2+h)-1} - 3}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{9+5h} - 3}{h} \cdot \frac{\sqrt{9+5h}+3}{\sqrt{9+5h}+3} \\ &= \lim_{h\to 0} \frac{(\sqrt{9+5h})^2 - 3^2}{h(\sqrt{9+5h}+3)} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{9+5h-9}{h(\sqrt{9+5h}+3)} \\
&= \lim_{h\to 0} \frac{5h}{h(\sqrt{9+5h}+3)} \\
&= \lim_{h\to 0} \frac{5}{\sqrt{9+5h}+3} \\
&= \frac{5}{\sqrt{9+5\cdot 0}+3} \\
&= \boxed{\frac{5}{6}}.
\end{aligned}
$$

So the tangent line passes through $(2,3)$ and has slope $\frac{5}{6}$. Therefore, it has equation $\boxed{y = \frac{5}{6}(x-2)+3}$.

(c) $f(x) = 18x^{-2}$ at $x = -3$.

Solution. We have $f(-3) = 2$ and

$$
\begin{aligned}
f'(-3) &= \lim_{h\to 0} \frac{f(-3+h)-f(-3)}{h} \\
&= \lim_{h\to 0} \frac{\frac{18}{(-3+h)^2}-2}{h} \\
&= \lim_{h\to 0} \frac{18-2(-3+h)^2}{h(-3+h)^2} \\
&= \lim_{h\to 0} \frac{18-2(9-6h+h^2)}{h(-3+h)^2} \\
&= \lim_{h\to 0} \frac{18-18+12h-2h^2}{h(-3+h)^2} \\
&= \lim_{h\to 0} \frac{12h-2h^2}{h(-3+h)^2} \\
&= \lim_{h\to 0} \frac{12-2h}{(-3+h)^2} \\
&= \frac{12-2\cdot 0}{(-3+0)^2} \\
&= \boxed{\frac{4}{3}}.
\end{aligned}
$$

So the tangent line passes through $(-3,2)$ and has slope $\frac{4}{3}$. Therefore, it has equation $\boxed{y = \frac{4}{3}(x+3)+2}$.

(d) $f(x) = 2x^3 + 5x + 3$ at $x = -1$.

Solution. We have $f(-1) = -4$ and

$$
f'(-1) = \lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h}
$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{2(-1+h)^3+5(-1+h)+3-(-4)}{h} \\
&= \lim_{h\to 0} \frac{2(-1+3h-3h^2+h^3)-5+5h+3+4}{h} \\
&= \lim_{h\to 0} \frac{-2+6h-6h^2+2h^3-5+5h+3+4}{h} \\
&= \lim_{h\to 0} \frac{11h-6h^2+2h^3}{h} \\
&= \lim_{h\to 0} 11-6h+2h^2 \\
&= \boxed{11}.
\end{aligned}
$$

So the tangent line passes through $(-1,-4)$ and has slope 11. Therefore, it has equation $\boxed{y = 11(x+1)-4}$.

[Advanced]

(e) $f(x) = 3\tan(4x)$ at $x = 0$.

Solution. We have $f(0) = 0$ and

$$
\begin{aligned}
f'(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \\
&= \lim_{h\to 0} \frac{3\tan(4h)-0}{h} \\
&= \lim_{h\to 0} \frac{3\sin(4h)}{h\cos(4h)} \cdot \frac{4h}{4h} \\
&= \lim_{h\to 0} \frac{3\sin(4h)}{4h} \cdot \frac{4h}{h\cos(4h)} \\
&= 3\left(\lim_{h\to 0} \frac{\sin(4h)}{4h}\right)\left(\lim_{h\to 0} \frac{4}{\cos(4h)}\right) \\
&= 3\cdot 1\cdot 4 \\
&= \boxed{12}.
\end{aligned}
$$

So the tangent line passes through $(0,0)$ and has slope 12. Therefore, it has equation $\boxed{y = 12x}$.

(f) $f(x) = x^{2/3}$ at $x = 8$.

Solution. We have $f(8) = 4$. To compute $f'(8)$, we will make use of the identity $(a-b)(a^2+ab+b^2) = a^3-b^3$. We get

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(8+h)-f(8)}{h} \\
&= \lim_{h\to 0} \frac{(8+h)^{2/3}-4}{h} \cdot \frac{(8+h)^{4/3}+4(8+h)^{2/3}+16}{(8+h)^{4/3}+4(8+h)^{2/3}+16} \\
&= \lim_{h\to 0} \frac{\left((8+h)^{2/3}\right)^3-4^3}{h\left((8+h)^{4/3}+4(8+h)^{2/3}+16\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{(8+h)^2 - 64}{h\left((8+h)^{4/3} + 4(8+h)^{2/3} + 16\right)} \\
&= \lim_{h\to 0} \frac{64 + 16h + h^2 - 64}{h\left((8+h)^{4/3} + 4(8+h)^{2/3} + 16\right)} \\
&= \lim_{h\to 0} \frac{16h + h^2}{h\left((8+h)^{4/3} + 4(8+h)^{2/3} + 16\right)} \\
&= \lim_{h\to 0} \frac{16 + h}{(8+h)^{4/3} + 4(8+h)^{2/3} + 16} \\
&= \frac{16}{8^{4/3} + 4(8)^2 + 16} \\
&= \frac{1}{3}.
\end{aligned}
$$

So the tangent line passes through $(8, 4)$ and has slope $\frac{1}{3}$. Therefore, it has equation $\boxed{y = \frac{1}{3}(x - 8) + 4}$.

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Sections 3.3, 3.5: Differentiation Rules - Worksheet Solutions

1. Calculate the derivatives of the following functions.

(a) $f(x) = 5x^4 - 8\sqrt[5]{x} - e^4$.

Solution.

$$f'(x) = 5\frac{d}{dx}(x^4) - 8\frac{d}{dx}(x^{1/5}) - \frac{d}{dx}(e^4) = \boxed{20x^3 - \frac{8}{5}x^{-4/5} - 0}.$$

(b) $f(x) = 7x\cos(x)e^x$.

Solution.

$$f'(x) = \frac{d}{dx}(7x)\cos(x)e^x + 7x\frac{d}{dx}(\cos(x))\, e^x + 7x\cos(x)\frac{d}{dx}(e^x) = \boxed{7\cos(x)e^x - 7x\sin(x)e^x + 7\cos(x)e^x}.$$

(c) $f(x) = ex^e + 4\frac{\sqrt{x}}{\sin(x)}$.

Solution.

$$f'(x) = e\frac{d}{dx}(x^e) + 4\frac{d}{dx}\left(\frac{\sqrt{x}}{\sin(x)}\right) = \boxed{e^2x^{e-1} + 4\frac{\frac{\sin(x)}{2\sqrt{x}} - \sqrt{x}\cos(x)}{\sin(x)^2}}.$$

(d) $f(x) = \frac{3}{5+x^4}$.

Solution.

$$f'(x) = \frac{\frac{d}{dx}(3)\,(5+x^4) - 3\frac{d}{dx}\left(5+x^4\right)}{(5+x^4)^2} = \boxed{\frac{-12x^3}{(5+x^4)^2}}.$$

(e) $f(x) = 3\sin(1)7^x - x^{4/3}$.

Solution.

$$f'(x) = 3\sin(1)\frac{d}{dx}(7^x) - \frac{d}{dx}\left(x^{4/3}\right) = \boxed{3\sin(1)\ln(7)7^x - \frac{4}{3}x^{1/3}}.$$

(f) $f(x) = \dfrac{x^2}{xe^x - 1}$.

Solution.

$$f'(x) = \frac{\frac{d}{dx}\left(x^2\right)(xe^x-1) - x^2\frac{d}{dx}(xe^x-1)}{(xe^x-1)^2} = \boxed{\frac{2x(xe^x-1) - x^2(e^x+xe^x)}{(xe^x-1)^2}}.$$

(g) $f(x) = 2^x x^2$.

Solution.

$$f'(x) = \frac{d}{dx}(2^x)\,x^2 + 2^x\frac{d}{dx}\left(x^2\right) = \ln(2)2^x x^2 + 2^x(2x) = \boxed{\ln(2)2^x x^2 + 2^{x+1}x}.$$

(h) $f(x) = \dfrac{\cos(x)}{\sin(x)+1}$.

Solution.

$$f'(x) = \frac{\frac{d}{dx}(\cos(x))(\sin(x)+1) - \cos(x)\frac{d}{dx}(\sin(x)+1)}{(\sin(x)+1)^2} = \boxed{\frac{-\sin(x)(\sin(x)+1) - \cos(x)^2}{(\sin(x)+1)^2}}.$$

(i) $f(x) = \dfrac{x\cos(x)\sin(x)}{5^x}$.

Solution.

$$f'(x) = \frac{\frac{d}{dx}(x\cos(x)\sin(x))\,5^x - x\cos(x)\sin(x)\frac{d}{dx}(5^x)}{(5^x)^2}$$

$$= \boxed{\frac{(\cos(x)\sin(x) - x\sin(x)^2 + x\cos(x)^2)5^x - \ln(5)x\cos(x)\sin(x)5^x}{5^{2x}}}$$

2. (a) Find the points on the graph of $f(x) = 2\sec(x) + \tan(x)$, $-\dfrac{\pi}{2} < x < \frac{\pi}{2}$, where the tangent line is horizontal.

Solution. The tangent line is horizontal when $f'(x) = 0$. Here, we have $f'(x) = 2\sec(x)\tan(x) + \sec(x)^2 = \sec(x)(2\tan(x) + \sec(x))$. So we get the equation

$$\sec(x)(2\tan(x) + \sec(x)) = 0$$

which produces $\sec(x) = 0$ or $2\tan(x) + \sec(x) = 0$. The equation $\sec(x) = 0$ has no solution, while the other equation gives

$$2\tan(x) + \sec(x) = 0$$

$$\frac{2\sin(x)+1}{\cos(x)} = 0$$
$$\sin(x) = -\frac{1}{2}$$
$$x = -\frac{\pi}{6} \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

For this value of x, we have $y = 2\sec\left(-\frac{\pi}{6}\right) + \tan\left(-\frac{\pi}{6}\right) = \frac{4}{\sqrt{3}} - \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$, so we have obtained the point $\boxed{\left(-\frac{\pi}{6}, \sqrt{3}\right)}$.

(b) Find the points on the graph of $f(x) = \dfrac{1}{1-2x}$ where the tangent line passes through the origin.

Solution. The tangent line to the graph of f at $x = a$ passes through $\left(a, \frac{1}{1-2a}\right)$ and has slope

$$f'(a) = -\frac{-2}{(1-2a)^2} = \frac{2}{(1-2a)^2}.$$

So the equation of the tangent is $y - \frac{1}{1-2a} = \frac{2}{(1-2a)^2}(x-a)$. The tangent line passes through the origin if this equation is satisfied for $(x, y) = (0, 0)$, which gives the condition

$$0 - \frac{1}{1-2a} = \frac{2}{(1-2a)^2}(0-a)$$
$$\frac{1}{1-2a} = \frac{2a}{(1-2a)^2}$$
$$1 - 2a = 2a$$
$$4a = 1$$
$$a = \frac{1}{4}.$$

For this value of a, we have $y = \frac{1}{1-2(1/4)} = 2$, so we have obtained the point $\boxed{\left(\frac{1}{4}, 2\right)}$.

(c) [**Advanced**] Find the values of the constant a for which the tangent lines to the graph of $f(x) = x^3 + 3x^2 + 5x$ at $x = a$ and $x = a + 1$ are parallel.

Solution. We have $f'(x) = 3x^2 + 6x$. The tangent lines to f at $x = a$ and $x = a + 1$ are parallel when $f'(a) = f'(a + 1)$, which gives

$$3a^2 + 6a = 3(a+1)^2 + 6(a+1)$$
$$3a^2 + 6a = 3a^2 + 6a + 3 + 6a + 6$$
$$6a = -9$$
$$\boxed{a = -\frac{3}{2}}.$$

3. Find the second derivative of the functions below.

(a) $f(x) = x^3 e^x$.

Solution.

$$f'(x) = 3x^2 e^x + x^3 e^x = (3x^2 + x^3)e^x,$$
$$f''(x) = (6x + 3x^2)e^x + (3x^2 + x^3)e^x = \boxed{(6x + 6x^2 + x^3)e^x}.$$

(b) $f(x) = \dfrac{3x+5}{2x+7}$.

Solution.

$$f'(x) = \frac{3(2x+7) - 2(3x+5)}{(2x+7)^2} = -\frac{1}{(2x+5)^2},$$
$$f''(x) = -\frac{(0)(2x+5)^2 - 1(2(2x+5) + (2x+5)(2))}{(2x+5)^4} = \boxed{\frac{4}{(2x+5)^3}}.$$

(c) $f(x) = \dfrac{7\cos(x)}{x}$.

Solution.

$$f'(x) = 7\frac{-\sin(x)x - \cos(x)}{x^2},$$
$$f''(x) = 7\frac{(-\cos(x)x - \sin(x) + \sin(x))x^2 - (-\sin(x)x - \cos(x))(2x)}{x^4} = \boxed{-7\frac{\cos(x)x^2 - 2\sin(x)x + 2\cos(x)}{x^3}}.$$

4. Suppose that f is a differentiable function such that $y = -2x + 1$ is tangent to the graph of f at $x = 3$. Evaluate the following

(a) $f(3)$.

Solution. $\boxed{f(1) = -2(3) + 1 = -5}$.

(b) $f'(3)$.

Solution. $\boxed{f'(3) = -2}$.

(c) $\dfrac{d}{dx}\left(2f(x) - x^3\right)_{|x=3}$.

Solution.

$$\frac{d}{dx}\left(2f(x) - x^3\right)_{|x=3} = \left(2f'(x) - 3x^2\right)_{|x=3} = 2f'(3) - 3 \cdot 3^2 = \boxed{-37}.$$

(d) $\frac{d}{dx}\left(\frac{f(x)}{x}\right)_{|x=3}$.

Solution.

$$\frac{d}{dx}\left(\frac{f(x)}{x}\right)_{|x=3} = \frac{f'(x)x - f(x)}{x^2}_{|x=3} = \frac{f'(3)3 - f(3)}{3^2} = -\frac{1}{9}$$

(e) **[Advanced]** $\frac{d}{dx}\left(e^x f(x)^2\right)_{|x=3}$.

Solution.

$$\begin{aligned}
\frac{d}{dx}\left(e^x f(x)^2\right)_{|x=3} &= \frac{d}{dx}\left(e^x f(x)f(x)\right)_{|x=3} \\
&= \left(e^x f(x)f(x) + e^x f'(x)f(x) + e^x f(x)f'(x)\right)_{x=3} \\
&= e^3 f(3)^2 + 2e^3 f'(3)f(3) \\
&= \boxed{45e^3}.
\end{aligned}$$

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Section 3.4: Rates of Change - Worksheet Solutions

1. The position of a body moving an axis is given by $s(t) = \dfrac{t^4}{4} - 2t^3 + 8$.

   (a) Find the body's displacement and average velocity on the time interval $[0, 2]$.

   Solution. The displacement is $\Delta s = s(2) - s(0) = -12 - 0 = \boxed{-12}$. The average velocity is $\dfrac{\Delta s}{\Delta t} = \dfrac{s(2) - s(0)}{2 - 0} = \dfrac{-12}{2} = \boxed{-6}$.

   (b) Find the velocity and acceleration of the body.

   Solution. The velocity is $v(t) = \dfrac{ds}{dt} = \boxed{t^3 - 6t^2}$. The acceleration is $a(t) = \dfrac{dv}{dt} = \boxed{3t^2 - 12t}$.

   (c) When does the body change direction?

   Solution. The body changes direction when the velocity changes sign. Factoring the velocity, we get $v(t) = t^2(t - 6)$. This polynomial changes sign at $\boxed{t = 6}$.

2. A projectile is thrown at $t = 0$ straight up in the air from an altitude of 99 m at a speed of 24 m/sec. The projectile being subject to gravity only, physicists tell us that the elevation of the projectile is subject to a law of the form $h(t) = at^2 + bt + c$, where a, b, c are unspecified constants.

   (a) Find b and c using the information given.

   Solution. We know that $h(0) = 99$, so $a \cdot 0^2 + b \cdot 0 + c = 99$, giving us $\boxed{c = 99}$. Also, $v(0) = 24$, which gives $(2at + b)_{|t=0} = 24$, so $\boxed{b = 24}$.

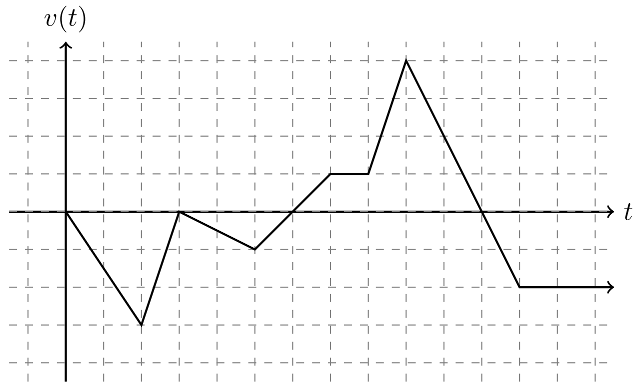
   (b) Suppose that the projectile reaches its maximum elevation 4 seconds after being thrown. Find the value of the constant a.

   Solution. We know that $v(4) = 0$ since the velocity is 0 at the instant the projectile reaches its maximum elevation. Therefore, $(2at + 24)_{|t=4} = 0$, that is $8a + 24 = 0$, so $\boxed{a = -3}$.

   (c) When will the projectile hit the ground?

   Solution. The projectile hits the ground when $h(t) = 0$, that is $-3t^2 + 24t + 99 = 0$. This gives the solutions $t = -3, 11$. Since the motion of the object is for $t \geqslant 0$, the only solution that makes sense is $\boxed{t = 11 \text{ sec}}$.

3. The graph below shows the velocity v of an object moving along an axis.



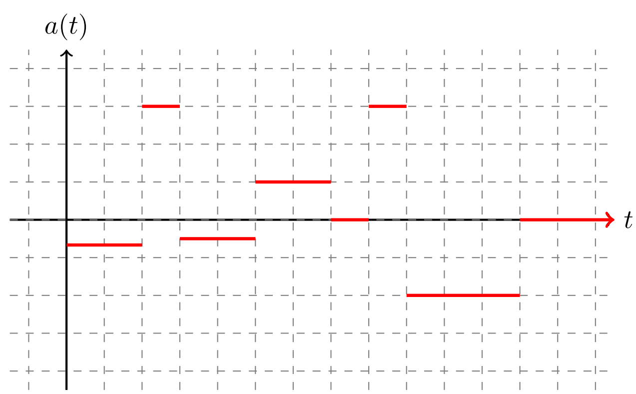
(a) When is the object moving forward? backward?

Solution. The object moves forward when $v(t) > 0$, which happens on the interval $\boxed{(6, 11)}$. The object moves backward when $v(t) < 0$, which happens on the intervals $\boxed{(0, 3), (3, 6), (11, \infty)}$.

(b) When does the object reverse direction?

Solution. The object reverses direction when the velocity changes sign, that is at $\boxed{t = 6, 11}$.

(c) Sketch the graph of the acceleration of the object.
Solution. Recall that the value of the acceleration gives the slope of the graph of the velocity



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Section 3.6: Chain Rule - Worksheet Solutions

1. Calculate the derivatives of the following functions.

   (a) $f(x) = 2\sec(4x^3+7)$

   Solution.

$$f'(x) = 2\sec(4x^3+7)\tan(4x^3+7)(12x^2) = \boxed{24x^2\sec(4x^3+7)\tan(4x^3+7)(12x^2)}.$$

   (b) $f(x) = 14\sqrt[7]{4x-\sin(5x)}$

   Solution.

$$f'(x) = 14\frac{1}{7}(4x-\sin(5x))^{1/7-1}(4-5\cos(5x)) = \boxed{\frac{2(4-5\cos(5x))}{(4x-\sin(5x))^{6/7}}}.$$

   (c) $f(x) = \cos(x^2) - \cos(x)^2$

   Solution.

$$f'(x) = -\sin(x^2)(2x) - 2\cos(x)(-\sin(x)) = \boxed{-2x\sin(x^2)+2\cos(x)\sin(x)}.$$

   (d) $f(x) = 3\left(\tan\left(\frac{x}{7}\right)+1\right)^{21}$

   Solution.

$$f'(x) = 3\cdot 21\left(\tan\left(\frac{x}{7}\right)+1\right)^{20}\sec\left(\frac{x}{7}\right)^2\frac{1}{7} = \boxed{9\left(\tan\left(\frac{x}{7}\right)+1\right)^{20}\sec\left(\frac{x}{7}\right)^2}.$$

   (e) $f(x) = \sqrt{25-4x^2}$

   Solution.

$$f'(x) = \frac{1}{2}(25-4x^2)^{1/2-1}(-8x) = \boxed{-\frac{4x}{\sqrt{25-4x^2}}}.$$

   (f) $f(x) = e^{5\cos(3x)}$

   Solution.

$$f'(x) = e^{5\cos(3x)}5(-3\sin(3x)) = \boxed{-15e^{5\cos(3x)}\sin(3x)}.$$

(g) $f(x) = x5^{3x^2}$

Solution.

$$f'(x) = 1 \cdot 5^{3x^2} + x \cdot \ln(5)5^{3x^2}(6x) = \boxed{5^{3x^2}\left(1 + 6\ln(5)x^2\right)}.$$

(h) $f(x) = 6\cos(x^3 \sin(1-2x))$

Solution.

$$\begin{aligned} f'(x) &= -6\sin(x^3\sin(1-2x))\left(3x^2\sin(1-2x) + x^3\cos(1-2x)(-2)\right) \\ &= \boxed{-6\sin(x^3\sin(1-2x))\left(3x^2\sin(1-2x) - 2x^3\cos(1-2x)\right)}. \end{aligned}$$

(i) $f(x) = \frac{2x}{\sqrt{\cos(3x)}}$

Solution.

$$f'(x) = \frac{2\sqrt{\cos(3x)} - 2x\frac{1}{2}(\cos(3x))^{-1/2}(-3\sin(3x))}{(\sqrt{\cos(3x)})^2} = \boxed{\frac{2\cos(3x) + 3x\sin(3x)}{\cos(3x)^{3/2}}}.$$

2. Find the x-values of the points on the graph of $f(x) = (2x+1)e^{-x^2}$ where the tangent line is horizontal.

Solution. We have

$$f'(x) = 2e^{-x^2} + (2x+1)e^{-x^2}(-2x) = 2e^{-x^2}(1 + (2x+1)(-x)) = 2e^{-x^2}(1 - x - 2x^2).$$

The tangent line to the graph of f is horizontal when $f'(x) = 0$. The equation $2e^{-x^2}(1 - x - 2x^2) = 0$ gives $1 - x - 2x^2 = 0$ since $2e^{-x^2}$ is never zero. The solutions of this quadratic equation are

$$x = \frac{-1 \pm \sqrt{1 + 4 \cdot 2}}{4} = \frac{-1 \pm 3}{4} = \boxed{-1, \frac{1}{2}}.$$

3. **[Advanced]** Suppose that f is a differentiable function such that

$$\begin{aligned} &f(0) = -1, \quad f(1) = 3, \quad f(2) = -5, \quad f(4) = 7, \\ &f'(0) = -2, \quad f'(1) = 4, \quad f(2) = 3, \quad f'(4) = -1. \end{aligned}$$

Find an equation of the tangent lines to each of the following functions at the given point.

(a) $g(x) = f(-2x)$ at $x = -1$.

Solution. We have $g(-1) = f(2) = -5$ and

$$g'(-1) = \frac{d}{dx}\left(f(-2x)\right)_{|x=-1} = -2f'(-2x)_{|x=-1} = -2f'(2) = -6.$$

So the tangent line has equation $\boxed{y = -6(x+1) - 5}$.

(b) $g(x) = f(x^2)$ at $x = 2$.

Solution. We have $g(2) = f(2^2) = f(4) = 7$ and

$$g'(2) = -\frac{d}{dx}\left(f(x^2)\right)_{|x=2} = 2xf'(x^2)_{|x=2} = 4f'(4) = -4.$$

So the tangent line has equation $\boxed{y = -4(x-2) - 7}$.

(c) $g(x) = \sec\left(\frac{\pi f(x)}{12}\right)$ at $x = 1$.

Solution. We have $g(1) = \sec\left(\frac{\pi f(1)}{12}\right) = \sec\left(\frac{3\pi}{12}\right) = \sec\left(\frac{\pi}{4}\right) = \sqrt{2}$ and

$$g'(1) = -\frac{d}{dx}\left(\sec\left(\frac{\pi f(x)}{12}\right)\right)_{|x=1} = \frac{\pi f'(x)}{12}\sec\left(\frac{\pi f(x)}{12}\right)\tan\left(\frac{\pi f(x)}{12}\right)_{|x=1} = \frac{4\pi}{12}\sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = \frac{\pi\sqrt{2}}{3}.$$

So the tangent line has equation $\boxed{y = \frac{\pi\sqrt{2}}{3}(x-1) - \sqrt{2}}$.

(d) $g(x) = f(4x)e^{3x}$ at $x = 0$.

Solution. We have $g(0) = f(4\cdot 0)e^{3\cdot 0} = -1$ and

$$g'(0) = -\frac{d}{dx}\left(f(4x)e^{3x}\right)_{|x=0} = \left(4f'(4x)e^{3x} + 3f(4x)e^{3x}\right)_{|x=0} = 4f'(0) + 3f(0) = -11.$$

So the tangent line has equation $\boxed{y = -11x - 1}$.

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Section 3.7: Implicit Differentiation - Worksheet Solutions

1. Calculate $\dfrac{dy}{dx}$ for the following curves.

(a) $e^{5xy} + 11\tan(x) = y^2$

Solution. Differentiating both sides with respect to x gives

$$\begin{aligned}
5e^{5xy}(y + xy') + 11\sec(x)^2 &= 2yy' \\
5e^{5xy}xy' - 2yy' &= -11\sec(x)^2 - 5e^{5xy}y \\
(5e^{5xy}x - 2y)y' &= -11\sec(x)^2 - 5e^{5xy}y \\
y' &= \boxed{\frac{-11\sec(x)^2 - 5e^{5xy}y}{5e^{5xy}x - 2y}}
\end{aligned}$$

(b) $x^3 - x\sin(y) = 3xy$

Solution. Differentiating both sides with respect to x gives

$$\begin{aligned}
3x^2 - \sin(y) - x\cos(y)y' &= 3y + 3xy' \\
3xy' + x\cos(y)y' &= 3x^2 - \sin(y) - 3y \\
(3x + x\cos(y))y' &= 3x^2 - \sin(y) - 3y \\
y' &= \boxed{\frac{3x^2 - \sin(y) - 3y}{3x + x\cos(y)}}
\end{aligned}$$

(c) $\sqrt{x^2 + y^2} = 3^y$

Solution. Differentiating both sides with respect to x gives

$$\begin{aligned}
\frac{2x + 2yy'}{2\sqrt{x^2 + y^2}} &= \ln(3)3^y y' \\
x + yy' &= \ln(3)3^y\sqrt{x^2 + y^2}y' \\
\ln(3)3^y\sqrt{x^2 + y^2}y' - yy' &= x \\
\left(\ln(3)3^y\sqrt{x^2 + y^2} - y\right)y' &= x \\
y' &= \boxed{\frac{x}{\ln(3)3^y\sqrt{x^2 + y^2} - y}}
\end{aligned}$$

(d) $x^4 + 6xy^2 + 5y^3 = 0$

Solution. Differentiating both sides with respect to x gives

$$
\begin{aligned}
&4x^3 + 6y^2 + 12xyy' + 15y^2y' = 0 \\
&\left(12xy + 15y^2\right) y' = -4x^3 - 6y^2 \\
&y' = \boxed{\frac{-4x^3 - 6y^2}{12xy + 15y^2}}
\end{aligned}
$$

2. Consider the curve of equation $x^2+6xy-y^2 = 40$. Find the points on the curve, if any, where the tangent line is (a) horizontal, (b) vertical, (c) [**Advanced**] perpendicular to $y = 2x + 9$.

Solution. First, let us differentiate the relation with respect to x:

$$
\begin{aligned}
&2x + 6y + 6xy' - 2yy' = 0 \\
&x + 3y + 3xy' - yy' = 0.
\end{aligned}
$$

(a) The tangent line is horizontal when $y' = 0$. Using this in the previous equation, we get $x + 3y = 0$, or $x = -3y$. Plugging this in the equation of the curve gives $(-3y)^2 + 6(-3y)y - y^2 = 40$, or $-10y^2 = 40$. This equation has no solution, so there are no points on the curve where the tangent line is horizontal.

(b) Solving for y' in the previous equation gives $y' = -\dfrac{x+3y}{3x-y}$, so the tangent line is vertical when $y = 3x$. Plugging this in the equation of the curve gives $x^2 + 6x(3x) - (3x)^2 = 40$, or $10x^2 = 40$. We get $x^2 = 4$, that is $x = 2$ (which gives $y = 6$) and $x = -2$ (which gives $y = -6$). So the points where the tangent line is vertical are $\boxed{(2, 6), (-2, -6)}$.

(c) The tangent line is perpendicular to $y = 2x+9$ when $y' = -\frac{1}{2}$. Plugging this in $2x+6y+6xy'-2yy' = 0$ gives $2x + 6y - 3x + y = 0$, or $x = 7y$. Substituting $x = 7y$ in the equation of the curve gives

$$
\begin{aligned}
&(7y)^2 + 6(7y)y - y^2 = 40 \\
&90y^2 = 40 \\
&y^2 = \frac{4}{9} \\
&y = \pm\frac{2}{3}.
\end{aligned}
$$

For $y = \frac{2}{3}$, we get $x = \frac{14}{3}$ and for $y = -\frac{2}{3}$, we get $x = -\frac{14}{3}$. Therefore, the points on the curve where the tangent line is perpendicular to $y = 2x + 9$ are $\boxed{\left(\frac{14}{3}, \frac{2}{3}\right), \left(-\frac{14}{3}, -\frac{2}{3}\right)}$.

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Sections 3.8-9: Derivatives of Inverse Functions - Worksheet Solutions

1. Calculate the derivatives of the following functions.

(a) $f(x) = \sin^{-1}(4x)$

Solution.

$$f'(x) = \frac{1}{\sqrt{1-(4x)^2}} \cdot 4 = \boxed{\frac{4}{\sqrt{1-16x^2}}}.$$

(b) $f(x) = \ln(2\arctan(5x) + 1)$

Solution.

$$f'(x) = \frac{1}{2\arctan(5x)+1} \cdot 2\frac{1}{1+(5x)^2} \cdot 5 = \boxed{\frac{10}{(2\arctan(5x)+1)(1+25x^2)}}.$$

(c) $f(x) = x\sec^{-1}(7x)$

Solution.

$$f'(x) = \sec^{-1}(7x) + x\frac{1}{|7x|\sqrt{(7x)^2-1}} \cdot 7 = \boxed{\sec^{-1}(7x) + \frac{7x}{|7x|\sqrt{49x^2-1}}}.$$

(d) $f(x) = \ln(x)^2 + 8\arccos(-x)$

Solution.

$$f'(x) = 2\ln(x) \cdot \frac{1}{x} + 8 \cdot -\frac{1}{\sqrt{1-(-x)^2}} \cdot -1 = \boxed{\frac{2\ln(x)}{x} + \frac{8}{\sqrt{1-x^2}}}$$

(e) $f(x) = \cot^{-1}(e^{3x})$

Solution.

$$f'(x) = -\frac{1}{1+(e^{3x})^2} \cdot e^{3x} \cdot 3 = \boxed{-\frac{3e^{3x}}{1+e^{6x}}}.$$

(f) $f(x) = \cos(x)\log_7(\sec(x))$

Solution.

$$f'(x) = -\sin(x)\log_7(\sec(x)) + \cos(x) \cdot \frac{1}{\ln(7)\sec(x)} \cdot \sec(x)\tan(x) = \boxed{\sin(x)\log_7(\sec(x)) + \frac{\sin(x)}{\ln(7)}}.$$

(g) $f(x) = x^{3\tan^{-1}(2x)}$

Solution. With $y = x^{3\tan^{-1}(2x)}$ we have

$$\ln(y) = \ln\left(x^{3\tan^{-1}(2x)}\right) = 3\tan^{-1}(2x)\ln(x).$$

Differentiating with respect to x, we obtain

$$\begin{aligned} \frac{y'}{y} &= \frac{6\ln(x)}{1+4x^2} + \frac{3\tan^{-1}(2x)}{x} \\ \Rightarrow\ y' &= y\left(\frac{6\ln(x)}{1+4x^2} + \frac{3\tan^{-1}(2x)}{x}\right) \\ &= \boxed{x^{3\tan^{-1}(2x)}\left(\frac{6\ln(x)}{1+4x^2} + \frac{3\tan^{-1}(2x)}{x}\right)} \end{aligned}$$

(h) $f(x) = \cos(x)^{\ln(x)}$

Solution. With $y = \cos(x)^{\ln(x)}$ we have

$$\ln(y) = \ln\left(\cos(x)^{\ln(x)}\right) = \ln(x)\ln(\cos(x)).$$

Differentiating with respect to x, we obtain

$$\begin{aligned} \frac{y'}{y} &= \frac{\ln(\cos(x))}{x} + \ln(x)\frac{-\sin(x)}{\cos(x)} \\ &= \frac{\ln(\cos(x))}{x} - \ln(x)\tan(x) \\ \Rightarrow\ y' &= y\left(\frac{\ln(\cos(x))}{x} - \ln(x)\tan(x)\right) \\ &= \boxed{\cos(x)^{\ln(x)}\left(\frac{\ln(\cos(x))}{x} - \ln(x)\tan(x)\right)} \end{aligned}$$

(i) $f(x) = (1-5x)^{x^2}$

Solution. With $y = (1-5x)^{x^2}$ we have

$$\ln(y) = \ln\left((1-5x)^{x^2}\right) = x^2\ln(1-5x).$$

Differentiating with respect to x, we obtain

$$\begin{aligned}\frac{y'}{y} &= 2x\ln(1-5x) + x^2\frac{-5}{1-5x} \\ &= 2x\ln(1-5x) - \frac{5x^2}{1-5x} \\ \Rightarrow\ y' &= y\left(2x\ln(1-5x) - \frac{5x^2}{1-5x}\right) \\ &= \boxed{(1-5x)^{x^2}\left(2x\ln(1-5x) - \frac{5x^2}{1-5x}\right)}\end{aligned}$$

2. Simplify each of the following. Your answer should not contain any trigonometric or inverse trigonometric functions.

   (a) $\cos\left(\sin^{-1}(x+1)\right)$

   Solution. We use the Pythagorean identity $\cos(\theta)^2+\sin(\theta)^2 = 1$ with $\theta = \sin^{-1}(x+1)$. By definition of $\sin^{-1}$, we know that $\sin(\theta) = x+1$ and θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We get

$$\begin{aligned}&\cos(\theta)^2 + (x+1)^2 = 1 \\ &\cos(\theta)^2 = 1-(x+1)^2 = -2x-x^2 \\ &\sqrt{\cos(\theta)^2} = \sqrt{-2x-x^2} \\ &|\cos(\theta)| = \sqrt{-2x-x^2} \\ &\cos(\theta) = \pm\sqrt{-2x-x^2}\end{aligned}$$

   To determine which sign is appropriate, recall that $\theta = \sin^{-1}(x+1)$ is an angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\cos(\theta) \geqslant 0$. Hence

$$\boxed{\cos(\sin^{-1}(x)) = \sqrt{-2x-x^2}}.$$

   (b) $\sin(2\cos^{-1}(3x))$

   Solution. We start by using the identity

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

   with $\theta = \cos^{-1}(3x)$. This means that $\cos(\theta) = 3x$ and that θ is in $[0, \pi]$. To find $\sin(\theta)$, we use the Pythagorean identity $\cos(\theta)^2 + \sin(\theta)^2 = 1$, which gives

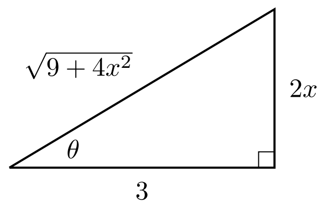
$$\begin{aligned}&(3x)^2 + \sin(\theta)^2 = 1 \\ &\sin(\theta)^2 = 1-9x^2 \\ &\sqrt{\sin(\theta)^2} = \sqrt{1-9x^2} \\ &|\sin(\theta)| = \sqrt{1-9x^2} \\ &\sin(\theta) = \sqrt{1-9x^2} \text{ since } \sin(\theta) > 0 \text{ as } 0 \leqslant \theta \leqslant \pi.\end{aligned}$$

   Therefore

$$\sin(2\cos^{-1}(3x)) = 2\sqrt{1-9x^2}(3x) = \boxed{6x\sqrt{1-9x^2}}.$$

(c) $\csc\left(\tan^{-1}\left(\frac{2x}{3}\right)\right)$

Solution. Let us solve this one with a right triangle. Consider a right triangle with base angle $\theta = \tan^{-1}\left(\frac{2x}{3}\right)$. Then $\tan(\theta) = \frac{2x}{3}$, so we can take the opposite side to be $2x$ and the adjacent to be 3. By the Pythagorean identity, the hypotenuse is $\sqrt{9+4x^2}$.



We get

$$\boxed{\csc\left(\tan^{-1}\left(\frac{2x}{3}\right)\right) = \frac{\sqrt{9+4x^2}}{2x}}.$$

Remark: in general, this method only yields the correct answer up to a sign. Here however, there is no sign issue as θ is an angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and csc and tan have the same sign on this interval, which is also that of x.

(d) $\sec(\theta)$ given that $\cot(\theta) = 5$ and $\sin(\theta) < 0$

Solution. We use the Pythagorean identity $\sec(\theta)^2 = 1 + \tan(\theta)^2$, which gives

$$\begin{aligned}
\sec(\theta)^2 &= 1 + \frac{1}{\cot(\theta)^2} = 1 + \frac{1}{25} = \frac{26}{25} \\
\sqrt{\sec(\theta)^2} &= \sqrt{\frac{26}{25}} \\
|\sec(\theta)| &= \frac{2\sqrt{6}}{25} \\
\sec(\theta) &= \pm\frac{2\sqrt{6}}{25}
\end{aligned}$$

To find the appropriate sign, observe that $\cot(\theta) > 0$ and $\sin(\theta) < 0$, which means that θ is an angle in quadrant III. Therefore, $\sec(\theta) < 0$. So

$$\boxed{\sec(\theta) = -\frac{2\sqrt{6}}{5}}.$$

3. Suppose that f is a one-to-one function and that the tangent line to the graph of $y = f(x)$ at $x = 3$ is $y = -4x + 5$. Find an equation of the tangent line to the graph of $y = f^{-1}(x)$ at $x = f(3)$.

Solution. We have $f(3) = -4 \cdot 3 + 5 = -7$ and $f'(3) = -4$. So

$$f^{-1}(-7) = 3, \qquad (f^{-1})'(-7) = \frac{1}{f'(f^{-1}(-7))} \frac{1}{f'(3)} = -\frac{1}{4}.$$

Hence, the tangent line has equation $\boxed{y = -\frac{1}{4}(x+7) - 3}$.

4. Consider the one-to-one function $f(x) = 3xe^{x^2-4}$. Calculate $f(2)$ and find an equation of the tangent line to the graph of $y = f^{-1}(x)$ at $x = f(2)$.

Solution. We have
$$f(2) = 3 \cdot 2e^{2^2-4} = \boxed{6}.$$
So $f^{-1}(6) = 2$. To find $(f^{-1})'(6)$, we will need $f'(2)$. We have
$$f'(2) = \left[3e^{x^2-4} + 3xe^{x^2-4}2x\right]_{|x=2} = 27.$$
So
$$f^{-1}(6) = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(2)} = \frac{1}{27}.$$
Hence the tangent line has equation $\boxed{y = \frac{1}{27}(x-6)+2}$.

5. Suppose that f and g are differentiable functions such that
$$\begin{array}{lll} f(-1) = 9, & f(0) = 2, & f(1) = 4, \\ f'(-1) = 3, & f'(0) = -5, & f'(1) = 8, \\ g(-1) = 2, & g(0) = 3, & g(1) = -2, \\ g'(-1) = 7, & g'(0) = -4, & g'(1) = 6. \end{array}$$

(a) For $F(x) = \ln\big(f(x^2) + g(x)\big)$, evaluate $F'(-1)$.

Solution. We have
$$F'(x) = \frac{1}{f(x^2)+g(x)} \cdot (f'(x^2)2x + g'(x)) = \frac{2f'(x^2)x + g'(x)}{f(x^2)+g(x)}.$$
So
$$F'(-1) = \frac{2f'((-1)^2)(-1) + g'(-1)}{f((-1)^2) + g(-1)} = \frac{-2f'(1) + g'(-1)}{f(1) + g(-1)} = \frac{-2 \cdot 8 + 7}{4+2} = \boxed{-\frac{3}{2}}.$$

(b) For $G(x) = \arctan\left(3\sqrt{f(x)}\right)$, evaluate $G'(1)$.

Solution. We have
$$G'(x) = \frac{1}{1 + \left(3\sqrt{f(x)}\right)^2} \cdot \frac{3}{2\sqrt{f(x)}} \cdot f'(x) = \frac{3f'(x)}{2\sqrt{f(x)}(1+9f(x))}.$$
So
$$G'(1) = \frac{3f'(1)}{2\sqrt{f(1)}(1+9f(1))} = \frac{3 \cdot 8}{2\sqrt{4}(1+9\cdot 4)} = \boxed{\frac{6}{37}}.$$

(c) For $H(x) = 2^{f(x)}g(3x+1)$, evaluate $H'(0)$.

Solution. We have

$$H'(x) = \ln(2)2^{f(x)}f'(x)g(3x+1)+2^{f(x)}g'(3x+1)\cdot 3 = 2^{f(x)}\left(\ln(2)f'(x)g(3x+1)+3f(x)g'(3x+1)\right).$$

So

$$H'(0) = 2^{f(0)}\left(\ln(2)f'(0)g(1)+3f(0)g'(1)\right) = 2^2\left(\ln(2)(-5)(-2)+3\cdot 2\cdot 6\right) = \boxed{8\left(5\ln(2)+18\right)}.$$

(d) **[Advanced]** For $K(x) = f(2x)^{g(x)}$, evaluate $K'(0)$.

Solution. We have

$$\ln(K(x)) = \ln\left(f(2x)^{g(x)}\right) = g(x)\ln(f(2x)).$$

Taking derivatives with respect to x, we get

$$\begin{aligned}
\frac{K'(x)}{K(x)} &= g'(x)\ln(f(2x)) + g(x)\frac{1}{f(2x)}\cdot f'(2x)\cdot 2 \\
&= g'(x)\ln(f(2x)) + \frac{2g(x)f'(2x)}{f(2x)} \\
\Rightarrow\ K'(x) &= K(x)\left(g'(x)\ln(f(2x)) + \frac{2g(x)f'(2x)}{f(2x)}\right) \\
&= f(2x)^{g(x)}\left(g'(x)\ln(f(2x)) + \frac{2g(x)f'(2x)}{f(2x)}\right).
\end{aligned}$$

So

$$K'(0) = f(0)^{g(0)}\left(g'(0)\ln(f(0)) + \frac{2g(0)f'(0)}{f(0)}\right) = 2^3\left(-4\ln(2) + \frac{2\cdot 3(-5)}{2}\right) = \boxed{-8(4\ln(2)+15)}.$$

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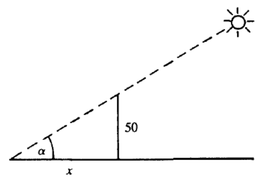
Section 3.10: Related Rates - Worksheet Solutions

1. How fast is the shadow cast on level ground by a pole 50 feet tall lengthening when the angle a of elevation of the sun is 45° and is decreasing by $\frac{1}{4}$ radian per hour? (See figure below.)



Solution. If we call x the length of the shadow and α the elevation angle of the sun, we have

$$\cot(\alpha) = \frac{x}{50}.$$

Differentiating this relation with respect to the time t, we get

$$-\csc(\alpha)^2 \frac{d\alpha}{dt} = \frac{1}{50}\frac{dx}{dt}.$$

We will now substitute the given information, $\alpha = \frac{\pi}{4}$ and $\frac{d\alpha}{dt} = -\frac{1}{4}$, in these equations. This gives us

$$\begin{cases} \cot\left(\frac{\pi}{4}\right) = \frac{x}{50}, \\ -\csc\left(\frac{\pi}{4}\right)^2 \left(-\frac{1}{4}\right) = \frac{1}{50}\frac{dx}{dt}. \end{cases}$$

We want to solve for $\frac{dx}{dt}$. For this, the second equation is enough, and we get

$$\frac{dx}{dt} = -50\csc\left(\frac{\pi}{4}\right)^2\left(-\frac{1}{4}\right) = \frac{50(\sqrt{2})^2}{4} = \boxed{25 \text{ ft/hr}}.$$

2. A sphere of radius 5 in fills with water at a rate of 4 in^3/min. When the water level inside the sphere is 6 in, how fast is it increasing? *(Hint: the volume of a spherical cap of height h in a sphere of radius r is $V = \frac{\pi}{3}(3rh^2 - h^3)$.)*

Solution. If we call h the height of water inside the sphere and V the volume of water, we have the relation $V = \frac{\pi}{3}(15h^2 - h^3)$ (the formula given with the radius of the sphere being $r = 5$, a constant). Differentiating the relation with respect to the time t, we obtain

$$\frac{dV}{dt} = \frac{\pi}{3}\left(30h\frac{dh}{dt} - 3h^2\frac{dh}{dt}\right) = \pi\left(10h - h^2\right)\frac{dh}{dt}.$$

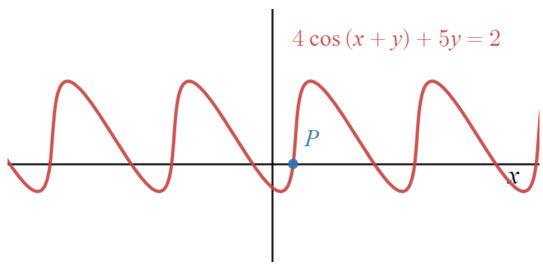
If we now substitute the information, $\frac{dV}{dt} = 4$ and $h = 6$, we get

$$\begin{cases} V = \frac{\pi}{3}(15(6)^2 - 6^3), \\ 4 = \pi\left(10(6) - (6)^2\right)\frac{dh}{dt}. \end{cases}$$

We want to solve for $\frac{dh}{dt}$. For this, we need only the second equation and we get

$$\frac{dh}{dt} = \frac{4}{\pi\left(10(6) - (6)^2\right)} = \frac{4}{24\pi} = \boxed{\frac{1}{6\pi}\ \text{in/min}}.$$

3. A particle travels toward the right on the graph of the implicit function $4\cos(x + y) + 5y = 2$, see the figure below.



When the particle first crosses the positive x-axis (at the point P on the figure), its x-coordinate increases at 6 units/sec. At what rate is the y-coordinate of the particle changing at that time?

Solution. We have the relation $4\cos(x + y) + 5y = 2$. Differentiating this with respect to the time t, we get

$$-4\sin(x + y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right) + 5\frac{dy}{dt} = 0.$$

We now need to plug in the information given. When the particle passes through the point P, we have $y = 0$ and we also know that $\frac{dx}{dt} = 6$. This gives us

$$\begin{cases} 4\cos(x + 0) + 5(0) = 2, \\ -4\sin(x + 0)\left(6 + \frac{dy}{dt}\right) + 5\frac{dy}{dt} = 0. \end{cases}$$

We need to solve these relations for $\frac{dy}{dt}$. We will start by solving the first equation for x, and we'll then use this to solve for $\frac{dy}{dt}$ in the second equation. The first equation gives $4\cos(x) = 2$, that is $\cos(x) = \frac{1}{2}$. The first positive solution to this equation is $x = \frac{\pi}{3}$. Plugging this in the second equation gives

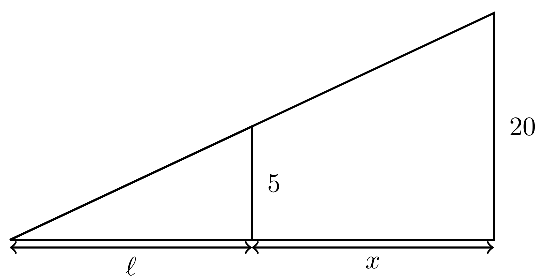
$$\begin{aligned} &-4\sin\left(\frac{\pi}{3}\right)\left(6 + \frac{dy}{dt}\right) + 5\frac{dy}{dt} = 0 \\ &-4\frac{\sqrt{3}}{2}\left(6 + \frac{dy}{dt}\right) + 5\frac{dy}{dt} = 0 \\ &-2\sqrt{3}\left(6 + \frac{dy}{dt}\right) + 5\frac{dy}{dt} = 0 \\ &-12\sqrt{3} - 2\sqrt{3}\frac{dy}{dt} + 5\frac{dy}{dt} = 0 \end{aligned}$$

$$(5-2\sqrt{3})\frac{dy}{dt} = 12\sqrt{3}$$

$$\boxed{\frac{dy}{dt} = \frac{12\sqrt{3}}{5-2\sqrt{3}} \text{ units/sec}}.$$

4. A 5-foot person is walking toward a 20-foot lamppost at the rate of 6 feet per second. How fast is the length of their shadow (cast by the lamp) changing?

   Solution. We call ℓ the length of the shadow and x the distance between the person and the lamppost, see figure below.



   Similar triangles give us the relation $\frac{\ell}{5} = \frac{\ell + x}{20}$. Differentiating this relation with respect to t gives

$$\frac{1}{5}\frac{d\ell}{dt} = \frac{1}{20}\left(\frac{d\ell}{dt} + \frac{dx}{dt}\right).$$

   We can now plug in the information, that is $\frac{dx}{dt} = -6$, and solve for $\frac{d\ell}{dt}$. We get

$$\frac{1}{5}\frac{d\ell}{dt} = \frac{1}{20}\left(\frac{d\ell}{dt} - 6\right)$$

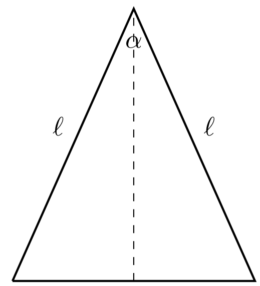
$$\frac{1}{5}\frac{d\ell}{dt} - \frac{1}{20}\frac{d\ell}{dt} = -\frac{3}{10}$$

$$\frac{3}{20}\frac{d\ell}{dt} = -\frac{3}{10}$$

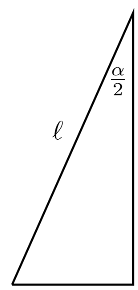
$$\boxed{\frac{d\ell}{dt} = -2 \text{ ft/sec}}.$$

5. The legs of an isosceles triangle of base 6 cm are increasing at a rate of 14 cm/hour, causing the vertex angle to decrease. When the legs are 4 cm, how fast is the vertex angle decreasing?

   Solution. Call ℓ the length of the legs of the triangle and α the vertex angle, see figure below.



Let us consider the right triangle formed by the height, one of the legs and half of the base of the isosceles triangle, see figure below.



Then we have the relation $\sin\left(\frac{\alpha}{2}\right) = \frac{3}{\ell}$. Differentiating with respect to the time t gives

$$\cos\left(\frac{\alpha}{2}\right)\frac{1}{2}\frac{d\alpha}{dt} = -\frac{3}{\ell^2}\frac{d\ell}{dt}$$

We can now plug in the information, $\ell = 6$ and $\dfrac{d\ell}{dt} = 14$ in these equations to get

$$\begin{cases} \sin\left(\frac{\alpha}{2}\right) = \frac{3}{4}, \\ \cos\left(\frac{\alpha}{2}\right)\frac{1}{2}\frac{d\alpha}{dt} = -\frac{3}{4^2}(14). \end{cases}$$

We solve for $\frac{d\alpha}{dt}$. We have

$$\cos\left(\frac{\alpha}{2}\right)^2 + \sin\left(\frac{\alpha}{2}\right)^2 = 1 \;\Rightarrow\; \cos\left(\frac{\alpha}{2}\right) = \sqrt{1 - \sin\left(\frac{\alpha}{2}\right)^2} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}.$$

Using this in the second equation, we get

$$\begin{aligned} \frac{\sqrt{7}}{8}\frac{d\alpha}{dt} &= -\frac{21}{8} \\ \frac{d\alpha}{dt} &= -\frac{21}{\sqrt{7}} = \boxed{-3\sqrt{7}\text{ rad/sec}}. \end{aligned}$$

6. **[Advanced]** An object moves along the graph of a function $y = f(x)$. At a certain point, the slope of the graph is -4 and the y-coordinate of the object is increasing at the rate of 3 units per second. At that

point, how fast is the x-coordinate of the object changing?

Solution. Differentiating the relation $y = f(x)$ with respect to the time t gives

$$\frac{dy}{dt} = f'(x)\frac{dx}{dt}.$$

We are given the information $f'(x) = -4$ and $\frac{dy}{dt} = 3$. Plugging this in the previous equation gives

$$3 = -4\frac{dx}{dt} \quad \Rightarrow \quad \boxed{\frac{dx}{dt} = -\frac{3}{4} \text{ units/sec}}.$$

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Section 3.11: Linear Approximations - Worksheet Solutions

1. Use a well-chosen linear approximation to estimate the following quantities.

(a) $\sqrt[3]{62}$

Solution. For $f(x) = \sqrt[3]{x}$ at $x = 64$, we have $f(64) = 4$ and $f'(64) = \frac{1}{3}64^{-2/3} = \frac{1}{48}$, so

$$L(x) = \frac{1}{48}(x - 64) + 4.$$

Therefore, $\sqrt[3]{62} \simeq L(62) = \frac{1}{48}(62 - 64) + 4 = \boxed{\frac{95}{24}}$.

(b) $e^{-0.8}$

Solution. For $f(x) = e^x$ at $x = 0$, we have $f(0) = 1$ and $f'(0) = e^0 = 1$, so $L(x) = x + 1$. Therefore, $e^{-0.8} \simeq L(-0.8) = -0.8 + 1 = \boxed{0.2}$.

(c) $\sqrt{49.6}$

Solution. For $f(x) = \sqrt{x}$ at $x = 49$, we have $f(49) = 7$ and $f'(64) = \frac{1}{2}49^{-1/2} = \frac{1}{14}$, so

$$L(x) = \frac{1}{14}(x - 49) + 7.$$

Therefore, $\sqrt{49.6} \simeq L(49.6) = \frac{1}{14}(49.6 - 49) + 7 = \boxed{\frac{493}{70}}$.

(d) $\ln(1 + 5\sin(0.06))$

Solution. For $f(x) = \ln(1+5\sin(x))$ at $x = 0$, we have $f(0) = 0$ and $f'(0) = \left(\frac{5\cos(x)}{1 + 5\sin(x)}\right)_{|x=0} = 5$, so

$$L(x) = 5x.$$

Therefore, $\ln(1 + 5\sin(0.06)) \simeq L(0.06) = 5(0.06) = \boxed{0.3}$.

(e) $\cot\left(\frac{\pi}{6} + 0.02\right) - \sqrt{3}$

Solution. For $f(x) = \cot(x)$ at $x = \frac{\pi}{6}$, we have $f\left(\frac{\pi}{6}\right) = \sqrt{3}$ and $f'\left(\frac{\pi}{6}\right) = -\csc^2\left(\frac{\pi}{6}\right) = -4$. So

$$\cot\left(\frac{\pi}{6} + 0.02\right) - \sqrt{3} = \Delta f \simeq f'\left(\frac{\pi}{6}\right)\Delta x = -4(0.02) = \boxed{-0.08}.$$

(f) $\sqrt[4]{17} - \sqrt[4]{16}$

Solution. For $f(x) = \sqrt[4]{x}$ at $x = 16$, we have $f(16) = 2$ and $f'(4) = \frac{1}{4}16^{-3/4} = \frac{1}{32}$. So

$$\sqrt[4]{17} - \sqrt[4]{16} = \Delta f \simeq f'(16)\,\Delta x = \frac{1}{32} \cdot 1 = \boxed{\frac{1}{32}}.$$

2. Suppose that f is a function such that $f(3) = -7$ and $f'(3) = 2$. Use a linear approximation to estimate the following quantities.

(a) $f(3.07)$

Solution. The linearization of f at $x = 3$ is $L(x) = f'(3)(x-3) + f(3) = 2(x-3) - 7$. So

$$f(3.07) \simeq L(3.07) = 2(3.07 - 3) - 7 = \boxed{-6.86}.$$

(b) **[Advanced]** $f(1 + \cos(0.1) + e^{0.2})$

Solution. Put $g(x) = f(1 + \cos(x) + e^{2x})$. We have $g(0) = f(1 + \cos(0) + e^0) = f(3) = -7$ and

$$g'(x) = f'(1 + \cos(x) + e^{2x})\left(-\sin(x) + 2e^{2x}\right),$$

so $g'(0) = f'(3)(2) = 4$. Hence the linearization of g at $x = 0$ is $L(x) = 4x - 7$. So

$$f(1 + \cos(0.1) + e^{0.2}) = g(0.1) \simeq L(0.1) = 4(0.1) - 7 = \boxed{-6.6}.$$

3. Find the differential dy of the following functions.

(a) $y = \arcsin(3x^2)$

Solution. $dy = \dfrac{1}{\sqrt{1-(3x^2)^2}}(6x)dx = \boxed{\dfrac{6x}{\sqrt{1-9x^4}}dx}$.

(b) $y = 4\sqrt[3]{x} - \dfrac{5}{x^2} + e^3$

Solution. $\boxed{dy = \left(\dfrac{4}{3}x^{-2/3} + 10x^{-3}\right)dx}$.

(c) $y = \csc(5\theta)$

Solution. $\boxed{dy = -5\csc(5\theta)\cot(5\theta)d\theta}$.

(d) $y = 5^{3-t^2}$

Solution. $\boxed{dy = \ln(5)5^{3-t^2}(-2t)dt}$.

(e) $y = x^{\cos(2x)}$

Solution. We have $y = e^{\cos(2x)\ln(x)}$ so

$$\boxed{dy = e^{\cos(2x)\ln(x)}\left(-2\sin(x)\ln(x) + \frac{\cos(2x)}{x}\right)dx}.$$

(f) $y = \sin(3e^{-7z})$

Solution. $\boxed{dy = -21\cos(3e^{-7z})e^{-7z}dz}$.

4. The volume of a sphere is computed by measuring its diameter.

(a) Suppose that the diameter of the sphere is measured at 5 cm with a precision of 0.2 cm. What is the percentage error propagated in the computation of the volume?

Solution. Call x the diameter of the sphere. We have $V = \frac{4}{3}\pi\left(\frac{x}{2}\right)^3 = \frac{\pi}{6}x^3$. From this we deduce

$$\begin{aligned} dV &= \frac{\pi}{2}x^2dx, \\ \Rightarrow \frac{dV}{V} &= \frac{\frac{\pi}{2}x^2dx}{\frac{\pi}{6}x^3} = 3\frac{dx}{x}. \end{aligned}$$

The relative error propagated is

$$\frac{\Delta V}{V} \simeq \frac{dV}{V} = 3\frac{dx}{x} = 3\frac{0.2}{5} = 0.12,$$

which means that the pencentage error is $\boxed{12\%}$.

(b) **[Advanced]** Suppose that we want a measurement of the volume with an error of at most 1.5%. What is the maximum percentage error that can be made measuring the diameter?

Solution. We want to have $\frac{\Delta V}{V} = 0.015$, so

$$\frac{dx}{x} \simeq \frac{1}{3}\frac{\Delta V}{V} = 0.005.$$

Therefore, the maximum percentage error that can be made measuring the diameter is $\boxed{0.5\%}$.

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Section 4.1: Extreme Values - Worksheet Solutions

1. Find the absolute extrema of the following functions on the given interval.

(a) $f(x) = 2x^3 + 3x^2 - 12x + 1$ on $[-1, 2]$.

Solution. First, we find the critical points of f in $[-1, 2]$. We have $f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1)$.

- $f'(x) = 0$ gives $x = -2, 1$.
- $f'(x)$ undefined: no x-values.

So the critical point in $[-1, 2]$ is $x = 1$. Now, we evaluate $f(x)$ at the endpoints and the critical point.

x	-1	1	2
$f(x)$	14	-6	5

Therefore, the absolute maximum of $f(x)$ on $[-1, 2]$ is $\boxed{14}$ (reached at $x = -1$) and the absolute minimum is $\boxed{-6}$ (reached at $x = -1$).

(b) $f(x) = x(7 - x)^{2/5}$ on $[1, 6]$.

Solution. First, we find the critical points of f in $[1, 6]$. We have

$$f'(x) = (7 - x)^{2/5} - \frac{2x}{5(7 - x)^{3/5}} = \frac{5(7 - x) - 2x}{5(7 - x)^{3/5}} = \frac{35 - 7x}{5(7 - x)^{3/5}}.$$

- $f'(x) = 0$ gives $35 - 7x = 0$, so $x = 5$.
- $f'(x)$ undefined gives $x = 7$.

So the critical point in $[1, 6]$ is $x = 5$. Now, we evaluate $f(x)$ at the endpoints and the critical point.

x	1	5	6
$f(x)$	$6^{2/5}$	$5 \cdot 4^{1/5}$	6

We need to determine which of these is the largest and which is the least. First, observe that $6 > 6^{2/5}$ since $\frac{2}{5} < 1$. Next, we have $6 < 5 \cdot 4^{1/5}$. To see this, we can compare the 5th power of these numbers to see that $6^5 = 7776 < 5^5 \cdot 4 = 12500$. Therefore, the absolute maximum of $f(x)$ on $[1, 6]$ is $\boxed{5 \cdot 4^{1/5}}$ (reached at $x = 5$) and the absolute minimum is $\boxed{6^{2/5}}$ (reached at $x = 1$).

(c) $f(x) = 3x^4 - 10x^3 + 6x^2 - 7$ on $[-2, 1]$.

Solution. First, we find the critical points of f in $[-2, 1]$. We have $f'(x) = 12x^3 - 30x + 12x = 6x(2x - 1)(x - 2)$.

- $f'(x) = 0$ gives $x = 0, \frac{1}{2}, 2$.

- $f'(x)$ undefined: no x-values.

So the critical points in $[-2, 1]$ are $x = 0, \frac{1}{2}$. Now, we evaluate $f(x)$ at the endpoints and the critical point.

x	-2	0	$\frac{1}{2}$	1
$f(x)$	145	-7	$-\frac{105}{16}$	-8

Therefore, the absolute maximum of $f(x)$ on $[-2, 1]$ is $\boxed{145}$ (reached at $x = -2$) and the absolute minimum is $\boxed{-8}$ (reached at $x = 1$).

(d) $f(x) = (e^x - 2)^{4/7}$ on $[0, \ln(3)]$.

Solution. First, we find the critical points of f in $[0, \ln(3)]$. We have $f'(x) = \dfrac{4e^x}{7(e^x - 2)^{3/7}}$.

- $f'(x) = 0$ gives $4e^x = 0$, which has no solution.
- $f'(x)$ undefined gives $e^x - 2 = 0$, so $x = \ln(2)$.

So the critical point in $[0, \ln(3)]$ is $x = \ln(2)$. Now, we evaluate $f(x)$ at the endpoints and the critical point.

x	0	$\ln(2)$	$\ln(3)$
$f(x)$	1	0	1

Therefore, the absolute maximum of $f(x)$ on $[0, \ln(3)]$ is $\boxed{1}$ (reached at $x = 0$ and $x = \ln(3)$) and the absolute minimum is $\boxed{0}$ (reached at $x = \ln(2)$).

(e) $f(x) = \dfrac{\ln(x)}{\sqrt{x}}$ on $[1, e^4]$.

Solution. First, we find the critical points of f in $[1, e^4]$. We have $f'(x) = \dfrac{1}{x^{3/2}} - \dfrac{\ln(x)}{2x^{3/2}} = \dfrac{2 - \ln(x)}{2x^{3/2}}$.

- $f'(x) = 0$ gives $2 - \ln(x) = 0$, so $x = e^2$.
- $f'(x)$ undefined gives no solution in the domain of f, which is $(0, \infty)$.

So the critical point in $[1, e^4]$ is $x = e^2$. Now, we evaluate $f(x)$ at the endpoints and the critical point.

x	1	e^2	2^4
$f(x)$	0	$\frac{2}{e}$	$\frac{4}{e^2}$

It is clear that the smallest of these values is $\boxed{0}$, which is the absolute minimum of f on $[1, e^4]$. To find the largest value, observe that $\frac{4}{e^2} = \left(\frac{2}{e}\right)^2$. Since $\frac{2}{e} < 1$, $\left(\frac{2}{e}\right)^2 < \frac{2}{e}$. Therefore, the absolute maximum of $f(x)$ on $[1, e^4]$ is $\boxed{\dfrac{2}{e}}$.

(f) **[Advanced]** $f(x) = 2\arctan(3x) - 3x$ on $\left[0, \frac{1}{\sqrt{3}}\right]$. *(Hint: use the approximations $\pi \simeq 3.1$ and $\sqrt{3} \simeq 1.7$).)*

Solution. First, we find the critical points of f in $\left[0, \frac{1}{\sqrt{3}}\right]$. We have $f'(x) = \dfrac{6}{1 + 9x^2} - 3 = \dfrac{3 - 27x^2}{1 + 9x^2}$.

- $f'(x) = 0$ gives $3 - 27x^2 = 0$, so $x = \frac{1}{3}, -\frac{1}{3}$.
- $f'(x)$ undefined gives $1 + 9x^2 = 0$, which has no solution.

So the critical point in $\left[0, \frac{1}{\sqrt{3}}\right]$ is $x = \frac{1}{3}$. Now, we evaluate $f(x)$ at the endpoints and the critical point.

x	0	$\frac{1}{3}$	$\frac{1}{\sqrt{3}}$
$f(x)$	0	$\frac{\pi}{2} - 1$	$\frac{2\pi}{3} - \sqrt{3}$

It is clear that the smallest of these values is $\boxed{0}$, which is the absolute minimum of f on $\left[0, \frac{1}{\sqrt{3}}\right]$. To find the largest value, observe that $\frac{\pi}{2} - 1 \simeq 0.5$ using the approximation $\pi \simeq 3.1$, and $\frac{2\pi}{3} - \sqrt{3} \simeq 2 - 1.7 = 0.3$. So the absolute maximum of f on $\left[0, \frac{1}{\sqrt{3}}\right]$ is $\boxed{\dfrac{\pi}{2} - 1}$.

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Sections 4.2-3: Mean Value Theorem and First Derivative Test - Worksheet Solutions

1. Find the values of the constants A, B for which the following function satisfies the assumptions of the Mean Value Theorem on the interval $[-2, 2]$.

$$f(x) = \begin{cases} e^{5x+B} & \text{if } x \geqslant 0 \\ \arctan(Ax+1) & \text{if } x < 0 \end{cases}$$

Solution. To satisfy the assumptions of the MVT, f must be continuous on $[-2, 2]$ and differentiable on $(-2, 2)$. Each piece of f is differentiable (therefore also continuous) so we only need to check for continuity and differentiability at $x = 0$. For continuity, we will want

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0).$$

This gives $e^B = \arctan(1) = \frac{\pi}{4}$, so $\boxed{B = \ln\left(\frac{\pi}{4}\right)}$.

For differentiability at $x = 0$, we start by computing the derivative of each piece of f:

$$f'(x) = \begin{cases} 5e^{5x+B} & \text{if } x > 0 \\ \dfrac{A}{1 + (Ax+1)^2} & \text{if } x < 0 \end{cases}$$

For f to be differentiable at 0, we need $\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^-} f'(x)$, which gives $5e^B = \frac{A}{2}$. Therefore, $A = 10e^B = 10\frac{\pi}{4} = \boxed{\frac{5\pi}{2}}$.

2. Suppose that f is continuous on $[-2, 4]$, that $f(4) = 1$ and that $f'(x) \geqslant 3$ for x in $(-2, 4)$. Find the largest possible value of $f(-2)$.

Solution. We use the MVT: there exists a point c in $(-2, 4)$ such that

$$\frac{f(4) - f(-2)}{4 - (-2)} = f'(c).$$

This gives $\frac{1 - f(-2)}{6} = f'(c)$, which is $f(-2) = 1 - 6f'(c)$. By assumption, we have $f'(c) \geqslant 3$, so $6f'(c) \geqslant 18$ and $f'(2) = 1 - 6f'(c) \leqslant -17$. So the largest possible value of $f'(2)$ is $\boxed{-17}$.

3. Find and classify the critical points of the following functions.

(a) $f(x) = x^{4/7}(72 - x^2)$

Solution. We have

$$f'(x) = \frac{4(72 - x^2)}{7x^{3/7}} - 2x^{11/7} = \frac{4(72 - x^2) - 14x^2}{7x^{3/7}} = \frac{288 - 18x^2}{7x^{3/7}} = \frac{18(4 - x)(4 + x)}{7x^{3/7}}.$$

The critical points of f are $x = 4, -4$ (where f' is 0) and $x = 0$ (where f' is undefined). We now test the sign of f' between each critical point.

- On $(-\infty, -4)$, f' is $\dfrac{(+)(-)}{(-)} = (+)$.
- On $(-4, 0)$, f' is $\dfrac{(+)(+)}{(-)} = (-)$.
- On $(0, 4)$, f' is $\dfrac{(+)(+)}{(+)} = (+)$.
- On $(-\infty, -4)$, f' is $\dfrac{(-)(+)}{(+)} = (-)$.

So f has local maxima at $x = -4, 4$ (f' changes from + to -) and a local minimum at $x = 0$ (f' changes from - to +).

(b) $f(x) = x^5 \ln(x)$

Solution. Note that the domain of f is $(0, \infty)$. We have

$$f'(x) = 5x^4 \ln(x) + x^5 \cdot \frac{1}{x} = 5x^4 \ln(x) + x^4 = x^4(5 \ln(x) + 1).$$

The critical point of f is $x = e^{-1/5}$ (where f' is 0). We now test the sign of f' on either side of the critical point.

- On $(0, e^{-1/5})$, f' is negative.
- On $(e^{-1/5}, \infty)$, f' is positive.

So f has a local minimum at $x = e^{-1/5}$ (f' changes from - to +).

(c) $f(x) = x + \cos(2x)$ on $\left[0, \dfrac{\pi}{2}\right]$

Solution. We have

$$f'(x) = 1 - 2\sin(2x).$$

Solving $f'(x) = 0$ gives $\sin(2x) = \frac{1}{2}$, which gives the solutions $x = \frac{\pi}{12}, \frac{5\pi}{12}$ on the interval $\left[0, \dfrac{\pi}{2}\right]$. We now test the sign of f' between each critical point.

- On $\left(0, \frac{\pi}{12}\right)$, f' is positive since $\sin(2x) < \frac{1}{2}$.
- On $\left(\frac{\pi}{12}, \frac{5\pi}{12}\right)$, f' is negative since$\sin(x) > \frac{1}{2}$.
- On $\left(\frac{5\pi}{12}, \frac{\pi}{2}\right)$, f' is positive since $\sin(2x) < \frac{1}{2}$.

So f has a local maximum at $x = \frac{\pi}{12}$ (f' changes from + to -) and a local minimum at $x = \frac{5\pi}{12}$ (f' changes from - to +).

(d) $f(x) = \sin^{-1}\left(e^{-x^2}\right)$

Solution. We have

$$f'(x) = \frac{1}{\sqrt{1-(e^{-x^2})^2}} e^{-x^2}(-2x) = \frac{-2xe^{-x^2}}{\sqrt{1-e^{-2x^2}}}.$$

The critical point of f is $x = 0$ (where f' is undefined). We now test the sign of f' on either side of the critical point.

- On $(-\infty, 0)$, f' is $\frac{(+)(+)}{(+)} = (+)$.
- On $(0, \infty)$, f' is $\frac{(-)(+)}{(+)} = (-)$.

So f has a local maximum at $x = 0$ (f' changes from + to -).

4. Suppose that f is continuous on $(-\infty, \infty)$ and that $f'(x) = \frac{(x+3)(x-5)^2}{x^{2/3}(x-1)^{1/5}}$.

(a) Find the critical points of f.

Solution. We have $f'(x) = 0$ when $x = -3, 5$. We have $f'(x)$ undefined when $x = 0, -$. Therefore, the critical points of f are $\boxed{x = -3, 0, 1, 5}$.

(b) Find the intervals where f is increasing and the intervals where f is decreasing.

Solution. We test for the sign of f' between the critical points.

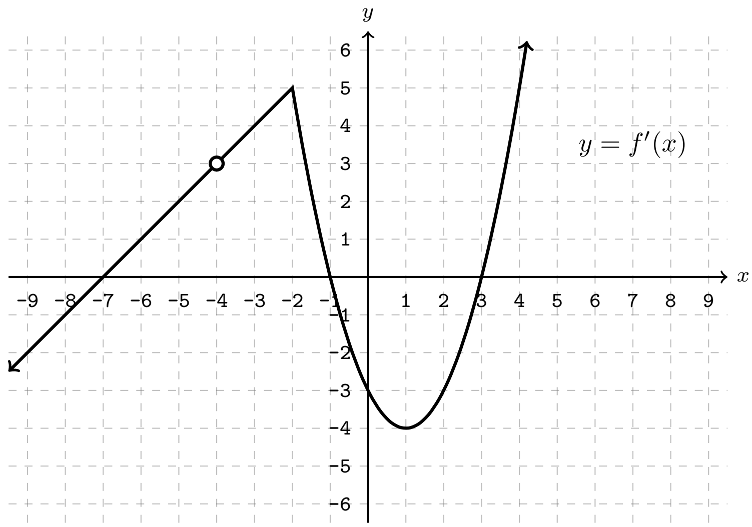
- On $(-\infty, -3)$: $\frac{(-)(+)}{(+)(-)} = (+)$
- On $(-3, 0)$: $\frac{(+)(+)}{(+)(-)} = (-)$
- On $(0, 1)$: $\frac{(+)(+)}{(+)(-)} = (-)$
- On $(1, 5)$: $\frac{(+)(+)}{(+)(+)} = (+)$
- On $(5, \infty)$: $\frac{(+)(+)}{(+)(+)} = (+)$

So f is increasing on $(-\infty, -3]$ and $[1, \infty)$, and decreasing on $[-3, 1]$.

(c) Find the location of the local extrema of f.

Solution. Based on our previous analysis, we can conclude that f has a local maximum at $x = -3$ and a local minimum at $x = 1$.

5. Suppose that f is a differentiable function. The graph of the **derivative** of f, $y = f'(x)$, is sketched below.



(a) Find the critical points of f.

Solution. Using the graph, we see that $f'(x) = 0$ when $x = -7, -1, 3$ and $f'(x)$ is undefined when $x = -4$. Therefore, the critical points of f are $\boxed{x = -7, -4, -1, 3}$.

(b) Find the intervals where f is increasing and the intervals where f is decreasing.

Solution. Using the graph, we see that $f'(x) > 0$ on $(-7, -4), (-4, -1)$ and $(3, \infty)$. So f is increasing on $[-7, -1]$ and $[3, \infty)$. Likewise, $f'(x) < 0$ on $(-\infty, -7)$ and $(-1, 3)$. So f is decreasing on $(-\infty, -7]$ and $[-1, 3]$.

(c) Find the location of the local extrema of f.

Solution. f has a local maximum at $x = -1$ (f' changes from + to -) and local minima at $x = -7, 3$ (f' changes from - to +).

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Sections 4.4: Concavity and Curve Sketching - Worksheet Solutions

1. Find the intervals where the functions below are concave up, concave down and find the inflection points.

(a) $f(x) = \dfrac{1}{x^2+12}$

Solution. We need to find the second derivative of f. We have

$$\begin{aligned} f'(x) &= -\frac{1}{(x^2+12)^2}(2x) = -2x(x^2+12)^{-2}, \\ f''(x) &= -2(x^2+12)^{-2} - 2x(-2)(x^2+12)^{-3}(2x) \\ &= -2(x^2+12)^{-3}\left((x^2+12-4x^2)\right) \\ &= -2\frac{12-3x^2}{(x^2+12)^3} \\ &= -6\frac{(2-x)(2+x)}{(x^2+12)^3}. \end{aligned}$$

We now use a sign analysis to determine the intervals on which $f''(x)$ is positive and negative.

- On $(-\infty,-2)$, the sign of $f''(x)$ is $(-)\frac{(+)(-)}{(+)} = (+)$.
- On $(-2,2)$, the sign of $f''(x)$ is $(-)\frac{(+)(+)}{(+)} = (-)$.
- On $(2,\infty)$, the sign of $f''(x)$ is $(-)\frac{(-)(+)}{(+)} = (+)$.

Therefore f is $\boxed{\text{concave up on } (-\infty,-2),(2,\infty)}$ and $\boxed{\text{concave down on } (-2,2)}$. The inflection points of f are $\boxed{\left(-2,\frac{1}{16}\right),\left(2,\frac{1}{16}\right)}$.

(b) $f(x) = x^4e^{-3x}$

Solution. We need to find the second derivative of f. We have

$$\begin{aligned} f'(x) &= 4x^3e^{-3x} - 3x^4e^{-3x} = e^{-3x}(4x^3-3x^4), \\ f''(x) &= e^{-3x}(12x^2-12x^3) - 3e^{-3x}(4x^3-3x^4) \\ &= e^{-3x}(12x^2-12x^3-12x^3+9x^4) \\ &= e^{-3x}(12x^2-24x^3+9x^4) \\ &= 3x^2e^{-3x}(4-8x+3x^2) \\ &= 3x^2e^{-3x}(3x-2)(x-2). \end{aligned}$$

We now use a sign analysis to determine the intervals on which $f''(x)$ is positive and negative.

- On $(-\infty,0)$, the sign of $f''(x)$ is $(+)$.

- On $\left(0, \frac{2}{3}\right)$, the sign of $f''(x)$ is $(+)$.
- On $\left(\frac{2}{3}, 2\right)$, the sign of $f''(x)$ is $(-)$.
- On $(2, \infty)$, the sign of $f''(x)$is $(+)$.

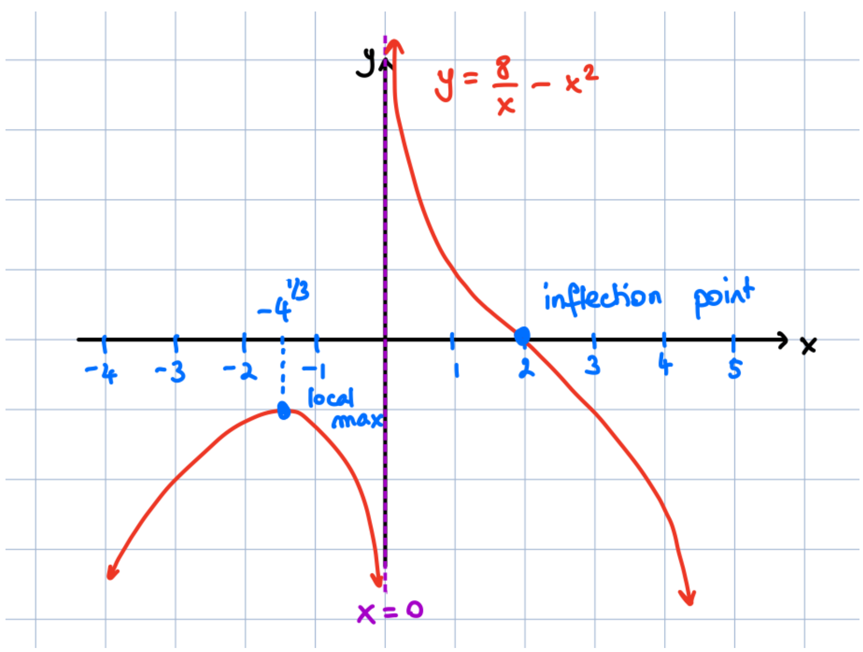
Therefore f is $\boxed{\text{concave up on } (-\infty, 0), \left(0, \dfrac{2}{3}\right), (2, \infty)}$ and $\boxed{\text{concave down on } \left(\dfrac{2}{3}, 2\right)}$. The inflection points of f are $\boxed{\left(\dfrac{2}{3}, \dfrac{16e^{-2}}{81}\right), (2, 16e^{-6})}$.

2. Sketch the graphs of the following functions. Your graph should clearly show any asymptotes, local extrema and inflection points of the functions.

(a) $f(x) = \frac{8}{x} - x^2$

Solution.

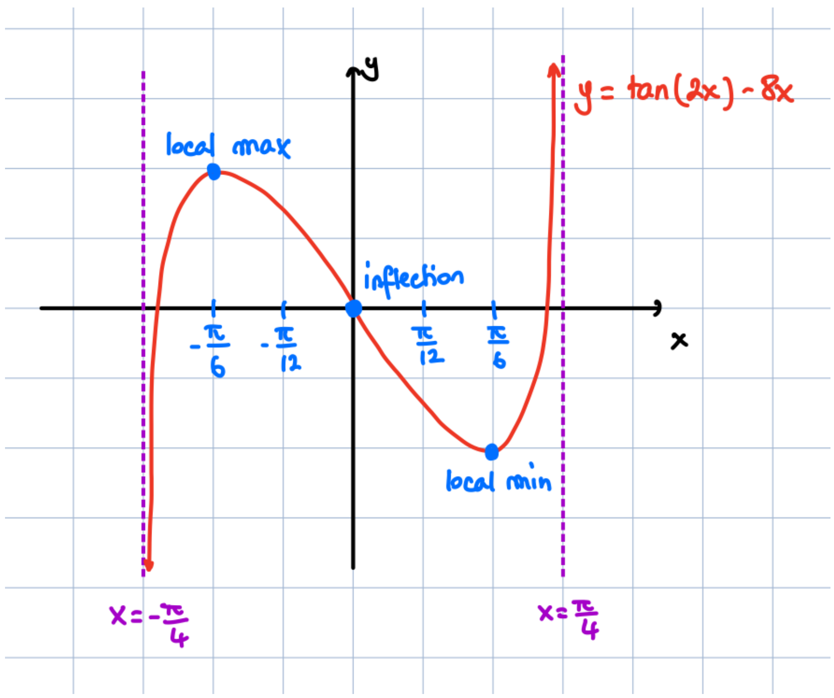
- Horizontal asymptotes: none since $\lim_{x\to\infty} f(x) = -\infty = \lim_{x\to-\infty} f(x)$.
- Vertical asymptotes: $x = 0$.
- Info from first derivative: $f'(x) = -\frac{8}{x^2} - 2x = -2\frac{4+x^3}{x^2}$. The critical point of f is $x = -4^{1/3}$, and $f'(x)$ is positive on $(-\infty, -4^{1/3})$ and negative on $(-4^{1/3}, 0)$, $(0, \infty)$. So f is increasing on $(-\infty, -4^{1/3})$ and decreasing on $(-4^{1/3}, 0)$, $(0, \infty)$. Therefore, f has a local maximum at $x = -4^{1/3}$.
- Info from second derivative: $f''(x) = \frac{16}{x^3} - 2 = 2\frac{8-x^3}{x^3}$. The sign of $f''(x)$ is positive on $(0, 2)$ and negative on $(-\infty, 0)$, $(2, \infty)$. Therefore, f is concave up on $(0, 2)$ and concave down on $(-\infty, 0)$, $(2, \infty)$, and f has an inflection point at $x = 2$.



(b) $f(x) = \tan(2x) - 8x$ on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

Solution.

- Horizontal asymptotes: none since we are graphing on a bounded interval.
- Vertical asymptotes: $x = \frac{\pi}{4}$, $x = -\frac{\pi}{4}$ (since $\tan(x)$ has infinite discontinuities at $x = \pm\frac{\pi}{2}$).
- Info from first derivative: $f'(x) = 2\sec^2(2x) - 8 = 2(\sec(2x) - 2)(\sec(2x) + 2)$. To find the critical points of f, we need to solve $f'(x) = 0$, which gives $\sec(2x) = -2$ (no solution in the interval) and $\sec(2x) = 2$ (solutions in the interval are $x = \pm\frac{\pi}{6}$). The sign of $f'(x)$ is positive (so f is increasing) on $\left(-\frac{\pi}{4}, -\frac{\pi}{6}\right)$ and $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$, and negative (so f is decreasing) on $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$. So f has a local maximum at $x = -\frac{\pi}{6}$ and a local minimum at $x = -\frac{\pi}{6}$.
- Info from second derivative: $f''(x) = 4\sec(2x)\sec(2x)\tan(2x)(2) = 8\sec^2(2x)\tan(2x)$. We see that $f''(x) > 0$ (so f is concave up) on $\left(-\frac{\pi}{4}, 0\right)$ and $f''(x) < 0$ (so f is concave down) on $\left(0, \frac{\pi}{4}\right)$. Therefore, f has an inflection point at $x = 0$.



3. Suppose that f is continuous on $(-\infty, \infty)$, that $f'(x) = \dfrac{x}{(x+4)^{1/3}}$ and that $f''(x) = \dfrac{2x+12}{(x+4)^{4/3}}$.

(a) Find the critical points of f.

Solution.

- $f'(x) = 0$ when $x = 0$.
- $f'(x)$ is undefined when $x = -4$.

Therefore, the critical points of f are $\boxed{x = -4, 0}$.

(b) Find the intervals where f is increasing and the intervals where f is decreasing.

Solution. $f'(x) > 0$ on $(-\infty, -4)$, $(0, \infty)$, and $f'(x) < 0$ on $(-4, 0)$. Therefore, f is increasing on $\boxed{(-\infty, -4], [0, \infty)}$ and decreasing on $\boxed{[-4, 0]}$.

(c) Find the location of the local extrema of f.

Solution. Based on our previous answer, f has a local maximum at $\boxed{x = -4}$ and a local minimum at $\boxed{x = 0}$.

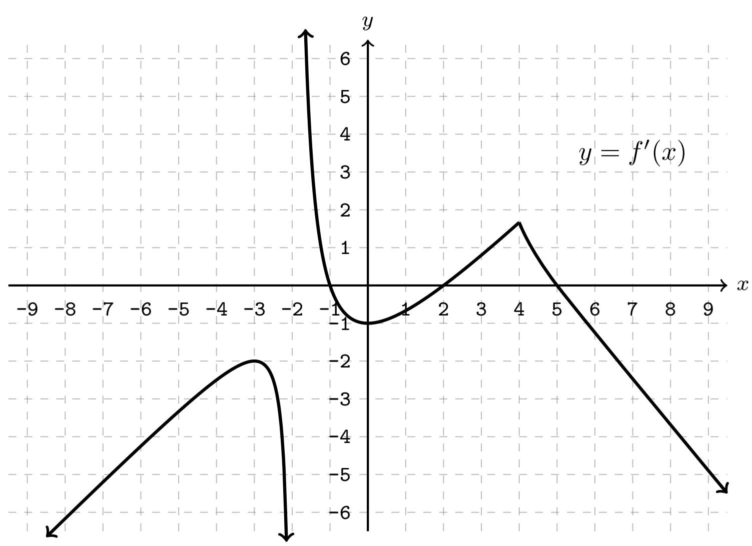
(d) Find the intervals where f is concave up and the intervals where f is concave down.

Solution. $f''(x) > 0$ on $(-6, \infty)$ and $f''(x) < 0$ on $(-\infty, -6)$. So f is concave up on $\boxed{[-6, \infty)}$ and concave down on $\boxed{(-\infty, -6]}$.

(e) Find the x-coordinates of the inflection points of f.

Solution. The only place where f changes concavity is $\boxed{x = -6}$.

4. Suppose that f is a differentiable function. The graph of the **derivative** of f, $y = f'(x)$, is sketched below.



(a) Find the critical points of f.

Solution.

- $f'(x) = 0$ when $x = -1, 2, 5$.
- $f'(x)$ is undefined when $x = -2$.

Therefore, the critical points of f are $\boxed{x = -2, -1, 2, 5}$.

(b) Find the intervals where f is increasing and the intervals where f is decreasing.

Solution. $f'(x) > 0$ is positive on $(-2, -1)$ and $(2, 5)$, so f is increasing on $\boxed{[-2, -1], [2, 5]}$. $f'(x) < 0$ on $(-\infty, -2), (-1, 2)$ and $(5, \infty)$, so f is decreasing on $\boxed{(-\infty, -2], [-1, 2], [5, \infty)}$.

(c) Find the location of the local extrema of f.

Solution. f has local maxima at $\boxed{x = -1, 5}$ (f' changes from positive to negative) and local minima at $\boxed{x = -2, 2}$ (f' changes from negative to positive).

(d) Find the intervals where f is concave up and the intervals where f is concave down.

Solution. f is concave up when f' is increasing, which happens on $\boxed{(-\infty, -3], [0, 4]}$. f is concave down when f' is decreasing, which happens on $\boxed{[-3, -2], [-2, 0], [4, \infty)}$.

(e) Find the x-coordinates of the inflection points of f.

Solution. f has inflection points at $\boxed{x = -3, 0, 4}$.

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Section 4.5: L'Hôpital's Rule - Worksheet Solutions

1. Evaluate the following limits. **Note:** L'Hôpital's Rule is not possible/necessary for every limit.

(a) $\lim_{x\to 8} \frac{\sqrt[3]{x}-2}{64-x^2}$

Solution. We can compute this limit using L'Hôpital's Rule twice with the indeterminate form $\frac{0}{0}$. This gives

$$\begin{aligned}\lim_{x\to 8} \frac{\sqrt[3]{x}-2}{64-x^2} &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to 8} \frac{\frac{1}{3}x^{-2/3}}{-2x} \\ &= \frac{\frac{1}{3}\cdot 8^{-2/3}}{-16} \\ &= \boxed{-\frac{1}{192}}.\end{aligned}$$

(b) $\lim_{x\to\infty} \frac{\ln(x)^2}{\sqrt{x}}$

Solution. We can compute this limit using L'Hôpital's Rule twice with the indeterminate form $\frac{\infty}{\infty}$. This gives

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln(x)^2}{\sqrt{x}} &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{2\ln(x)\frac{1}{x}}{\frac{1}{2\sqrt{x}}} \\ &= \lim_{x\to\infty} \frac{4\ln(x)}{\sqrt{x}} \\ &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{\frac{4}{x}}{\frac{1}{2\sqrt{x}}} \\ &= \lim_{x\to\infty} \frac{8}{\sqrt{x}} \\ &= \boxed{0}.\end{aligned}$$

(c) $\lim_{x\to 0} \frac{5^x-3^x}{\sin(2x)}$

Solution. This limit is an indeterminate form $\frac{0}{0}$. We can resolve the indeterminate form using L'Hôpital's Rule, remembering that for a positive constant a, we have

$$\frac{d}{dx}a^x = \ln(a)a^x.$$

We obtain

$$\begin{aligned}
\lim_{x\to 0} \frac{5^x - 3^x}{\sin(2x)} &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to 0} \frac{\ln(5)5^x - \ln(3)3^x}{2\cos(2x)} \\
&= \frac{\ln(5)5^0 - \ln(3)^0}{2\cos(2\cdot 0)} \\
&= \boxed{\frac{\ln(5)-\ln(3)}{2}}.
\end{aligned}$$

(d) $\displaystyle\lim_{\theta\to\frac{\pi}{2}} \frac{1-\csc(\theta)}{1-\sec(4\theta)}$

Solution. Solution. We can compute this limit using L'Hôpital's Rule twice with the indeterminate form $\frac{0}{0}$. This gives

$$\begin{aligned}
\lim_{\theta\to\frac{\pi}{2}} \frac{1-\csc(\theta)}{1-\sec(4\theta)} &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{\theta\to\frac{\pi}{2}} \frac{\csc(\theta)\cot(\theta)}{-4\sec(4\theta)\tan(4\theta)} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{\theta\to\frac{\pi}{2}} \frac{-\csc(\theta)\cot(\theta)\cot(\theta) + \csc(\theta)(-\csc^2(\theta))}{-16\sec(4\theta)\tan(4\theta)\tan(4\theta) - 16\sec(4\theta)\sec^2(4\theta)} \\
&= \lim_{\theta\to\frac{\pi}{2}} \frac{-\csc(\theta)\cot^2(\theta) - \csc^3(\theta)}{-16\sec(4\theta)\tan^2(4\theta) - 16\sec^3(4\theta)} \\
&= \frac{-1\cdot 0^2 - 1^3}{-16\cdot 1\cdot 0^2 - 16\cdot 1^3} \\
&= \boxed{\frac{1}{16}}.
\end{aligned}$$

(e) $\displaystyle\lim_{x\to\infty} \ln(5x+1) - \ln(x)$

Solution. This limit is an indeterminate form $\infty - \infty$. It can be evaluated by combining the logarithms and evaluating the limit of the inside. This gives

$$\begin{aligned}
\lim_{x\to\infty} \ln(5x+1) - \ln(x) &= \lim_{x\to\infty} \ln\left(\frac{5x+1}{x}\right) \\
&= \lim_{x\to\infty} \ln\left(5 + \frac{1}{x}\right) \\
&= \ln(5+0) \\
&= \boxed{\ln(5)}.
\end{aligned}$$

(f) $\displaystyle\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^x$

Solution. This limit is an indeterminate power 1^∞. **Warning:** limits of the form 1^∞ need not be equal to 1! This is because the base is not equal to 1, it is *approaching* 1. We can resolve the indeterminate form by rewriting the power with an exponential using the formula

$$a^b = e^{b\ln(a)}$$

and applying L'Hôpital's Rule in the resulting exponent. This gives

$$\begin{aligned}
\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x &= \lim_{x\to\infty} e^{x\ln\left(1+\frac{2}{x}\right)} \\
&= e^{\lim_{x\to\infty}\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} e^{\lim_{x\to\infty}\frac{-\frac{2}{x^2}\cdot\frac{1}{1+\frac{2}{x}}}{-\frac{1}{x^2}}} \\
&= e^{\lim_{x\to\infty}\frac{2}{1+\frac{2}{x}}} \\
&= \boxed{e^2}.
\end{aligned}$$

(g) $\lim_{x\to 0}\dfrac{2^{\sin(x)}-1}{\sin^{-1}(5x)}$

Solution. This limit is a $\frac{0}{0}$ indeterminate form, which we can evaluate using L'Hôpital's Rule. We obtain

$$\begin{aligned}
\lim_{x\to 0}\frac{2^{\sin(x)}-1}{\sin^{-1}(5x)} &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to 0}\frac{\ln(2)2^{\sin(x)}\cos(x)}{\frac{5}{\sqrt{1-(5x)^2}}} \\
&= \boxed{\frac{\ln(2)}{5}}.
\end{aligned}$$

(h) $\lim_{x\to-\infty}\dfrac{2x+3\cos(x)}{5x}$

Solution. This limit is an indeterminate form $\frac{\infty}{\infty}$. However, we **cannot use L'Hôpital's Rule** here. This is because L'Hôpital's Rule only applies if the resulting limit exists or is infinite, but here, the resulting limit

$$\lim_{x\to-\infty}\frac{2-3\sin(x)}{5}$$

does not exist. The Squeeze (or Sandwich) Theorem will work for this limit. Since $-1 \leqslant \cos(x) \leqslant 1$ for all x, we have

$$\frac{2x-3}{5x} \leqslant \frac{2x+3\cos(x)}{5x} \leqslant \frac{2x+3}{5x}$$

for any $x \neq 0$. Furthermore, we have

$$\begin{aligned}
\lim_{x\to-\infty}\frac{2x-3}{5x} &= \lim_{x\to\infty}\frac{2}{5}-\frac{3}{5x} = \frac{2}{5}, \\
\lim_{x\to-\infty}\frac{2x+3}{5x} &= \lim_{x\to\infty}\frac{2}{5}+\frac{3}{5x} = \frac{2}{5}.
\end{aligned}$$

Since the two limits are equal, we conclude that

$$\boxed{\lim_{x\to-\infty}\frac{2x+3\cos(x)}{5x} = \frac{2}{5}}.$$

(i) $\lim_{x\to\infty} x^{1/x}$

Solution. This limit is an indeterminate power ∞^0. **Warning:** limits of the form ∞^0 need not be equal to 1! We can resolve the indeterminate form by rewriting the power with an exponential using the formula

$$a^b = e^{b\ln(a)}$$

and applying L'Hôpital's Rule in the resulting exponent. This gives

$$\begin{aligned}
\lim_{x\to\infty} x^{1/x} &= \lim_{x\to\infty} e^{\frac{\ln(x)}{x}} \\
&= e^{\lim_{x\to\infty} \frac{\ln(x)}{x}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} e^{\lim_{x\to\infty} \frac{1/x}{1}} \\
&= e^0 \\
&= \boxed{1}.
\end{aligned}$$

(j) $\lim_{x\to-\infty} x^3 e^{5x+2}$

Solution. This limit is an indeterminate form $\infty \cdot 0$. We can resolve the indeterminate form by rewriting the expression as a fraction of the form $\frac{\infty}{\infty}$ and applying L'Hôpital's Rule 3 times. This gives

$$\begin{aligned}
\lim_{x\to-\infty} x^3 e^{5x+2} &= \lim_{x\to-\infty} \frac{x^3}{e^{-5x-2}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to-\infty} \frac{3x^2}{-5e^{-5x-2}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to-\infty} \frac{6x}{25e^{-5x-2}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to-\infty} \frac{6}{-125e^{-5x-2}} \\
&= \boxed{0}.
\end{aligned}$$

(k) $\lim_{x\to0^+} \sqrt[3]{x}\log_2(x)$

Solution. This limit is an indeterminate form $0 \cdot \infty$. We can resolve the indeterminate form by rewriting the expression as a fraction of the form $\frac{\infty}{\infty}$ and applying L'Hôpital's Rule. This gives

$$\begin{aligned}
\lim_{x\to0^+} \sqrt[3]{x}\log_2(x) &= \lim_{x\to0^+} \frac{\log_2(x)}{x^{-1/3}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to0^+} \frac{\frac{1}{\ln(2)x}}{-\frac{1}{3}x^{-4/3}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to0^+} \frac{-3x^{1/3}}{\ln(2)} \\
&= \boxed{0}.
\end{aligned}$$

(l) $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+4}}$

Solution. This limit is an indeterminate form $\frac{\infty}{\infty}$, but using L'Hôpital's Rule would result in an infinite loop and would not help evaluate the limit. Instead, we use algebra to cancel out the highest powers of x. We have

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+4}} &= \lim_{x\to-\infty} \frac{x}{\sqrt{x^2\left(1+\frac{4}{x^2}\right)}} \\
&= \lim_{x\to-\infty} \frac{x}{|x|\sqrt{1+\frac{4}{x^2}}} \\
&= \lim_{x\to-\infty} \frac{x}{-x\sqrt{1+\frac{4}{x^2}}} \qquad (x<0) \\
&= \lim_{x\to-\infty} \frac{1}{-\sqrt{1+\frac{4}{x^2}}} \\
&= \boxed{-1}.
\end{aligned}$$

(m) $\lim_{x\to0} \cos(3x)^{1/x^2}$

Solution. This limit is an indeterminate power 1^∞. **Warning:** limits of the form 1^∞ need not be equal to 1! This is because the base is not equal to 1, it is *approaching* 1. We can resolve the indeterminate form by rewriting the power with an exponential using the formula

$$a^b = e^{b\ln(a)}$$

and applying L'Hôpital's Rule in the resulting exponent. This gives

$$\lim_{x\to0} \cos(3x)^{1/x^2} = \lim_{x\to0} e^{\ln(\cos(3x))/x^2}$$

Now we calculate the limit of the exponent using L'Hôpital's Rule and we obtain

$$\begin{aligned}
\lim_{x\to0} \frac{\ln(\cos(3x))}{x^2} &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to0} \frac{\frac{1}{\cos(3x)}(-\sin(3x))3}{2x} \\
&= \lim_{x\to0} \frac{-3\tan(3x)}{2x} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to0} \frac{-9\sec^2(3x)}{2} \\
&= -\frac{9}{2}.
\end{aligned}$$

Going back to the original limit, we obtain

$$\lim_{x\to0} \cos(3x)^{1/x^2} = \lim_{x\to0} e^{\ln(\cos(3x))/x^2} = \boxed{e^{-9/2}}.$$

(n) $\lim_{x\to\infty} \left(\frac{x+5}{x+3}\right)^{4x}$

Solution. This limit is an indeterminate power 1^∞. **Warning:** limits of the form 1^∞ need not be equal to 1! This is because the base is not equal to 1, it is *approaching* 1. We can resolve the indeterminate form by rewriting the power with an exponential using the formula

$$a^b = e^{b\ln(a)}$$

and applying L'Hôpital's Rule in the resulting exponent. This gives

$$\lim_{x\to\infty} \left(\frac{x+5}{x+3}\right)^{4x} = \lim_{x\to\infty} e^{4x\ln\left(\frac{x+5}{x+3}\right)}.$$

We now compute the limit of the exponent using L'Hôpital's Rule:

$$\begin{aligned}
\lim_{x\to\infty} 4x\ln\left(\frac{x+5}{x+3}\right) &= \lim_{x\to\infty} 4\frac{\ln(x+5)-\ln(x+3)}{\frac{1}{x}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} 4\frac{\frac{1}{x+5}-\frac{1}{x+3}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} -4x^2\frac{(x+3)-(x+5)}{(x+5)(x+3)} \\
&= \lim_{x\to\infty} \frac{8x^2}{(x+5)(x+3)} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{8}{(1+5/x)(1+3/x)} \\
&= 8.
\end{aligned}$$

So

$$\lim_{x\to\infty} e^{4x\ln\left(\frac{x+5}{x+3}\right)} = \boxed{e^8}.$$

(o) $\lim_{x\to\infty} x^{1/\ln(x+1)}$

Solution. This limit is an indeterminate power ∞^0. **Warning:** limits of the form ∞^0 need not be equal to 1! We can resolve the indeterminate form by rewriting the power with an exponential using the formula

$$a^b = e^{b\ln(a)}$$

and applying L'Hôpital's Rule in the resulting exponent. This gives

$$\lim_{x\to\infty} x^{1/\ln(x+1)} = \lim_{x\to\infty} e^{\frac{\ln(x)}{\ln(x+1)}}$$

We now compute the limit of the exponent using L'Hôpital's Rule, an we obtain

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(x)}{\ln(x+1)} &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x+1}} \\
&= \lim_{x\to\infty} \frac{x+1}{x} \\
&= 1.
\end{aligned}$$

Therefore

$$\lim_{x\to\infty} x^{1/\ln(x+1)} = \lim_{x\to\infty} e^{\frac{\ln(x)}{\ln(x+1)}} = e^1 = \boxed{e}.$$

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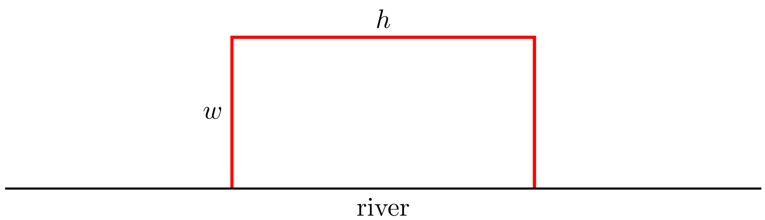
Section 4.6: Optimization - Worksheet Solutions

1. Farmer Brown wants to enclose rectangular pens for the animals on her farm. The three parts of this problem are independent.

   (a) Suppose that Farmer Brown wants to enclose a single pen alongside a river with 300 ft of fencing. The side of the pen alongside the river needs no fencing. What dimensions (length and width) would produce the pen with largest surface area?

   Solution. Call w the width and h the height of the pen, see figure below.



   The objective function is the area of the pen $A = wh$. To express this function in terms of a single variable, we use the constraint given by the fact that the amount of fencing is 300. This gives the equation $2w + h = 300$, so $h = 300 - 2w$. Therefore, the objective function in terms of the variable w is $A(w) = w(300 - 2w) = 300w - 2w^2$.

   To find the feasible interval, observe that lengths cannot be negative, so we need $w \geqslant 0$ and $h \geqslant 0$. This last inequality gives $300 - 2w \geqslant 0$, so $w \leqslant 150$. Therefore, the interval is $[0, 150]$.

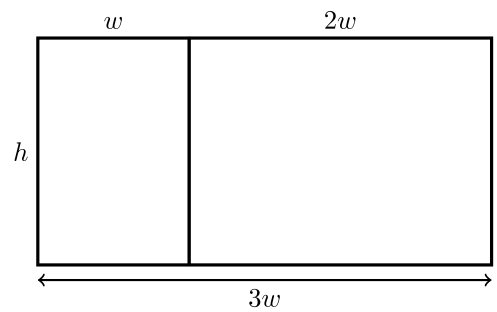
   We now use calculus to find the absolute maximum of $A(w) = 300w - 2w^2$ on $[0, 75]$. First, we find the critical points. We have $A'(w) = 300 - 4w$. The equation $A'(w) = 0$ gives the solution $w = 75$, which is the only critical point. To find the absolute maximum, we now evaluate $A(w)$ at the critical point and the endpoints.

   - $A(0) = 0$
   - $A(75) = 11250$
   - $A(150) = 0$

   Hence, the area of the pen is maximal when its width is $\boxed{w = 75 \text{ ft}}$ and its height is $h = 300 - 2w = \boxed{150 \text{ ft}}$.

   (b) Suppose that Farmer Brown has 360 ft of fencing to enclose 2 adjacent pens. Both pens have the same height, but the second one is twice as wide as the first. What is the largest total area that can be enclosed?

   Solution. Call h the height of the pens and w the width of the smaller one, see figure below.



The objective function is the total area of both pens $A = 3wh$. To express this function in terms of a single variable, we use the constraint given by the fact that the amount of fencing is 360. This gives the equation $6w + 3h = 360$, so $h = 120 - 2w$. Therefore, the objective function in terms of the variable w is $A(w) = 3w(120 - 2w) = 360w - 6w^2$.

To find the feasible interval, observe that lengths cannot be negative, so we need $w \geqslant 0$ and $h \geqslant 0$. This last inequality gives $120 - 2w \geqslant 0$, so $w \leqslant 60$. Therefore, the interval is $[0, 60]$.

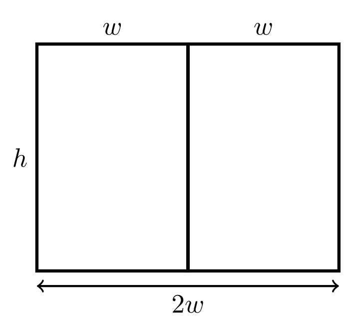
We now use calculus to find the absolute maximum of $A(w) = 360w - 6w^2$ on $[0, 60]$. First, we find the critical points. We have $A'(w) = 360 - 12w$. The equation $A'(w) = 0$ gives the solution $w = 30$, which is the only critical point. To find the absolute maximum, we now evaluate $A(w)$ at the critical point and the endpoints.

- $A(0) = 0$
- $A(75) = 5400$
- $A(60) = 0$

Therefore, the maximal total area that can be enclosed is $\boxed{5,400 \text{ ft}^2}$.

(c) Suppose that Farmer Brown wants to enclose a total of 2,400 ft^2 in two adjacent pens having the same dimensions. What is the minimal amount of fencing needed?

Solution. Call h the height and w the width of the pens, see figure below.



The objective function is the amount of fencing used (or perimeter of the figure), $P = 4w + 3h$. To express this function in terms of a single variable, we use the constraint given by the fact that the total area is 2,400 ft^2. This gives $2wh = 2400$, so $h = \frac{1200}{w}$. Therefore, the objective function in terms of the variable w is $P(w) = 4w + \frac{3600}{w}$.

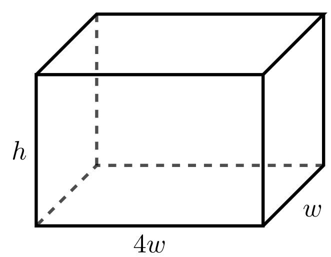
To find the feasible interval, observe that lengths cannot be negative, so we need $w \geqslant 0$ and $h \geqslant 0$. This last inequality gives $\frac{1200}{w} \geqslant 0$, so $w > 0$. Therefore, the interval is $(0, \infty)$.

We now use calculus to find the absolute minimum of $P(w) = 4w + \frac{1200}{w}$ on $(0, \infty)$. First, we find the critical points. We have $P'(w) = 4 - \frac{3600}{w^2}$. The equation $P'(w) = 0$ gives $w^2 = 900$, so $w = \pm 30$. The only critical point in the feasible interval is $w = 30$.

To determine if $w = 30$ gives a local maximum or minimum of $P(w)$, we use the SDT. We have $P''(w) = \frac{7200}{w^3}$. Since $P''(w) > 0$ on $(0, \infty)$, $P(w)$ is concave up on $0, \infty$, and therefore $w = 30$ gives a local minimum of $P(w)$. Hence, the minimal amount of fencing needed is $P(30) = \boxed{240 \text{ ft}}$.

2. A rectangular box has total surface area 216 in^2, and the length of its base is 4 times its width. Find the dimensions of such a box with largest volume.

   Solution. Call w the width of the box and h its height, see figure below.



The objective function is the volume of the box $V = h(4w)w = 4hw^2$. The constraint equation is given by the surface area being 216, which gives $2h(4w) + 2w(4w) + 2hw = 216$, or $2(5wh + 4w^2) = 216$. Solving this for h gives $5wh = 108 - 4w^2$, or $h = \frac{108-4w^2}{5w}$. Therefore, the objective function in terms of the variable w is $V(w) = 4\frac{108-4w^2}{5w}w^2 = \frac{16}{5}(27w - w^3)$.

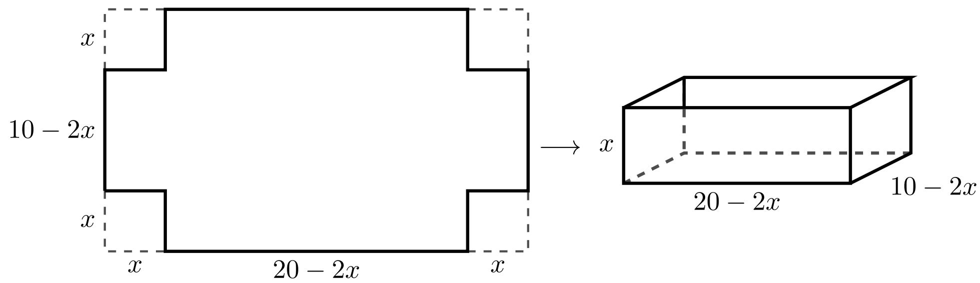
To find the feasible interval, we observe that lengths cannot be negative, so we need $w \geqslant 0$ and $h \geqslant 0$. This last inequality gives $\frac{108-4w^2}{5w} \geqslant 0$, which gives $0 < w \leqslant \sqrt{27}$. So the interval is $(0, \sqrt{27}]$.

We now use calculus to find the absolute maximum of $V(w) = 4\frac{108-4w^2}{5w}w^2 = \frac{16}{5}(27w - w^3)$ on the interval $(0, \sqrt{27}]$. We have $V'(w) = \frac{16}{5}(27 - 3w^2)$. The equation $V'(w) = 0$ gives $w^2 = 9$, so $w = \pm 3$. The only critical point in $(0, \sqrt{27}]$ is $w = 3$.

Let us use the SDT to determine whether $V(w)$ has a maximum or a minimum at $w = 3$. We have $V''(w) = \frac{16}{5}(-6w)$. Since $V''(w) < 0$ on the interval $(0, \sqrt{27}]$, $V(w)$ is concave down in $(0, \sqrt{27}]$, and therefore reaches its absolute maximum at $w = 3$. Hence, the box with largest volume has width $\boxed{3 \text{ ft}}$, height $h = \frac{108-4w^2}{5w} = \boxed{\frac{25}{4} \text{ ft}}$ and length $4w = \boxed{12 \text{ ft}}$.

3. A rectangular box is created by cutting equal size squares from the corners of a 10 in by 20 in cardboard rectangle and folding the sides. What size should the cut squares be for the resulting box to have the largest possible volume?

Solution. Call x the side length of the squares cut from the corners of the rectangle. The resulting box has height x, length $20-2x$ and width $10-2x$, see figure below.



The objective function is the volume of the box $V(x) = x(10-2x)(20-2x) = 4(x^3 - 15x^2 + 50x)$. To find the interval of interest, observe that all lengths must be postive, so we need $x \geqslant 0$, $10-2x \geqslant 0$ and $20-2x \geqslant 0$. This gives the interval $[0,5]$. We now use calculus to find the absolute maximum of $V(x) = 4(x^3 - 15x^2 + 50x)$ on $[0,5]$.

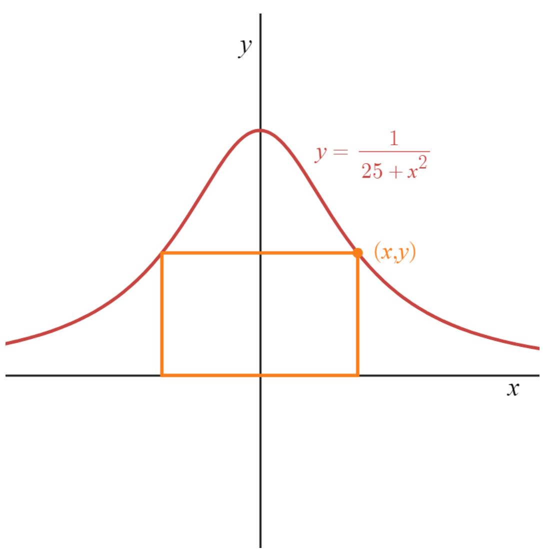
First, we find the critical points of $V(x)$ we have $V'(x) = 4(3x^2 - 30x + 50)$. Using the quadratic formula, the solutions of $V'(x) = 0$ are $x = \frac{30 \pm \sqrt{300}}{6} = 5 \pm \frac{5\sqrt{3}}{3}$. The only critical point in the interval of interest is $x = 5 - \frac{5\sqrt{3}}{3}$. We now evaluate $V(x)$ at the critical point and the endpoint.

- $V(0) = 0$
- $V\left(5 - \frac{5\sqrt{3}}{3}\right)$ is some positive value.
- $V(5) = 0$

Therefore, the volume is maximal when the square cut off from the the corners has base $x = \boxed{5 - \dfrac{5\sqrt{3}}{3} \text{ in}}$.

4. A rectangle has base on the x-axis and its two other vertices on the graph of $y = \frac{1}{25+x^2}$. Find the dimensions of such a rectangle with largest possible area.

   Solution. Call (x, y) the vertex of the rectangle on the graph in the first quadrant, see figure below.



The rectangle has base $2x$ and height y. The objective function is the area $A = 2xy$. The constraint is given by the fact that (x, y) is a point on the graph, which gives the equation $y = \frac{1}{25+x^2}$. Therefore, the objective function in terms of the variable x only is $A = \frac{2x}{25+x^2}$.

To find the feasible interval, observe that (x, y) can be any point on the graph in the first quadrant, so we have $x \geqslant 0$. This gives the interval $[0, \infty)$.

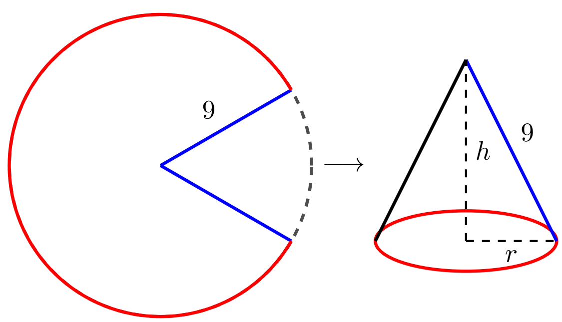
We now use calculus to find the absolute maximum of $A(x) = \frac{2x}{25+x^2}$ on the interval $[0, \infty)$. First, we find the critical points. We have

$$A'(x) = \frac{2(25 + x^2) - 2x(2x)}{(25 + x^2)^2} = \frac{50 - 2x^2}{(25 + x^2)^2} = \frac{2(5 - x)(5 + x)}{(25 + x^2)^2}.$$

The equation $A'(x) = 0$ gives the solutions $x = 5, -5$, and there are no values of x for which $A'(x)$ is undefined. Therefore, the only critical point in $[0, \infty)$ is $x = 5$. We can use the FDT to classify the critical point. When $0 \leq x < 5$, we have $A'(x) > 0$, so $A(x)$ is increasing on $[0, 5]$. When $x > 5$, we have $A'(x) < 50$, so $A(x)$ is decreasing on $[5, \infty)$. Therefore, we can conclude that the absolute maximum of $A(x)$ occurs when $x = 5$. For this value of x, the rectangle has width $2x = \boxed{10}$ and height $y = \dfrac{1}{25 + x^2} = \boxed{\dfrac{1}{50}}$.

5. A circular cone is created by cutting a circular sector from a disk of radius 9in and sealing the resulting open wedge together. What is the largest possible volume of such a cone?

Solution. Call h and r the height and radius of the resulting cone, see figure below.



The objective function is the volume of the cone $V = \frac{1}{3}\pi r^r h$. The constraint is given by the fact that the slant height of the cone is 9 - the radius of the original disk. Therefore, $r^2 + h^2 = 81$, which gives $r^2 = 81 - h^2$. Hence, the volume in terms of the variable h only is $V(h) = \frac{1}{3}\pi(81 - h^2)h = \frac{1}{3}\pi(81h - h^3)$.

To find the feasible interval, observe that the height can take any value between 0 (which occurs when we don't cut anything from the disk and get a flat cone) and 9 (which occurs when we cut off the entire disk and get just a line segment as cone).Therefore, the interval is $[0, 9]$.

We now use calculus the find the absolute maximum of $V(h) = \frac{1}{3}\pi(81h - h^3)$ on $[0, 9]$. First, we find the critical points. We have

$$V'(h) = \frac{1}{3}\pi \left(81 - 3h^2\right).$$

So $V'(h) = 0$ when $3h^2 = 81$, which gives the solutions $h = \pm\sqrt{27}$. The only solution in the feasible interval is $h = \sqrt{27}$. We now evaluate $V(h)$ at the endpoints and the critical points.

- $V(0) = 0$
- $V(\sqrt{27}) = \frac{1}{3}\pi\left(81\sqrt{27} - (\sqrt{27})^3\right) = 54\pi\sqrt{3}$
- $V(9) = 0$

Hence, the largest possible value of the volume of such a cone is $\boxed{54\pi\sqrt{3}\ \text{in}^3}$.

6. The parts of this problem are independent.

   (a) Find the point on the line $2x + y = 5$ that is closest to the origin.

   Solution. We find the absolute minimum of the square of the distance between the origin and a point (x, y) on the line. The objective function is therefore $F = x^2+y^2$, subject to the constraint $2x+y = 5$. The constraint gives $y = 5-2x$, so the objective function in terms of x only is $F(x) = x^2+(5-2x)^2$. The feasible interval is $(-\infty, \infty)$ as the point can be anywhere on the line.

   We now use calculus to find the absolute minimum of $F(x) = x^2 + (5 - 2x)^2$ on $(-\infty, \infty)$. We have $F'(x) = 2x + 2(5 - 2x)(-2) = 10x - 20$. Therefore, the only critical point of $F(x)$ is $x = 2$. Since $F''(x) = 10 > 0$, $F(x)$ is concave up on $(-\infty, \infty)$, and thus $F(x)$ reaches its absolute minimum at $x = 2$. For this value of x, we have $y = 5 - 2x = 1$. Hence, the point on the line $2x + y = 5$ closest to the origin is $\boxed{(2, 1)}$.

   (b) Find the point on the graph of $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

   Solution. We find the absolute minimum of the square of the distance between the point $(3, 0)$ and a point (x, y) on the curve. The objective function is therefore $F = (x - 3)^2 + y^2$, subject to the constraint $y = \sqrt{x}$. Using the constraint, the objective function in terms of x only is $F(x) = (x-3)^2+x$. The feasible interval is $[0, \infty)$ as the point can be anywhere on the graph of $y = \sqrt{x}$.

   We now use calculus to find the absolute minimum of $F(x) = (x - 3)^2 + x$ on $[0, \infty)$. We have $F'(x) = 2(x-3)+1 = 2x-5$. Therefore, the only critical point of $F(x)$ is $x = \frac{5}{2}$. Since $F''(x) = 2 > 0$, $F(x)$ is concave up on $[0, \infty)$, and thus $F(x)$ reaches its absolute minimum at $x = \frac{5}{2}$. For this value of x, we have $y = \sqrt{\frac{5}{2}}$. Hence, the point on the curve $y = \sqrt{x}$ closest to $(3, 0)$ is $\boxed{\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)}$.

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Section 4.8: Antiderivatives - Worksheet Solutions

1. Evaluate the following antiderivatives.

(a) $\displaystyle\int \frac{7}{1+x^2}\,dx$

Solution. $\displaystyle\int \frac{7}{1+x^2}\,dx = 7\int \frac{1}{1+x^2}\,dx = \boxed{7\tan^{-1}(x)+C}.$

(b) $\displaystyle\int \frac{3}{\sqrt{16-x^2}}\,dx$

Solution. $\displaystyle\int \frac{3}{\sqrt{16-x^2}}\,dx = 3\int \frac{1}{\sqrt{4^2-x^2}} = \boxed{3\sin^{-1}\left(\frac{x}{4}\right)+C}.$

(c) $\displaystyle\int (3x+1)\left(x^2-\frac{5}{x}\right)dx$

Solution. We fully distribute the integrand, then use the power rule. This gives

$$\begin{aligned}\int (3x+1)\left(x^2-\frac{5}{x}\right)dx &= \int \left(3x^3+x^2-15-\frac{5}{x}\right)dx \\ &= \boxed{\frac{3}{4}x^4+\frac{1}{3}x^3-15x-5\ln|x|+C}.\end{aligned}$$

(d) $\displaystyle\int \left(e^{5x}+\cos(1)\right)dx$

Solution. Warning: an antiderivative of $\cos(1)$ is **not** $\sin(1)$, because $\cos(1)$ is a constant. The correct way to integrate $\cos(1)$ with respect to x is $\cos(1)x$. With this in mind, we have

$$\int \left(e^{5x}+\cos(1)\right)dx = \boxed{\frac{1}{5}e^{5x}+\cos(1)x+C}.$$

(e) $\displaystyle\int \left(5\sqrt[7]{x^3}+\frac{4}{81+x^2}\right)dx$

Solution.

$$\begin{aligned}\int \left(5\sqrt[7]{x^3}+\frac{4}{81+x^2}\right)dx &= 5\int x^{3/7}\,dx + 4\int \frac{1}{9^2+x^2}\,dx \\ &= 5\frac{x^{10/7}}{10/7}+\frac{4}{9}\tan^{-1}\left(\frac{x}{9}\right)+C \\ &= \boxed{\frac{35}{10}x^{10/7}+\frac{4}{9}\tan^{-1}\left(\frac{x}{9}\right)+C}.\end{aligned}$$

(f) $\displaystyle\int \csc(5\theta)\left(\sin(5\theta) - \cot(5\theta)\right) d\theta$

Solution.

$$\begin{aligned}\int \csc(5\theta)\left(\sin(5\theta) - \cot(5\theta)\right) d\theta &= \int \left(\csc(5\theta)\sin(5\theta) - \csc(5\theta)\cot(5\theta)\right) d\theta \\ &= \int \left(1 - \csc(5\theta)\cot(5\theta)\right) d\theta \\ &= \boxed{\theta + \frac{1}{5}\csc(5\theta) + C}.\end{aligned}$$

(g) $\displaystyle\int \frac{7t - 11}{\sqrt{t}}\, dt$

Solution.

$$\begin{aligned}\int \frac{7t - 11}{\sqrt{t}}\, dt &= \int \left(\frac{7t}{\sqrt{t}} - \frac{11}{\sqrt{t}}\right) dt \\ &= \int \left(7t^{1/2} - 11t^{-1/2}\right) dt \\ &= 7\frac{t^{3/2}}{3/2} - 11\frac{t^{1/2}}{1/2} + C \\ &= \boxed{\frac{14}{3}t^{3/2} - 22t^{1/2} + C}.\end{aligned}$$

(h) $\displaystyle\int \left(2^x - \frac{1}{7x}\right) dx$

Solution. $\displaystyle\int \left(2^x - \frac{1}{7x}\right) dx = \boxed{\frac{2^x}{\ln(2)} - \frac{\ln|x|}{7} + C}.$

(i) $\displaystyle\int \frac{\tan(3x) + 5\sec(3x)}{\cos(3x)}\, dx$

Solution.

$$\begin{aligned}\int \frac{\tan(3x) + 5\sec(3x)}{\cos(3x)}\, dx &= \int \left(\frac{\tan(3x)}{\cos(3x)} + 5\frac{\sec(3x)}{\cos(3x)}\right) dx \\ &= \int \left(\tan(3x)\sec(3x) + 5\sec^2(3x)\right) dx \\ &= \boxed{\frac{1}{3}\sec(3x) + \frac{5}{3}\tan(3x) + C}.\end{aligned}$$

(j) $\displaystyle\int \left(\frac{1}{z^{7/4}} - \frac{3}{36 + z^2}\right) dz$

Solution.

$$\begin{aligned}\int \left(\frac{1}{z^{7/4}} - \frac{3}{36+z^2}\right) dz &= \int \left(z^{-7/4} - 3\frac{1}{6^2+z^2}\right) dz \\ &= \frac{z^{-3/4}}{-3/4} - \frac{3}{6}\tan^{-1}\left(\frac{z}{6}\right) + C \\ &= \boxed{-\frac{4}{3z^{3/4}} - \frac{1}{2}\tan^{-1}\left(\frac{z}{6}\right) + C}.\end{aligned}$$

2. Solve the following initial value problems.

(a) $\dfrac{dy}{dx} = 2 - 7x$ and $y(2) = 0$.

Solution. First, we find the general form of $y(x)$ by integrating $y'(x)$.

$$y(x) = \int (2-7x)dx = 2x - \frac{7}{2}x^2 + C.$$

Next, we find the value of the constant C by using the initial condition $y(2) = 0$. This gives

$$2 \cdot 2 - \frac{7}{2} \cdot 2^2 + C = 0 \;\Rightarrow\; -10 + C = 0 \;\Rightarrow\; C = 10.$$

Therefore, the solution of the initial value problem is $\boxed{y(x) = 2x - \frac{7}{2}x^2 + 10}$.

(b) $\dfrac{dy}{dx} = x^{-6} + \dfrac{6}{x}$ and $y(1) = 3$.

Solution. First, we find the general form of $y(x)$ by integrating $y'(x)$.

$$y(x) = \int \left(x^{-6} + \frac{6}{x}\right) dx = \frac{x^{-5}}{-5} + 6\ln|x| + C.$$

Next, we find the value of the constant C by using the initial condition $y(1) = 3$. This gives

$$\frac{1^{-5}}{-5} + 6\ln|1| + C = 3 \;\Rightarrow\; -\frac{1}{5} + C = 3 \;\Rightarrow\; C = \frac{16}{5}.$$

Therefore, the solution of the initial value problem is $\boxed{y(x) = \frac{x^{-5}}{-5} + 6\ln|x| + \frac{16}{5}}$.

(c) $\dfrac{dy}{dx} = \dfrac{5}{9+x^2}$ and $y(3) = -1$.

Solution. First, we find the general form of $y(x)$ by integrating $y'(x)$.

$$y(x) = \int \frac{5}{9+x^2} dx = \frac{5}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

Next, we find the value of the constant C by using the initial condition $y(3) = -1$. This gives

$$\frac{5}{3}\tan^{-1}(1) + C = -1 \;\Rightarrow\; \frac{5\pi}{12} + C = -1 \;\Rightarrow\; C = -1 - \frac{5\pi}{12}.$$

Therefore, the solution of the initial value problem is $\boxed{y(x) = \frac{5}{3}\tan^{-1}\left(\frac{x}{3}\right) - 1 - \frac{5\pi}{12}}$.

(d) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{64 - x^2}}$ and $y(-4) = 0$.

Solution. First, we find the general form of $y(x)$ by integrating $y'(x)$.

$$y(x) = \int \frac{1}{\sqrt{64 - x^2}} dx = \sin^{-1}\left(\frac{x}{8}\right) + C.$$

Next, we find the value of the constant C by using the initial condition $y(-4) = 0$. This gives

$$\sin^{-1}\left(-\frac{1}{2}\right) + C = 0 \Rightarrow -\frac{\pi}{6} + C = 0 \Rightarrow C = \frac{\pi}{6}.$$

Therefore, the solution of the initial value problem is $\boxed{y(x) = \sin^{-1}\left(\frac{x}{8}\right) + \frac{\pi}{6}}$.

(e) $\dfrac{d^2y}{dx^2} = 3 - e^{2x}$, $y'(0) = 1$ and $y(0) = 7$.

Solution. We first solve the initial value problem $\dfrac{dy'}{dx} = 3 - e^{2x}$, $y'(0) = 1$ to find $y'(x)$. The general form of $y'(x)$ is

$$y'(x) = \int \left(3 - e^{2x}\right) dx = 3x - \frac{e^{2x}}{2} + C.$$

To find the value of the constant C, we use the initial condition $y'(0) = 1$. This gives

$$3 \cdot 0 - \frac{e^0}{2} + C = 1 \Rightarrow -\frac{1}{2} + C = 1 \Rightarrow C = \frac{3}{2}.$$

Therefore, $y'(x) = 3x - \dfrac{e^{2x}}{2} + \dfrac{3}{2}$. We can now find $y(x)$ by solving the initial value problem $\dfrac{dy}{dx} = 3x - \dfrac{e^{2x}}{2} + \dfrac{3}{2}$, $y(0) = 7$. We have

$$y(x) = \int \left(3x - \frac{e^{2x}}{2} + \frac{3}{2}\right) = \frac{3x^2}{2} - \frac{e^{2x}}{4} + \frac{3x}{2} + D.$$

To find the value of the constant D, we use the initial condition $y(0) = 7$. This gives

$$\frac{\cdot 0}{2} - \frac{e^0}{4} + \frac{3 \cdot 0}{2} + D = 7 \Rightarrow -\frac{1}{4} + D = 7 \Rightarrow D = \frac{29}{4}.$$

Therefore $\boxed{y(x) = \frac{3x^2}{2} - \frac{e^{2x}}{4} + \frac{3x}{2} + \frac{29}{4}}$.

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Sections 5.1-2: Areas Estimations and Riemann Sums - Worksheet Solutions

1. (a) Approximate the net area between the graph of $f(x) = 9 - x^2$ and the x-axis on $[-1, 3]$ using 4 rectangles of equal width and (i) left endpoints, (ii) right endpoints.

Solution. Partitioning the interval $[-1, 3]$ into 4 subintervals of equal length will give us subintervals of length $\dfrac{3-(-1)}{4} = 1$. Therefore, we get the 4 subintervals $[-1, 0]$, $[0, 1]$, $[1, 2]$ and $[2, 3]$.

(i) Picking the value at the left endpoint for the height of the rectangles gives the sum

$$\begin{aligned} A &\simeq f(-1)\cdot 1 + f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 \\ &= (9-(-1)^2) + (9-0^2) + (9-1^2) + (9-2^2) \\ &= \boxed{30 \text{ square units}}. \end{aligned}$$

(ii) Picking the value at the right endpoint for the height of the rectangles gives the sum

$$\begin{aligned} A &\simeq f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 \\ &= (9-0^2) + (9-1^2) + (9-2^2) + (9-3^2) \\ &= \boxed{22 \text{ square units}}. \end{aligned}$$

(b) Approximate the net area between the graph of $f(x) = 2\cos(x)$ and the x-axis on $\left[0, \frac{\pi}{2}\right]$ using 3 rectangles of equal width and (i) left endpoints, (ii) right endpoints.

Solution. Partitioning the interval $\left[0, \frac{\pi}{2}\right]$ into 3 subintervals of equal length will give us subintervals of length $\frac{\frac{\pi}{2}-0}{3} = \frac{\pi}{6}$. Therefore, we get the 4 subintervals $\left[0, \frac{\pi}{6}\right]$, $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ and $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$.

(i) Picking the value at the left endpoint for the height of the rectangles gives the sum

$$\begin{aligned} A &\simeq f(0)\cdot\frac{\pi}{6} + f\left(\frac{\pi}{6}\right)\cdot\frac{\pi}{6} + f\left(\frac{\pi}{3}\right)\cdot\frac{\pi}{6} \\ &= \frac{\pi}{6}\left(2\cos(0) + 2\cos\left(\frac{\pi}{6}\right) + 2\cos\left(\frac{\pi}{3}\right)\right) \\ &= \frac{\pi}{6}\left(2 + 2\frac{\sqrt{3}}{2} + 2\frac{1}{2}\right) \\ &= \boxed{\frac{\pi(3+\sqrt{3})}{6} \text{ square units}}. \end{aligned}$$

(ii) Picking the value at the right endpoint for the height of the rectangles gives the sum

$$A \simeq f\left(\frac{\pi}{6}\right)\cdot\frac{\pi}{6} + f\left(\frac{\pi}{3}\right)\cdot\frac{\pi}{6} + f\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{6}$$

$$= \frac{\pi}{6}\left(2\cos\left(\frac{\pi}{6}\right) + 2\cos\left(\frac{\pi}{3}\right) + 2\cos\left(\frac{\pi}{2}\right)\right)$$

$$= \frac{\pi}{6}\left(2\frac{\sqrt{3}}{2} + 2\frac{1}{2} + 2\cdot 0\right)$$

$$= \boxed{\frac{\pi(1+\sqrt{3})}{6} \text{ square units}}.$$

2. Suppose that the function f has the following values.

$$f(0) = 3,\ f(1) = 7,\ f(2) = 5,\ f(3) = 1,\ f(4) = 2,\ f(5) = 8,$$
$$f(6) = 0,\ f(7) = 1,\ f(8) = 5,\ f(9) = 3,\ f(10) = 1.$$

Approximate the net area between the graph of $g(x) = f(8x+2)$ and the x-axis on the interval $[0,1]$ using a midpoint sum with 4 rectangles of equal width.

Solution. Dividing the interval $[0,1]$ into 4 subintervals of equal width gives the intervals $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{3}{4}\right]$ and $\left[\frac{3}{4}, 1\right]$. We will pick the height using the value at the midpoint of each interval, that is $x = \frac{1}{8}$, $x = \frac{3}{8}$, $x = \frac{5}{8}$ and $x = \frac{7}{8}$. We get the approximation

$$A \simeq g\left(\frac{1}{8}\right)\frac{1}{4} + g\left(\frac{3}{8}\right)\frac{1}{4} + g\left(\frac{5}{8}\right)\frac{1}{4} + g\left(\frac{7}{8}\right)\frac{1}{4}$$

$$= \frac{1}{4}\left(f(3) + f(5) + f(7) + f(9)\right)$$

$$= \boxed{\frac{13}{4} \text{ square units}}.$$

3. Evaluate the following sums.

(a) $\displaystyle\sum_{k=0}^{5} \frac{k(k-1)}{2}$.

Solution.

$$\sum_{k=0}^{5} \frac{k(k-1)}{2} = \frac{0(0-1)}{2} + \frac{1(1-1)}{2} + \frac{2(2-1)}{2} + \frac{3(3-1)}{2} + \frac{4(4-1)}{2} + \frac{5(5-1)}{2}$$

$$= \boxed{20}.$$

(b) $\displaystyle\sum_{j=1}^{4} \cos(j\pi)j$.

Solution.

$$\sum_{j=1}^{4} \cos(j\pi)j = \cos(\pi) + 2\cos(2\pi) + 3\cos(3\pi) + 4\cos(4\pi)$$

$$= -1 + 2 - 3 + 4$$

$$= \boxed{2}.$$

(c) $\sum_{n=1}^{5} \left(\frac{1}{n} - \frac{1}{n+1}\right)$.

Solution.

$$\begin{aligned}\sum_{n=1}^{5} \left(\frac{1}{n} - \frac{1}{n+1}\right) &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) \\ &= 1 - \frac{1}{6} \\ &= \boxed{\frac{5}{6}}.\end{aligned}$$

4. Consider the sum $2 + 4 + 8 + 16 + 32 + 64$.

(a) Write the sum in sigma notation with the index starting at the value 1.

(b) Write the sum in sigma notation with the index starting at the value 0.

(c) Write the sum in sigma notation with the index starting at the value 3.

Solution.

$$\boxed{2 + 4 + 8 + 16 + 32 + 64 = \sum_{k=1}^{6} 2^k = \sum_{k=0}^{5} 2^{k+1} = \sum_{k=3}^{8} 2^{k-2}}.$$

5. Use the common sum formulas

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4},$$

to evaluate the following sums.

(a) $\sum_{k=1}^{136} (2k - 3)$.

Solution.

$$\begin{aligned}\sum_{k=1}^{136} (2k-3) &= 2\sum_{k=1}^{136} k - \sum_{k=1}^{136} 3 \\ &= 2\frac{136(137)}{2} - 3 \cdot 136 \\ &= \boxed{18224}.\end{aligned}$$

(b) $\sum_{j=2}^{20} j^2(j-4)$.

Solution.

$$\sum_{j=2}^{20} j^2(j-4) = \sum_{j=2}^{20} (j^3 - 4j^2)$$

$$
\begin{aligned}
&= \sum_{j=2}^{20} j^3 - 4\sum_{j=2}^{20} j^2 \\
&= \sum_{j=1}^{20} j^3 - 1^3 - 4\left(\sum_{j=1}^{20} j^2 - 1^2\right) \\
&= \frac{20^2 \cdot 21^2}{4} - 1 - 4\frac{20 \cdot 21 \cdot 41}{6} + 4 \\
&= \boxed{55583}.
\end{aligned}
$$

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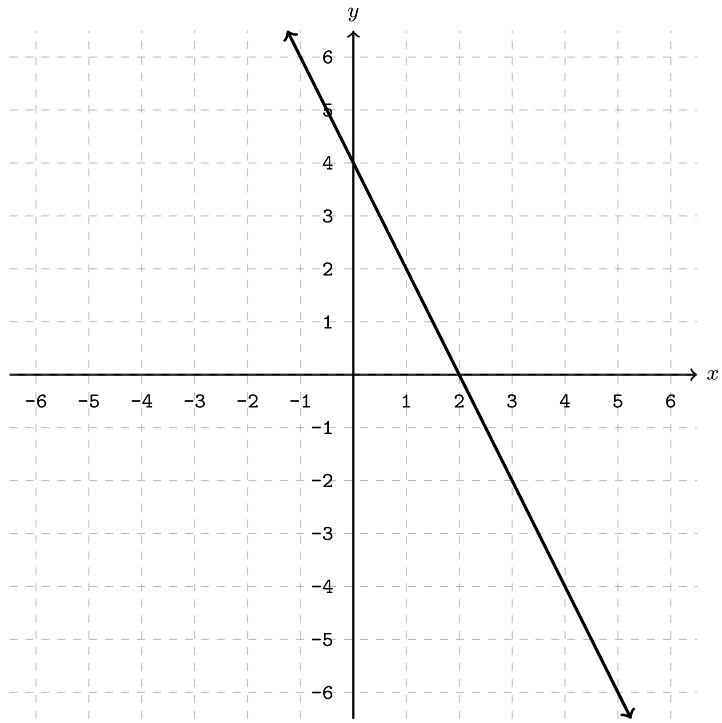
Section 5.3: Definite Integrals - Worksheet Solutions

1. Let $f(x) = 4 - 2x$. We are going to calculate $\int_0^2 f(x)dx$ using two methods.

   (a) Geometric method.

   (i) Sketch the graph of $y = f(x)$.

   Solution.



   (ii) Use your graph and a geometric formula to calculate $\int_0^2 f(x)dx$.

   Solution. $\int_0^2 f(x)dx$ is the area of a triangle with base 2 and height 4, so $\int_0^2 f(x)dx = \frac{1}{2} \cdot 2 \cdot 4 = \boxed{4}$

   (b) With Riemann sums.

   (i) Calculate R_n, the right-endpoint Riemann sum of f on $[0, 2]$ with n rectangles. Your answer should not contain the Σ or $\cdots$ symbols. *Hint: you will need to use the reference sum*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Solution. We have $\Delta x = \dfrac{b-a}{n} = \dfrac{2}{n}$ and

$$\begin{aligned}
R_n &= \sum_{k=1}^{n} f(a + k\Delta x)\Delta x \\
&= \sum_{k=1}^{n} \left(4 - 2\frac{2k}{n}\right)\frac{2}{n} \\
&= \sum_{k=1}^{n} \left(\frac{8}{n} - \frac{8k}{n}\right) \\
&= \left(\sum_{k=1}^{n} \frac{8}{n}\right) - \frac{8}{n}\sum_{k=1}^{n} k \\
&= \frac{8}{n} \cdot n - \frac{8}{n} \cdot \frac{n(n+1)}{2} \\
&= 8 - \frac{4(n+1)}{n} \qquad = \boxed{8 - 4\left(1 + \frac{1}{n}\right)}
\end{aligned}$$

(ii) Using your formula for R_n, calculate $\int_0^2 f(x)dx$.

Solution.

$$\int_0^2 f(x)dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} 8 - 4\left(1 + \frac{1}{n}\right) = 8 - 4 = \boxed{4}.$$

2. Write each limit below as the integral of a function $f(x)$ on an interval $[0, b]$.

(a) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{\frac{3k}{n} + 5}\ \frac{3}{n}$.

Solution.

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{\frac{3k}{n} + 5}\ \frac{3}{n} = \int_0^3 \sqrt{x+5}dx}.$$

(b) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} e^{12k/n}\frac{8}{n}$.

Solution.

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} e^{12k/n}\frac{8}{n} = \int_0^8 e^{3x/2}dx}.$$

(c) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \sin\left(\frac{k^3}{n^3}\right)\frac{2}{n}$.

Solution.

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} \sin\left(\frac{k^3}{n^3}\right)\frac{2}{n} = \int_0^2 \sin\left(\frac{x^3}{8}\right) dx}.$$

(d) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2n+5k}$.

Solution. We can write the sum in the format of a Riemann sum by factoring out an n from the denominator, which gives

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2n+5k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2+5\frac{k}{n}} \cdot \frac{1}{n} = \boxed{\int_0^1 \frac{dx}{2+5x}}.$$

3. Suppose that f and g are functions such that

$$\int_{-2}^{0} f(x)dx = 4, \quad \int_{-2}^{5} f(x)dx = -1, \quad \int_{-2}^{5} g(x)dx = 10.$$

Evaluate the following integrals.

(a) $\displaystyle\int_{-2}^{5} \frac{g(x)}{2} dx$

Solution.

$$\int_{-2}^{5} \frac{g(x)}{2} dx = \frac{1}{2}\int_{-2}^{5} g(x)dx = \boxed{5}.$$

(b) $\displaystyle\int_{-2}^{5} (2g(x) - 3f(x))dx$

Solution.

$$\int_{-2}^{5} (2g(x) - 3f(x))dx = 2\int_{-2}^{5} g(x)dx - 3\int_{-2}^{5} f(x)dx = 2 \cdot 10 - 3(-1) = \boxed{23}.$$

(c) $\displaystyle\int_{0}^{5} 7f(x)dx$

Solution.

$$\int_{0}^{5} 7f(x)dx = 7\int_{0}^{5} f(x)dx = 7\left(\int_{-2}^{5} f(x)dx - \int_{-2}^{0} f(x)dx\right) = 7(-1-4) = \boxed{-35}.$$

(d) $\int_5^{-2} (f(x) + 4g(x))\, dx$

Solution.

$$\begin{aligned}\int_5^{-2} (f(x) + 4g(x))\, dx &= \int_5^{-2} f(x)dx + 4\int_5^{-2} g(x)dx \\ &= -\int_{-2}^{5} f(x)dx - 4\int_{-2}^{5} g(x)dx \\ &= -(-1) - 4\cdot 10 \\ &= \boxed{-39}.\end{aligned}$$

(e) $\int_{-2}^{0} (2x + f(x) - 1)\, dx$

Solution. First, observe that

$$\int_{-2}^{0} (2x + f(x) - 1)\, dx = \int_{-2}^{0} 2xdx + \int_{-2}^{0} f(x)dx - \int_{-2}^{0} 1dx.$$

The integral $\int_{-2}^{0} 2xdx$ gives the area of a triangle of base 2 and height 4 located below the y-axis, therefore $\int_{-2}^{0} 2xdx = -4$. We know that $\int_{-2}^{0} f(x)dx = 4$ and $\int_{-2}^{0} 1dx = 2$. Hence

$$\int_{-2}^{0} (2x + f(x) - 1)\, dx = -4 + 4 - 2 = \boxed{-2}.$$

(f) $\int_5^{0} \left(f(x) - 4\sqrt{25 - x^2}\right) dx$

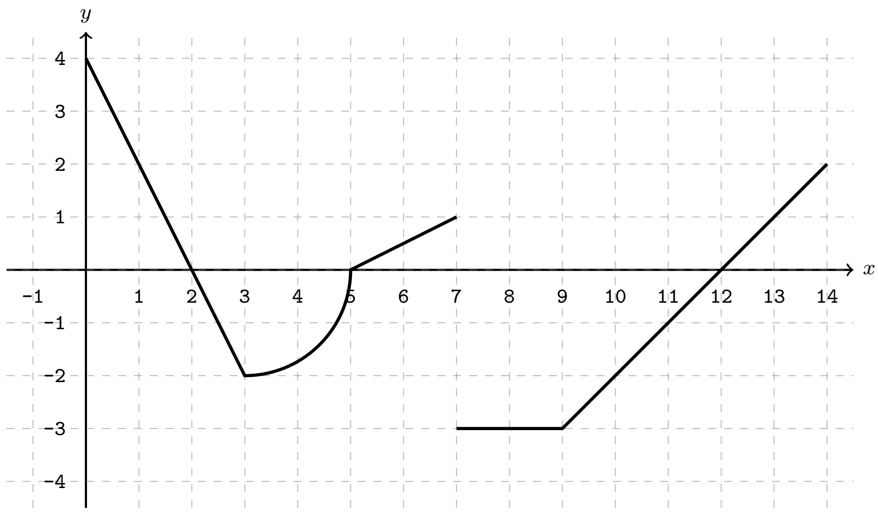
Solution. First, we have

$$\begin{aligned}\int_5^{0} \left(f(x) - 4\sqrt{25 - x^2}\right) dx &= \int_5^{0} f(x)dx - 4\int_5^{0} \sqrt{25 - x^2}dx \\ &= -\int_0^{5} f(x)dx + 4\int_0^{5} \sqrt{25 - x^2}dx.\end{aligned}$$

We know that $\int_0^5 f(x)dx = -5$. The graph of $y = \sqrt{25 - x^2}$ is a semi circle of radius 5 centered at $(0, 0)$. Thus, $\int_0^5 \sqrt{25 - x^2}dx$ is the area of a quarter disk of radius 5, that is $\frac{25\pi}{4}$. We obtain

$$\int_5^{0} \left(f(x) - 4\sqrt{25 - x^2}\right) dx = -(-5) + 4\frac{25\pi}{4} = \boxed{5 + 25\pi}.$$

4. Let f be the function whose graph is sketched below. You can assume that each piece of the graph of f is either a straight line or a circle arc.



Calculate the following integrals.

(a) $\int_0^5 f(x)dx$

Solution. $\int_0^5 f(x)dx = 4 - 1 - \pi = \boxed{3 - \pi}$.

(b) $\int_3^9 (3 - f(x))dx$

Solution. $\int_3^9 (3 - f(x))dx = \int_3^9 3dx - \int_3^9 f(x)dx = 3 \cdot 6 - (-\pi + 1 - 6) = \boxed{23 + \pi}$.

(c) $\int_{12}^5 f(x)dx$

Solution. $\int_{12}^5 f(x)dx = -\int_5^{12} f(x)dx = -\left(1 - 6 - \frac{9}{2}\right) = \boxed{\frac{19}{2}}$

(d) $\int_7^{14} |f(x)|dx$

Solution. $\int_7^{14} |f(x)|dx = 6 + \frac{9}{2} + 2 = \boxed{\frac{25}{2}}$

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Section 5.4: Fundamental Theorem of Calculus - Worksheet Solutions

1. Evaluate the following definite integrals.

(a) $\displaystyle\int_1^3 \frac{3x^2-2x+1}{x}\,dx$

Solution.

$$\begin{aligned}\int_1^3 \frac{3x^2-2x+1}{x}\,dx &= \int_1^3 \left(3x-2+\frac{1}{x}\right)dx \\ &= \left[\frac{3x^2}{2}-2x+\ln|x|\right]_1^3 \\ &= \left(\frac{3\cdot 3^2}{2}-2\cdot 3+\ln(3)\right)-\left(\frac{3}{2}-2+\ln(1)\right) \\ &= \boxed{\ln(3)-1}\end{aligned}$$

(b) $\displaystyle\int_0^{1/2} \frac{dt}{\sqrt{1-t^2}}$

Solution.

$$\begin{aligned}\int_0^{1/2} \frac{dt}{\sqrt{1-t^2}} &= \left[\sin^{-1}(t)\right]_0^{1/2} \\ &= \sin^{-1}\left(\frac{1}{2}\right)-\sin^{-1}(0) \\ &= \boxed{\frac{\pi}{6}}.\end{aligned}$$

(c) $\displaystyle\int_0^{\ln(2)} (e^x+1)^2\,dx$

Solution.

$$\begin{aligned}\int_0^{\ln(2)} (e^x+1)^2\,dx &= \int_0^{\ln(2)} \left(e^{2x}+2e^x+1\right)dx \\ &= \left[\frac{e^2x}{2}+2e^x+x\right]_0^{\ln(2)} \\ &= \left(\frac{e^{2\ln(2)}}{2}+2e^{\ln(2)}+\ln(2)\right)-\left(\frac{e^0}{2}+2e^0+0\right) \\ &= \boxed{\frac{7}{2}+\ln(2)}.\end{aligned}$$

(d) $\displaystyle\int_{\pi/30}^{\pi/20} \sec^2(5\theta)d\theta$

Solution.

$$\begin{aligned}\int_{\pi/30}^{\pi/20} \sec^2(5\theta)d\theta &= \left[\frac{1}{5}\tan(5\theta)\right]_{\pi/30}^{\pi/20} \\ &= \frac{1}{5}\tan\left(\frac{\pi}{4}\right) - \frac{1}{5}\tan\left(\frac{\pi}{6}\right) \\ &= \frac{1}{5} - \frac{\sqrt{3}}{15} \\ &= \boxed{\frac{3-\sqrt{3}}{15}}.\end{aligned}$$

(e) $\displaystyle\int_{-3}^{\sqrt{3}} \frac{4}{x^2+3}dx$

Solution.

$$\begin{aligned}\int_{-3}^{\sqrt{3}} \frac{4}{x^2+3}dx &= \left[\frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_{-3}^{\sqrt{3}} \\ &= \frac{4}{\sqrt{3}}\left(\tan^{-1}(1) - \tan^{-1}\left(-\sqrt{3}\right)\right) \\ &= \frac{4\sqrt{3}}{12}\left(\frac{\pi}{4} - \left(-\frac{\pi}{3}\right)\right) \\ &= \boxed{\frac{28\sqrt{3}\pi}{144}}.\end{aligned}$$

(f) $\displaystyle\int_0^5 \frac{dz}{4z+7}$

Solution.

$$\begin{aligned}\int_0^5 \frac{dz}{4z+7} &= \left[\frac{1}{4}\ln|4z+7|\right]_0^5 \\ &= \frac{1}{4}(\ln(27) - \ln(7)) \\ &= \boxed{\frac{1}{4}\ln\left(\frac{27}{7}\right)}.\end{aligned}$$

(g) $\displaystyle\int_1^4 \sqrt{x}\left(x - \frac{4}{x}\right)dx$

Solution.

$$\begin{aligned}\int_1^4 \sqrt{x}\left(x - \frac{4}{x}\right) dx &= \int_1^4 \left(x^{3/2} - 4x^{-1/2}\right) dx \\ &= \left[\frac{x^{5/2}}{5/2} - 4\frac{x^{1/2}}{1/2}\right]_1^4 \\ &= \left(\frac{4^{5/2}}{5/2} - 4\frac{4^{1/2}}{1/2}\right) - \left(\frac{1}{5/2} - \frac{4}{1/2}\right) \\ &= \left(\frac{64}{5} - 16\right) - \left(\frac{2}{5} - 8\right) \\ &= \boxed{\frac{22}{5}}.\end{aligned}$$

(h) $\displaystyle\int_0^{2\pi} \left(\sin\left(\frac{x}{3}\right) + 1\right) d\theta$

Solution.

$$\begin{aligned}\int_0^{2\pi} \left(\sin\left(\frac{x}{3}\right) + 1\right) d\theta &= \left[-3\cos\left(\frac{\theta}{3}\right) + \theta\right]_0^{2\pi} \\ &= \left(-3\cos\left(\frac{2\pi}{3}\right) + 2\pi\right) - (-3\cos(0) + 0) \\ &= \frac{3}{2} + 2\pi + 3 \\ &= \boxed{\frac{9}{2} + 2\pi}.\end{aligned}$$

(i) $\displaystyle\int_{\sqrt{2}}^2 \frac{5}{3x\sqrt{x^2-1}} dx$

Solution.

$$\begin{aligned}\int_{\sqrt{2}}^2 \frac{5}{3x\sqrt{x^2-1}} dx &= \left[\frac{5}{3}\sec^{-1}|x|\right]_{\sqrt{2}}^2 \\ &= \frac{5}{3}\left(\sec^{-1}(2) - \sec^{-1}(\sqrt{2})\right) \\ &= \frac{5}{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \boxed{\frac{5\pi}{36}}.\end{aligned}$$

2. Evaluate the following derivatives.

(a) $\displaystyle\frac{d}{dx}\left(\int_4^x \sqrt{t^4+1}dt\right)$

Solution. $\displaystyle\frac{d}{dx}\left(\int_4^x \sqrt{t^4+1}dt\right) = \boxed{\sqrt{x^4+1}}$

(b) $\dfrac{d}{dx}\left(\displaystyle\int_x^0 \sec(5t^2)dt\right)$

Solution. $\dfrac{d}{dx}\left(\displaystyle\int_x^0 \sec(5t^2)dt\right) = -\dfrac{d}{dx}\left(\displaystyle\int_0^x \sec(5t^2)dt\right) = \boxed{-\sec(5x^2)}.$

(c) $\dfrac{d}{dx}\left(\displaystyle\int_1^{2x} \frac{dt}{t^3+t+1}\right)$

Solution. $\dfrac{d}{dx}\left(\displaystyle\int_1^{2x} \frac{dt}{t^3+t+1}\right) = \dfrac{1}{(2x)^3+2x+1}(2) = \boxed{\dfrac{2}{8x^2+2x+1}}.$

(d) $\dfrac{d}{dx}\left(\displaystyle\int_{3x^2}^{7} (t^4+2)^{3/4}dt\right)$

Solution. $\dfrac{d}{dx}\left(\displaystyle\int_{3x^2}^{7} (t^4+2)^{3/4}dt\right) = -\dfrac{d}{dx}\left(\displaystyle\int_7^{3x^2} (t^4+2)^{3/4}dt\right) = -((3x^2)^4+2)^{3/4}(6x) = \boxed{-6x(81x^8+2)^{3/4}}.$

(e) $\dfrac{d}{dx}\left(\displaystyle\int_{\tan(2x)}^{\sec(2x)} \cos(\sqrt{t})dt\right)$

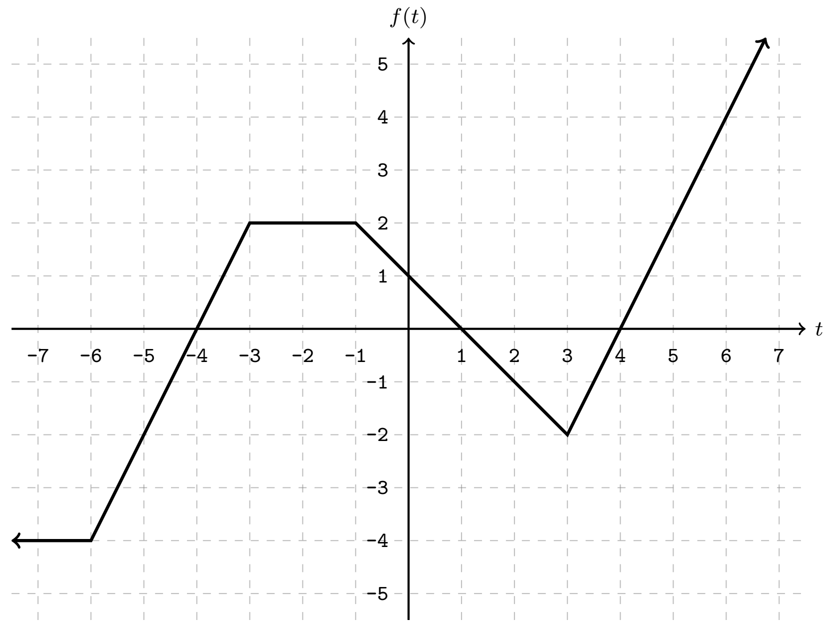
Solution.

$$\begin{aligned}\frac{d}{dx}\left(\int_{\tan(2x)}^{\sec(2x)} \cos(\sqrt{t})dt\right) &= \frac{d}{dx}\left(\int_0^{\sec(2x)} \cos(\sqrt{t})dt - \int_0^{\tan(2x)} \cos(\sqrt{t})dt\right)\\ &= \boxed{\cos(\sqrt{\sec(2x)})\sec(2x)\tan(2x)(2) - \cos(\sqrt{\tan(2x)})\sec^2(2x)(2)}.\end{aligned}$$

(f) $\dfrac{d}{dx}\left(\displaystyle\int_0^{\sin^{-1}(3x)} t^t dt\right)$

Solution. $\dfrac{d}{dx}\left(\displaystyle\int_0^{\sin^{-1}(3x)} t^t dt\right) = \boxed{\sin^{-1}(3x)^{\sin^{-1}(3x)}\dfrac{3}{\sqrt{1-(3x)^2}}}.$

3. For the function $f(t)$ sketched below, let $F(x) = \displaystyle\int_{-3}^{x} f(t)dt$.



(a) Evaluate the following.
Solution.

(i) $F(3) = \boxed{4}$ (ii) $F(-5) = \boxed{-3}$ (iii) $F'(-2) = \boxed{2}$ (iv) $F'(4) = \boxed{0}$

(b) Find an equation of the tangent line to the graph of $y = F(x)$ at $x = 6$.

Solution. We have $F(6) = 7$ and $F'(6) = f(6) = 4$, so an equation of the tangent line to the graph pf $y = F(x)$ at $x = 6$ is $\boxed{y - 7 = 4(x - 6)}$.

(c) Find the critical points of F.

Solution. We have $F'(x) = f(x)$. Observe that $f(x)$ is never undefined, and the solutions of $f(x) = 0$ are $\boxed{x = -4, 1, 4}$.

(d) Find the intervals on which F is increasing and the intervals on which F is decreasing.

Solution. F is increasing on $\boxed{[-4, 1], [4, \infty)}$. F is decreasing on $\boxed{(-\infty, -4], [1, 4]}$.

(e) Find the x-values at which $F(x)$ has a local maximum or a local minimum.

Solution. The location of the local maxima of F is $\boxed{x = 1}$. The location of the local minima of F are $\boxed{x = -4, 4}$.

(f) Find the intervals on which F is concave up and the intervals on which F is concave down.

Solution. F is concave up when $F' = f$ is increasing, which happens on $\boxed{[-6, -3], [3, \infty)}$. F is concave down when $F' = f$ is decreasing, which happens on $\boxed{[-1, 3]}$.

(g) Find the x-values at which $F(x)$ has an inflection point.

Solution. The inflection points of F are located where the concavity changes, which is at $\boxed{x = 3}$.

4. Let $f(x) = 7 + \int_{13}^{x} t(t-14)^{2/5} dt$.

(a) Find an equation of the tangent line to the graph of $y = f(x)$ at $x = 13$.

Solution. We have

$$f(11) = 7 + \int_{13}^{13} t(t-14)^{2/5} dt = 7 + 0 = 7$$

and

$$f'(11) = \frac{d}{dx}\left(\int_{13}^{x} t(t-14)^{2/5} dt\right)_{|x=13} = \left(x(x-14)^{2/5}\right)_{|x=13} = 13(13-12)^{2/5} = -13.$$

So an equation of the tangent line to the graph of $y = f(x)$ at $x = 13$ is $\boxed{y - 7 = -13(x - 13)}$.

(b) Find the critical points of f.

Solution. The derivative of f is $f'(x) = x(x-14)^{2/5}$ so the critical points of f are $\boxed{x = 0, 14}$.

(c) Find the intervals on which f is increasing and the intervals on which F is decreasing.

Solution. Let us test for the sign of $f'(x)$ in between the critical points.

- On $(-\infty, 0)$, the sign of $f'(x)$ is $(-)(+) = (-)$.
- On $(0, 14)$ the sign of $f'(x)$ is $(+)(+) = (+)$.
- On $(14, \infty)$, the sign of $f'(x)$ is $(+)(+) = (+)$.

Hence, f is increasing on $\boxed{[0, \infty)}$ and decreasing on $\boxed{(\infty, 0]}$.

(d) Find the x-values at which $f(x)$ has a local maximum or a local minimum.

Solution. Based on our findings in the previous question, we can deduce that f does not have a local maximum and has a local minimum at $\boxed{x = 0}$.

(e) Find the intervals on which f is concave up and the intervals on which F is concave down.

Solution. We'll need a sign chart for $f''(x)$ to determine this. We have

$$f''(x) = (x-14)^{2/5} + \frac{2x}{5(x-14)^{3/5}} = \frac{5(x-14)+2x}{5(x-14)^{3/5}} = \frac{7x-70}{(x-14)^{3/5}} = \frac{7(x-10)}{(x-14)^{3/5}}$$

Let us analyze the sign of $f''(x)$. The points where $f''(x)$ is zero or undefined are $x = 10, 14$.

- On $(-\infty, 10)$, the sign of $f''(x)$ is $\frac{(-)}{(-)} = (+)$.
- On $(10, 14)$, the sign of $f''(x)$ is $\frac{(+)}{(-)} = (-)$.

- On $(14, \infty)$, the sign of $f''(x)$ is $\frac{(+)}{(+)} = (+)$.

Hence, f is concave up on $\boxed{(-\infty, 10], [14, \infty)}$ and concave down on $\boxed{[10, 14]}$.

(f) Find the x-values at which $f(x)$ has an inflection point.

Solution. Based on our previous answer, we can deduce that f has inflection points at $\boxed{x = 10, 14}$.

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Sections 5.5-6: Substitution Method - Worksheet Solutions

1. Evaluate the following integrals.

(a) $\displaystyle\int (3x^4+6)\sec(x^5+10x)dx$

Solution. We substitute $u = x^5+10x$, so $du = (5x^4+10)dx = 5(x^4+2)dx$. We obtain

$$\begin{aligned}\int (3x^4+6)\sec(x^5+10x)dx &= \int 3\sec(x^5+10x)(x^4+2)dx \\ &= \int 3\sec(u)\frac{1}{5}du \\ &= \frac{3}{5}\int \sec(u)du \\ &= \frac{3}{5}\ln|\sec(u)+\tan(u)|+C \\ &= \boxed{\frac{3}{5}\ln|\sec(x^5+10x)+\tan(x^5+10x)|+C}.\end{aligned}$$

(b) $\displaystyle\int \frac{dx}{x\sqrt{3\ln(x)+5}}$

Solution 1. We substitute $u = 3\ln(x)+5$, so $du = \frac{3}{x}dx$, which gives $\frac{dx}{x} = \frac{du}{3}$. Therefore,

$$\begin{aligned}\int \frac{dx}{x\sqrt{3\ln(x)+5}} &= \int \frac{du}{3\sqrt{u}} \\ &= \frac{2}{3}\sqrt{u}+C \\ &= \boxed{\frac{2}{3}\sqrt{3\ln(x)+5}+C}.\end{aligned}$$

Solution 2. We substituted $u = \sqrt{3\ln(x)+5}$, so $du = \frac{1}{2\sqrt{3\ln(x)+5}}\cdot\frac{3}{x}dx$. This gives $\frac{dx}{x\sqrt{3\ln(x)+5}} = \frac{2}{3}du$, so

$$\begin{aligned}\int \frac{dx}{x\sqrt{3\ln(x)+5}} &= \int \frac{2}{3}du \\ &= \frac{2}{3}u+C \\ &= \boxed{\frac{2}{3}\sqrt{3\ln(x)+5}+C}.\end{aligned}$$

(c) $\int x^2\sqrt{x-1}dx$

Solution. We use the substitution $u = x-1$, so $du = dx$. Then we have $x = u+1$, so we get

$$\begin{aligned}
\int x^2\sqrt{x-1}dx &= \int (u+1)^2\sqrt{u}du \\
&= \int (u^2+2u+1)u^{1/2}du \\
&= \int (u^{5/2}+2u^{3/2}+u^{1/2})du \\
&= \frac{u^{7/2}}{7/2} + \frac{2u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2} + C \\
&= \boxed{\frac{2(x-1)^{7/2}}{7} + \frac{4(x-1)^{5/2}}{5} + \frac{2(x-1)^{1/2}}{3} + C}.
\end{aligned}$$

(d) $\int x^3\sin(x^4+2)dx$

Solution. We use the substitution $u = x^4+2$, $du = 4x^3dx$. Therefore, $x^3dx = \frac{du}{4}$, and we obtain

$$\begin{aligned}
\int x^3\sin(x^4+2)dx &= \int \frac{1}{4}\sin(u)du \\
&= -\frac{1}{4}\cos(u) + C \\
&= \boxed{-\frac{1}{4}\cos(x^4+2) + C}.
\end{aligned}$$

(e) $\int_0^1 \frac{x^3}{\sqrt{3+x^2}}dx$

Solution 1. We use the substitution $u = 3+x^2$. So $du = 2xdx$, that is $xdx = \frac{du}{2}$. The extraneous factor x^2 in the numerator can be expressed in terms of u using $x^2 = u-3$. Finally, the bounds become

$$\begin{aligned}
x = 0 &\Rightarrow u = 3+0^2 = 3, \\
x = 1 &\Rightarrow u = 3+1^2 = 4.
\end{aligned}$$

So the integral becomes

$$\begin{aligned}
\int_0^1 \frac{x^3}{\sqrt{3+x^2}}dx &= \int_0^1 \frac{x^2}{\sqrt{3+x^2}}xdx \\
&= \int_3^4 \frac{u-3}{2\sqrt{u}}du \\
&= \int_3^4 \left(\frac{1}{2}\sqrt{u} - \frac{3}{2\sqrt{u}}\right)du \\
&= \left[\frac{1}{2}\cdot\frac{2}{3}u^{3/2} - 3\sqrt{u}\right]_3^4
\end{aligned}$$

$$= \left(\frac{1}{3}4^{3/2} - 3\sqrt{4}\right) - \left(\frac{1}{3}3^{3/2} - 3\sqrt{3}\right)$$
$$= \boxed{2\sqrt{3} - \frac{10}{3}}.$$

Solution 2. We use the substitution $u = \sqrt{3+x^2}$. So $du = \frac{xdx}{\sqrt{3+x^2}}$. The extraneous factor x^2 in the numerator can be expressed in terms of u using $x^2 = u^2 - 3$. Finally, the bounds become

$$x = 0 \Rightarrow u = \sqrt{3+0^2} = \sqrt{3},$$
$$x = 1 \Rightarrow u = \sqrt{3+1^2} = 2.$$

So the integral becomes

$$\int_0^1 \frac{x^3}{\sqrt{3+x^2}}dx = \int_0^1 x^2 \frac{xdx}{\sqrt{3+x^2}}$$
$$= \int_{\sqrt{3}}^2 (u^2-3)du$$
$$= \left[\frac{1}{3}u^3 - 3u\right]_{\sqrt{3}}^2$$
$$= \left(\frac{1}{3}2^3 - 3\cdot 2\right) - \left(\frac{1}{3}\sqrt{3}^3 - 3\sqrt{3}\right)$$
$$= \boxed{2\sqrt{3} - \frac{10}{3}}.$$

(f) $\int t\sec^2\left(3t^2\right) e^{7\tan\left(3t^2\right)} dt$

Solution. We use the substitution $u = 7\tan\left(3t^2\right)$. This gives

$$du = 7\sec^2(3t^2)\cdot 6tdt = 42t\sec^2(3t^2)dt.$$

So

$$t\sec^2\left(3t^2\right)dt = \frac{du}{42}.$$

We get

$$\int t\sec^2\left(3t^2\right) e^{7\tan\left(3t^2\right)} dt = \int \frac{1}{42}e^u du$$
$$= \frac{1}{42}e^u + C$$
$$= \boxed{\frac{1}{42}e^{7\tan\left(3t^2\right)} + C}.$$

(g) $\int e^x(e^x-2)^{2/3}dx$

Solution. We use the substitution $u = e^x - 2$, so that $du = e^x dx$. This gives

$$\begin{aligned}\int e^x(e^x-2)^{2/3}dx &= \int u^{2/3}du \\ &= \frac{3}{5}u^{5/3} + C \\ &= \boxed{\frac{3}{5}(e^x-2)^{5/3} + C}.\end{aligned}$$

(h) $\int e^{2x}(e^x-2)^{2/3}dx$

Solution. We again use the substitution $u = e^x - 2$, $du = e^x dx$. But this time, we have an extraneous factor e^x since $e^{2x} = e^x e^x$. We can express this extraneous factor in terms of u as $e^x = u + 2$. Therefore

$$\begin{aligned}\int e^{2x}(e^x-2)^{2/3}dx &= \int e^x(e^x-2)^{2/3}e^x dx \\ &= \int (u+2)u^{2/3}du \\ &= \int \left(u^{5/3} + 2u^{2/3}\right) du \\ &= \frac{3}{8}u^{8/3} + \frac{6}{5}u^{5/3} + C \\ &= \boxed{\frac{3}{8}(e^x-2)^{8/3} + \frac{6}{5}(e^x-2)^{5/3} + C}.\end{aligned}$$

(i) $\int_{e^3}^{e^6} \frac{dt}{t\ln(t)}$

Solution. We use the substitution $u = \ln(t)$, so that $du = \frac{dt}{t}$. The bounds change as follows

$$\begin{aligned}t = e^3 &\Rightarrow u = \ln(e^3) = 3, \\ t = e^6 &\Rightarrow u = \ln(e^6) = 6.\end{aligned}$$

The integral becomes

$$\begin{aligned}\int_{e^3}^{e^6} \frac{dt}{t\ln(t)} &= \int_3^6 \frac{du}{u} \\ &= [\ln|u|]_3^6 \\ &= \ln(6) - \ln(3) \\ &= \ln\left(\frac{6}{3}\right) \\ &= \boxed{\ln(2)}.\end{aligned}$$

(j) $\displaystyle\int \frac{dx}{5x+4\sqrt{x}}$

Solution. We can first factor out a $\sqrt{x}$ from the denominator, which gives

$$\int \frac{dx}{5x+4\sqrt{x}} = \int \frac{dx}{\sqrt{x}\,(5\sqrt{x}+4)}.$$

We can then use the substitution $u = 5\sqrt{x}+4$, which gives $du = \frac{5dx}{2\sqrt{x}}$. This gives $\frac{dx}{\sqrt{x}} = \frac{2du}{5}$, and the integral becomes

$$\begin{aligned}\int \frac{dx}{\sqrt{x}\,(5\sqrt{x}+4)} &= \int \frac{2du}{5u} \\ &= \frac{2}{5}\ln|u| + C \\ &= \boxed{\frac{2}{5}\ln|5\sqrt{x}+4| + C}.\end{aligned}$$

(k) $\displaystyle\int \frac{dx}{\sqrt{2-x^2}}$

Solution. Recall the reference antiderivative

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u) + C.$$

We can use this antiderivative after factoring out a 2 from the square root and letting $u = \frac{x}{\sqrt{2}}$. This gives

$$\begin{aligned}\int \frac{dx}{\sqrt{2-x^2}} &= \int \frac{dx}{\sqrt{2\left(1-\frac{x^2}{2}\right)}} \\ &= \int \frac{dx}{\sqrt{2}\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}} \\ &= \int \frac{du}{\sqrt{1-u^2}} \\ &= \sin^{-1}(u) + C \\ &= \boxed{\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + C}.\end{aligned}$$

(l) $\displaystyle\int_0^1 \frac{xdx}{\sqrt{2-x^2}}$

Solution 1. This time, the numerator is (up to a constant factor) the derivative of the inside of the square root. Therefore, we can compute this integral with the substitution $u = 2-x^2$, $du = -2xdx$. Thus we have $xdx = -\frac{du}{2}$, and the bounds change to

$$x = 0 \;\Rightarrow\; u = 2 - 0^2 = 2,$$

$$x = 1 \;\Rightarrow\; u = 2 - 1^2 = 1.$$

We obtain

$$\begin{aligned} \int_0^1 \frac{x dx}{\sqrt{2-x^2}} &= \int_2^1 -\frac{du}{2\sqrt{u}} \\ &= \left[-\sqrt{u}\right]_2^1 \\ &= \boxed{1-\sqrt{2}}. \end{aligned}$$

Solution 2. We can be more ambitious with the substitution and let $u = \sqrt{2-x^2}$. The bounds change to

$$x = 0 \;\Rightarrow\; u = \sqrt{2-0^2} = \sqrt{2},$$
$$x = 1 \;\Rightarrow\; u = \sqrt{2-1^2} = 1.$$

Differentiating gives $du = -\frac{x dx}{\sqrt{2-x^2}}$, which is the entire integrand up to a negative sign. So the integral becomes

$$\int_0^1 \frac{x dx}{\sqrt{2-x^2}} = \int_{\sqrt{2}}^1 - du = \boxed{\sqrt{2}-1}.$$

(m) $\displaystyle\int_0^{2/3} \frac{dz}{4+9z^2}$

Solution. This integral will make use of the reference antiderivative

$$\int \frac{du}{1+u^2} = \tan^{-1}(u) + C.$$

To get to this form, we factor out a 4 from the denominator to obtain

$$\int_0^{2/3} \frac{dz}{4+9z^2} = \int_0^{2/3} \frac{dz}{4\left(1+\frac{9z^2}{4}\right)} = \int_0^{2/3} \frac{dz}{4\left(1+\left(\frac{3z}{2}\right)^2\right)}$$

We can then use the substitution $u = \frac{3z}{2}$, which gives $du = \frac{3dz}{2}$, so $dz = \frac{2du}{3}$. The bounds change to

$$x = 0 \;\Rightarrow\; u = 0,$$
$$x = \frac{2}{3} \;\Rightarrow\; u = 1.$$

We obtain

$$\begin{aligned} \int_0^{2/3} \frac{dz}{4\left(1+\left(\frac{3z}{2}\right)^2\right)} &= \int_0^1 \frac{1}{4} \cdot \frac{2}{3} \cdot \frac{du}{1+u^2} \\ &= \left[\frac{1}{6}\tan^{-1}(u)\right]_0^1 \\ &= \frac{1}{6}\left(\tan^{-1}(1) - \tan^{-1}(0)\right) \end{aligned}$$

$$= \frac{1}{6}\left(\frac{\pi}{4} - 0\right)$$
$$= \boxed{\frac{\pi}{24}}.$$

(n) $\displaystyle\int \frac{e^{4\arcsin(5x)}}{\sqrt{1-25x^2}}dx.$

Solution. We use the substitution $u = 4\arcsin(5x)$, so that $du = \frac{20dx}{\sqrt{1-25x^2}}$. Therefore, $\frac{dx}{\sqrt{1-25x^2}} = \frac{du}{20}$ and we obtain

$$\int \frac{e^{4\arcsin(5x)}}{\sqrt{1-25x^2}}dx = \int \frac{e^u}{20}du$$
$$= \frac{e^u}{20} + C$$
$$= \boxed{\frac{e^{4\arcsin(5x)}}{20} + C}.$$

(o) $\displaystyle\int_0^{\pi/10} \frac{\sin^3(5x)}{\cos(5x)+3}dx.$

Solution. We can write $\sin^3(5x) = \sin^2(5x)\sin(5x) = (1-\cos^2(5x))\sin(5x)$, which allows us to use the substitution $u = \cos(5x)$, $du = -5\sin(5x)dx$. The bounds of the integral change as follows:

$$x = 0 \;\Rightarrow\; u = \cos(0) = 1,$$
$$x = \frac{\pi}{10} \;\Rightarrow\; u = \cos\left(\frac{\pi}{2}\right) = 0.$$

Therefore we get

$$\int_0^{\pi/10} \frac{\sin^3(5x)}{\cos(5x)+3}dx = \int_0^{\pi/10} \frac{1-\cos^2(5x)}{\cos(5x)+3}\sin(5x)dx$$
$$= \int_1^0 -\frac{1-u^2}{5(u+3)}du$$
$$= \frac{1}{5}\int_0^1 \frac{1-u^2}{u+3}du.$$

To evaluate this last integral, we can substitute $w = u+3$, so that $dw = du$. The u in the numerator can be expressed in terms of w as $u = w-3$. We get

$$\frac{1}{5}\int_0^1 \frac{1-u^2}{u+3}du = \frac{1}{5}\int_3^4 \frac{1-(w-3)^2}{w}dw$$
$$= \frac{1}{5}\int_3^4 \frac{1-w^2+6w-9}{w}dw$$
$$= \frac{1}{5}\int_3^4 \frac{6w-w^2-8}{w}$$
$$= \frac{1}{5}\int_3^4 \left(6 - w - \frac{8}{w}\right)dw$$

$$
\begin{aligned}
&= \frac{1}{5}\left[6w - \frac{w^2}{2} - 8\ln|w|\right]_3^4 \\
&= \frac{1}{5}\left(24 - 8 - 8\ln(4) - 18 + \frac{9}{2} + 8\ln(3)\right) \\
&= \boxed{\frac{5 + 16\ln(3/4)}{10}}.
\end{aligned}
$$

(p) $\displaystyle\int \frac{\left(\tan^{-1}(t)\right)^3}{1+t^2}dt.$

Solution. We use the substitution $u = \tan^{-1}(t)$, so $du = \frac{dt}{1+t^2}$. This gives

$$
\begin{aligned}
\int \frac{\left(\tan^{-1}(t)\right)^3}{1+t^2}dt &= \int u^3 du \\
&= \frac{1}{4}u^4 + C \\
&= \boxed{\frac{1}{4}\left(\tan^{-1}(t)\right)^4 + C}.
\end{aligned}
$$

(q) $\displaystyle\int_e^{e^2} \frac{dx}{x\sqrt{\ln(x)}}dx.$

Solution. We use the substitution $u = \ln(x)$, so that $du = \frac{dx}{x}$. The bounds become

$$
\begin{aligned}
x = e &\Rightarrow u = \ln(e) = 1, \\
x = e^2 &\Rightarrow u = \ln(e^2) = 2.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\int_e^{e^2} \frac{dx}{x\sqrt{\ln(x)}}dx &= \int_1^2 \frac{du}{\sqrt{u}}du \\
&= \left[2\sqrt{u}\right]_1^2 \\
&= \boxed{2\left(\sqrt{2} - 1\right)}.
\end{aligned}
$$

(r) $\displaystyle\int \frac{\tan(3\ln(x))}{x}dx.$

Solution. We use the substitution $u = 3\ln(x)$, so $du = \frac{3dx}{x}$ and the integral becomes

$$
\begin{aligned}
\int \frac{\tan(3\ln(x))}{x}dx &= \int \frac{\tan(u)}{3}du \\
&= \frac{1}{3}\ln|\sec(u)| + C \\
&= \boxed{\frac{1}{3}\ln|\sec(3\ln(x))| + C}.
\end{aligned}
$$

(s) $\int \frac{x^3+1}{9+x^2}dx$.

Solution. We can start by splitting up the integral into a sum of two integrals:

$$\int \frac{x^3+1}{9+x^2}dx = \int \frac{x^3}{9+x^2}dx + \int \frac{1}{9+x^2}dx.$$

The first integral can be computed using the substitution $u = 9+x^2$, which gives $du = 2xdx$. The extraneous factor x^2 in the numerator can be replaced by $u-9$, which gives

$$\begin{aligned}
\int \frac{x^3}{9+x^2}dx &= \int \frac{x^2}{9+x^2}xdx \\
&= \int \frac{u-9}{2u}du \\
&= \frac{1}{2}\int \left(1-\frac{9}{u}\right)du \\
&= \frac{1}{2}\left(u - 9\ln|u|\right) + C \\
&= \frac{1}{2}\left(x^2+9-9\ln(x^2+9)\right) + C \\
&= \frac{1}{2}\left(x^2-9\ln(x^2+9)\right) + C.
\end{aligned}$$

For the second integral, we can factor out a 9 from the denominator and use the substitution $u = \frac{x}{3}$, which gives $du = \frac{dx}{3}$. We obtain

$$\begin{aligned}
\int \frac{dx}{9+x^2} &= \frac{1}{9}\int \frac{dx}{1+\left(\frac{x}{3}\right)^2} \\
&= \frac{1}{3}\int \frac{du}{1+u^2} \\
&= \frac{1}{3}\tan^{-1}(u) + C \\
&= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.
\end{aligned}$$

Putting the pieces together gives

$$\begin{aligned}
\int \frac{x^3+1}{9+x^2}dx &= \int \frac{x^3}{9+x^2}dx + \int \frac{1}{9+x^2}dx \\
&= \boxed{\frac{1}{2}\left(x^2-9\ln(x^2+9)\right) + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C}.
\end{aligned}$$

(t) $\int_0^{\pi/12} \tan^2(3\theta)\sec^2(3\theta)d\theta$.

Solution. We substitute $u = \tan(3\theta)$, so $du = 3\sec^2(3\theta)d\theta$. The bounds change as follows:

$$\begin{aligned}
x = 0 \ &\Rightarrow\ u = \tan(0) = 0, \\
x = \frac{\pi}{12} \ &\Rightarrow\ u = \tan\left(\frac{\pi}{4}\right) = 1.
\end{aligned}$$

Then we get

$$\begin{aligned}\int_0^{\pi/12} \tan^2(3\theta)\sec^2(3\theta)d\theta &= \int_0^1 \frac{u^2}{3}du \\ &= \left[\frac{u^2}{9}\right]_0^1 \\ &= \boxed{\frac{1}{9}}.\end{aligned}$$

(u) $\displaystyle\int \frac{e^{3x}}{\sqrt{49-e^{6x}}}dx.$

Solution. We use the substitution $u = e^3x$, so $du = 3e^{3x}dx$. Then we get

$$\begin{aligned}\int \frac{e^{3x}}{\sqrt{49-e^{6x}}}dx &= \int \frac{du}{3\sqrt{7^2-u^2}} \\ &= \frac{1}{3}\sin^{-1}\left(\frac{u}{7}\right) + C \\ &= \boxed{\frac{1}{3}\sin^{-1}\left(\frac{e^{3x}}{7}\right) + C}.\end{aligned}$$

(v) $\displaystyle\int_{-5/2}^{5/2} \frac{1+\sin(x)}{4x^2+25}dx.$

Solution. We can split up the integral into a sum of two integrals:

$$\int_{-5/2}^{5/2} \frac{1+\sin(x)}{4x^2+25}dx = \int_{-5/2}^{5/2} \frac{dx}{4x^2+25} + \int_{-5/2}^{5/2} \frac{\sin(x)}{4x^2+25}dx.$$

The second integral is the integral of an odd function on an interval symmetric about the origin since

$$\frac{\sin(-x)}{4(-x)^2+25} = \frac{-\sin(x)}{4x^2+25} = -\frac{\sin(x)}{4x^2+25}.$$

Therefore,

$$\int_{-5/2}^{5/2} \frac{\sin(x)}{4x^2+25}dx = 0.$$

For the other integral, we can use an arc tangent to get

$$\begin{aligned}\int_{-5/2}^{5/2} \frac{dx}{4x^2+25} &= \frac{1}{25}\int_{-5/2}^{5/2} \frac{dx}{\left(\frac{2x}{5}\right)^2+1} \\ &= \frac{1}{25}\left[\frac{5}{2}\tan^{-1}\left(\frac{2x}{5}\right)\right]_{-5/2}^{5/2} \\ &= \frac{1}{10}\left(\tan^{-1}(1) - \tan^{-1}(-1)\right) \\ &= \frac{\pi}{20}.\end{aligned}$$

Therefore

$$\boxed{\int_{-5/2}^{5/2} \frac{1+\sin(x)}{4x^2+25}dx = \frac{\pi}{20}}.$$

2. Suppose that f is an **even** function such that

$$\int_{-9}^{5} f(x)dx = -13 \quad \text{and} \quad \int_{0}^{9} f(x)dx = 4.$$

Evaluate the definite integrals below.

(a) $\int_{-9}^{9} f(x)dx$

Solution. Since f is even, by symmetry we have

$$\int_{-9}^{9} f(x)dx = 2\int_{0}^{9} f(x)dx = \boxed{8}.$$

(b) $\int_{0}^{5} (4x - 3f(x))dx$

Solution. Let us start by calculating $\int_{0}^{5} f(x)dx$. By additivity of the integral, we have

$$\int_{-9}^{5} f(x)dx = \int_{-9}^{0} f(x)dx + \int_{0}^{5} f(x)dx.$$

Since f is even, we have

$$\int_{-9}^{0} f(x)dx = \int_{0}^{9} f(x)dx = 4.$$

So we get

$$-13 = 4 + \int_{0}^{5} f(x)dx \Rightarrow \int_{0}^{5} f(x)dx = -17.$$

Now using the linearity of the integral, we obtain

$$\begin{aligned}\int_{0}^{5} (4x - 3f(x))dx &= 4\int_{0}^{5} xdx - 3\int_{0}^{5} f(x)dx \\ &= 4\left[\frac{1}{2}x^2\right]_0^5 - 3(-17) \\ &= 4\frac{1}{2}(25) + 51 \\ &= \boxed{101}.\end{aligned}$$

(c) $\int_{-3}^{3} xf(x)dx$

Solution. Since f is even, the function $g(x) = xf(x)$ is odd, as shown below:

$$g(-x) = (-x)f(-x) = -xf(x) = -g(x).$$

Since the interval of integration $[-3, 3]$ is centered at 0, we deduce

$$\boxed{\int_{-3}^{3} xf(x) = 0}.$$

(d) $\int_{0}^{3} xf(x^2)dx$

Solution. We can evaluate this integral using the substitution $u = x^2$, which gives $du = 2xdx$, or $xdx = \frac{du}{2}$. The bounds become

$$x = 0 \Rightarrow u = 0^2 = 0,$$
$$x = 3 \Rightarrow u = 3^2 = 9.$$

Therefore

$$\begin{aligned}\int_{0}^{3} xf(x^2)dx &= \int_{0}^{9} \frac{1}{2} f(u)du \\ &= \frac{1}{2}\int_{0}^{9} f(u)du \\ &= \frac{1}{2}4 \\ &= \boxed{2}.\end{aligned}$$

3. Find the average value of the following functions on the given interval.

(a) $f(x) = \dfrac{3}{\sqrt{100 - x^2}}$ on $[0, 5]$.

Solution. The average value is given by

$$\begin{aligned}\operatorname{av}(f) &= \frac{1}{5-0}\int_{0}^{5} \frac{3}{\sqrt{100 - x^2}}dx \\ &= \frac{3}{5}\int_{0}^{5} \frac{dx}{\sqrt{100\left(1 - \frac{x^2}{100}\right)}} \\ &= \frac{3}{5}\int_{0}^{5} \frac{dx}{10\sqrt{1 - \left(\frac{x}{10}\right)^2}} \\ &= \frac{3}{5}\int_{0}^{1/2} \frac{du}{\sqrt{1 - u^2}} \qquad \left(u = \frac{x}{10}\right) \\ &= \frac{3}{5}\left[\sin^{-1}(u)\right]_{0}^{1/2}\end{aligned}$$

$$= \frac{3}{5}\left(\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right)$$
$$= \frac{3}{5} \cdot \frac{\pi}{6}$$
$$= \boxed{\frac{\pi}{10}}.$$

(b) $f(x) = x\sqrt[3]{3x-7}$ on $[2,5]$.

Solution. The average value is given by

$$\text{av}(f) = \frac{1}{5-2}\int_2^5 x\sqrt[3]{3x-7}dx = \frac{1}{3}\int_2^5 x\sqrt[3]{3x-7}dx.$$

We can calculate the integral using the substitution $u = 3x-7$. This will give $du = 3dx$, or $dx = \frac{du}{3}$. The bounds will become

$$x = 2 \;\Rightarrow\; u = 3\cdot 2 - 7 = -1,$$
$$x = 5 \;\Rightarrow\; u = 3\cdot 5 - 7 = 8.$$

Finally, the extraneous factor x in the integrand can be expressed in terms of u as $x = \frac{u+7}{3}$. We obtain

$$\text{av}(f) = \frac{1}{3}\int_{-1}^{8} \frac{u+7}{3}\sqrt[3]{u}\frac{du}{3}$$
$$= \frac{1}{27}\int_{-1}^{8}\left(u^{4/3} + 7u^{1/3}\right)du$$
$$= \frac{1}{27}\left[\frac{3}{7}u^{7/3} + \frac{21}{4}u^{4/3}\right]_{-1}^{8}$$
$$= \frac{1}{27}\left(\left(\frac{3}{7}8^{7/3} + \frac{21}{4}8^{4/3}\right) - \left(\frac{3}{7}(-1)^{7/3} + \frac{21}{4}(-1)^{4/3}\right)\right)$$
$$= \boxed{\frac{139}{28}}.$$

Rutgers University
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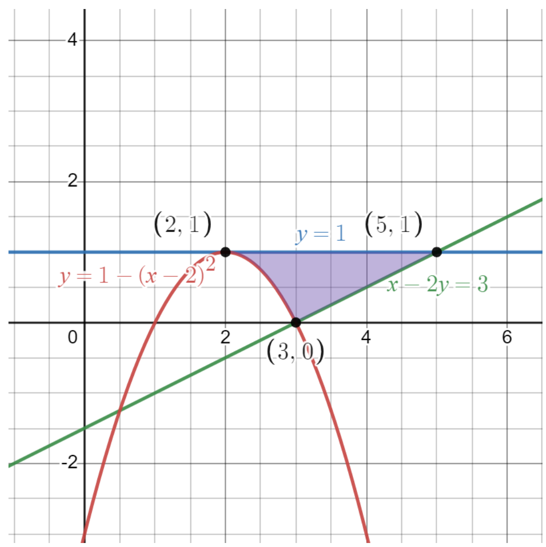
Section 5.6: Areas Between Curves - Worksheet Solutions

1. For each of the regions described below (i) sketch the region, clearly labeling the curves and their intersection points, (ii) calculate the area of the region using an x-integral and (iii) calculate the area of the region using a y-integral.

   (a) The region to the right of the parabola $y = 1 - (x-2)^2$, below the line $y = 1$ and to the left of the line $x - 2y = 3$.

   Solution. (i)



(ii) The region is not vertically simple, so we will need a sum of x-integrals. For $2 \leqslant x \leqslant 3$, the vertical strip at x is bounded by $y = 1$ on the top and $y = 1-(x-2)^2$ on the bottom. For $3 \leqslant x \leqslant 5$, the vertical strip at x is bounded by $y = 1$ on the top and the line $x - 2y = 3 \Rightarrow y = \frac{x-3}{2}$ on the bottom. Therefore the area is given by

$$\begin{aligned} A &= \int_2^3 \left(1 - (1 - (x-2)^2)\right) dx + \int_3^5 \left(1 - \frac{x-3}{2}\right) dx \\ &= \int_2^3 (x-2)^2 dx + \frac{1}{2}\int_3^5 (5-x)dx \\ &= \left[\frac{(x-2)^3}{3}\right]_2^3 + \frac{1}{2}\left[5x - \frac{x^2}{2}\right]_3^5 \\ &= \frac{(3-2)^3}{3} - \frac{(2-2)^3}{3} + \frac{1}{2}\left(5\cdot 5 - \frac{5^2}{2} - 5\cdot 3 + \frac{3^2}{2}\right) \\ &= \boxed{\frac{4}{3} \text{ square units}}. \end{aligned}$$

(iii) The region is horizontally simple. The horizontal strip at y is bounded on the right by the line $x - 2y = 3$, which gives $x = 2y + 3$ when expressed as a function of y. The curve bounding on the left is the right branch of the parabola $y = 1 - (x-2)^2$. Expressing this branch as a function of y gives

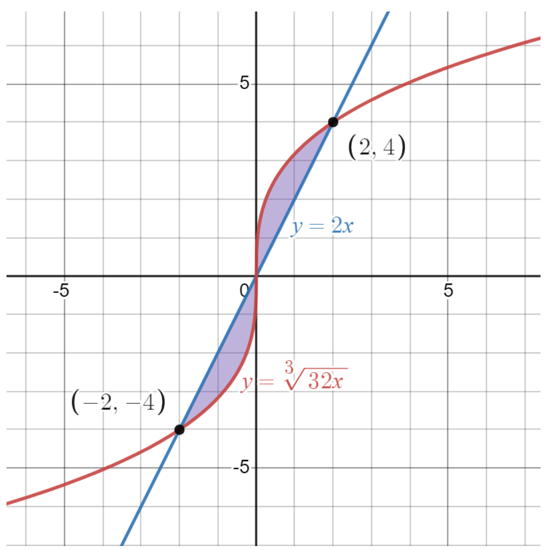
$$y = 1 - (x-2)^2 \Rightarrow (x-2)^2 = 1 - y \Rightarrow |x-2| = \sqrt{1-y} \Rightarrow x - 2 = \sqrt{1-y} \Rightarrow x = 2 + \sqrt{1-y}.$$

Note that $|x-2| = x - 2$ since $x - 2 \geqslant 0$ on the right branch of the parabola. Therefore the area is

$$\begin{aligned} A &= \int_0^1 \left((2y+3) - \left(2 + \sqrt{1-y}\right)\right) dy \\ &= \int_0^1 \left(2y + 1 - \sqrt{1-y}\right) dy \\ &= \left[y^2 + y + \frac{2}{3}(1-y)^{3/2}\right]_0^1 \\ &= 1 + 1 - \frac{2}{3} \\ &= \boxed{\frac{4}{3} \text{ square units}}. \end{aligned}$$

(b) The region bounded by the curves $y = 2x$ and $y = \sqrt[3]{32x}$.

Solution. (i)



(ii) The region is not vertically simple. For $0 \leqslant x \leqslant 2$, the vertical strip at x is bounded on the top by $y = \sqrt[3]{32x}$ and on the bottom by $y = 2x$. For $-2 \leqslant x \leqslant 0$, the vertical strip at x is bounded on the top by $y = 2x$ and on the bottom by $y = \sqrt[3]{32x}$. Therefore

$$\begin{aligned} A &= \int_{-2}^{0} \left(2x - \sqrt[3]{32x}\right) dx + \int_0^2 \left(\sqrt[3]{32x} - 2x\right) dx \\ &= \left[x^2 - 32^{1/3}\frac{3}{4}x^{4/3}\right]_{-2}^{0} + \left[32^{1/3}\frac{3}{4}x^{4/3} - x^2\right]_0^2 \end{aligned}$$

$$= -(-2)^2 + 32^{1/3}\frac{3}{4}(-2)^{4/3} + 32^{1/3}\frac{3}{4}2^{4/3} - 2^2$$
$$= \boxed{4 \text{ square units}}.$$

(iii) We need to express the curves as functions of y to use a y-integral.

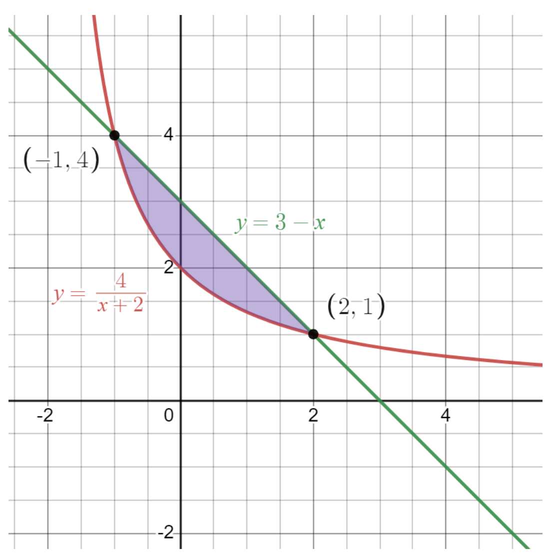
$$y = \sqrt[3]{32x} \;\Rightarrow\; 32x = y^3 \;\Rightarrow\; x = \frac{y^3}{32},$$
$$y = 2x \;\Rightarrow\; x = \frac{y}{2}.$$

The region is not horizontally simple. For $0 \leqslant y \leqslant 4$, the horizontal strip at y is bounded on the right by $x = \frac{y}{2}$ and on the left by $x = \frac{y^3}{32}$. For $-4 \leqslant y \leqslant 0$, the horizontal strip at y is bounded on the right by $x = \frac{y^3}{32}$ and on the left by $x = \frac{y}{2}$. Therefore

$$A = \int_{-4}^{0} \left(\frac{y^3}{32} - \frac{y}{2}\right) dy + \int_{0}^{4} \left(\frac{y}{2} - \frac{y^3}{32}\right) dy$$
$$= \left[\frac{y^4}{128} - \frac{y^2}{4}\right]_{-4}^{0} + \left[\frac{y^2}{4} - \frac{y^4}{128}\right]_{0}^{4}$$
$$= -\frac{(-4)^4}{128} + \frac{(-4)^2}{4} + \frac{4^2}{4} - \frac{4^4}{128}$$
$$= \boxed{4 \text{ square units}}.$$

(c) The region bounded by the curves $y = \dfrac{4}{x+2}$ and $y = 3 - x$.

Solution. (i)



(ii) The region is vertically simple. The vertical strip at x is bounded on the top by $y = 3 - x$ and on the bottom by $y = \frac{4}{x+2}$. Therefore the area is

$$A = \int_{-1}^{2} \left((3 - x) - \frac{4}{x+2}\right) dx$$

$$= \left[3x - \frac{1}{2}x^2 - 4\ln|x+2|\right]_{-1}^{2}$$
$$= \left(3\cdot 2 - \frac{1}{2}2^2 - 4\ln(4)\right) - \left(-3 - \frac{1}{2} - 4\ln(1)\right)$$
$$= \boxed{\frac{15}{2} - 8\ln(2) \text{ square units}}.$$

(iii) We need to express the curves as functions of y.

$$y = 3 - x \;\Rightarrow\; x = 3 - y,$$
$$y = \frac{4}{x+2} \;\Rightarrow\; x + 2 = \frac{4}{y} \;\Rightarrow\; x = \frac{4}{y} - 2.$$

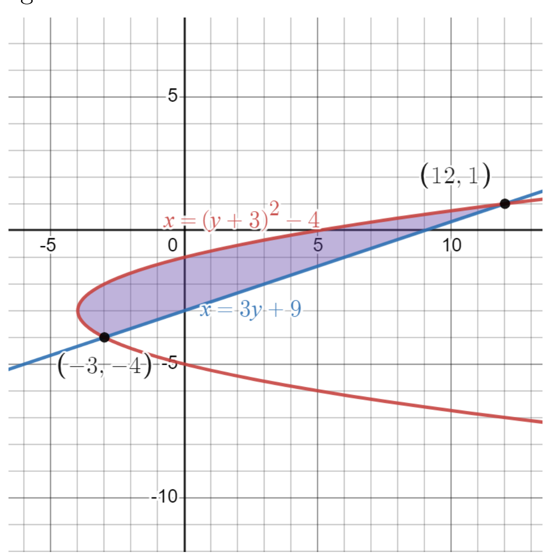
The region is horizontally simple. The horizontal strip at y is bounded on the right by $x = 3 - y$ and on the left by $x = \frac{4}{y} - 2$. Therefore

$$A = \int_1^4 \left((3-y) - \left(\frac{4}{y} - 2\right)\right) dy$$
$$= \int_1^4 \left(5 - y - \frac{4}{y}\right) dy$$
$$= \left[5y - \frac{1}{2}y^2 - 4\ln|y|\right]_1^4$$
$$= \left(5\cdot 4 - \frac{1}{2}4^2 - 4\ln(4)\right) - \left(5 - \frac{1}{2} - 4\ln(1)\right)$$
$$= \boxed{\frac{15}{2} - 8\ln(2) \text{ square units}}.$$

2. Calculate the area of the regions described below.

(a) The region bounded by the parabola $x = (y+3)^2 - 4$ and the line $x = 3y + 9$.

Solution. A sketch of the region is included below.

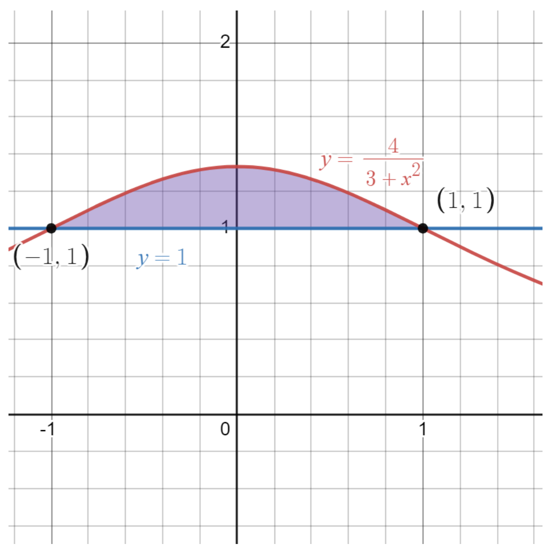


The region is horizontally simple, but not vertically simple. So computing the area using horizontal strips/integration with respect to y will be simpler than using vertical strips/integration with respect to x since we will only need one integral. The horizontal strip at y is bounded on the right by the line $x = 3y + 9$ and on the left by the parabola $x = (y+3)^2 - 4$. Therefore the area is given by

$$\begin{aligned} A &= \int_{-4}^{1} \big((3y+9) - ((y+3)^2 - 4)\big)\, dy \\ &= \int_{-4}^{1} \big(3y + 13 - (y^2 + 6y + 9)\big)\, dy \\ &= \int_{-4}^{1} \big(4 - 3y - y^2\big)\, dy \\ &= \left[4y - \frac{3}{2}y^2 - \frac{1}{3}y^3\right]_{-4}^{1} \\ &= \left(4 - \frac{3}{2} - \frac{1}{3}\right) - \left(4(-4) - \frac{3}{2}(-4)^2 - \frac{1}{3}(-4)^3\right) \\ &= \boxed{\frac{125}{6} \text{ square units}}. \end{aligned}$$

(b) The region bounded by $y = \dfrac{4}{3+x^2}$ and $y = 1$.

Solution. A sketch of the region is included below.



Note that the region is both vertically and horizontally simple. So we would need only one integral to compute the area using integration with respect to either x or y. However, integration with respect to x will be simpler here, both to set up the integral and compute the antiderivative. The vertical strip at x is bounded on the top by $y = \frac{4}{3+x^2}$ and on the bottom by $y = 1$. So the area is given by

$$A = \int_{-1}^{1} \left(\frac{4}{3+x^2} - 1\right) dx = 2\int_{0}^{1} \left(\frac{4}{3+x^2} - 1\right) dx,$$

the second equality holding because the integrand is even (or equivalently, because the region is symmetric with respect to the y-axis). To compute the antiderivative of the first term of the integrand, we can factor out a 3 from the denominator and use the reference antiderivative

$$\int \frac{du}{1+u^2} = \tan^{-1}(u) + C.$$

This gives

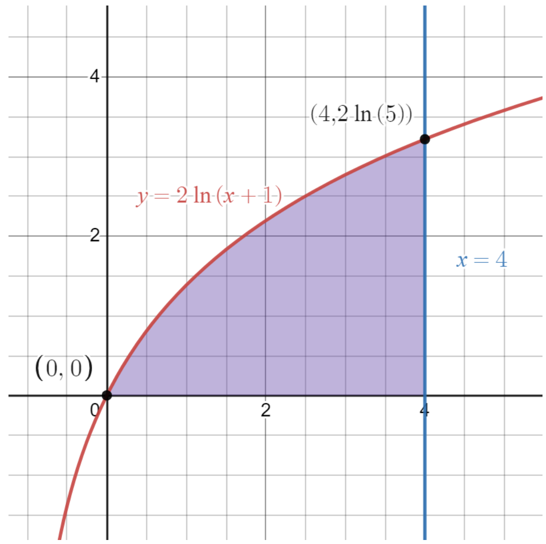
$$\begin{aligned}
\int \frac{4dx}{3+x^2} &= \int \frac{4dx}{3\left(1+\frac{x^2}{3}\right)} \\
&= \frac{4}{3}\int \frac{dx}{1+\left(\frac{x}{\sqrt{3}}\right)^2} \\
&= \frac{4}{3}\int \frac{\sqrt{3}du}{1+u^2} \qquad \left(u = \frac{x}{\sqrt{3}}\right) \\
&= \frac{4\sqrt{3}}{3}\tan^{-1}(u) + C \\
&= \frac{4\sqrt{3}}{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C.
\end{aligned}$$

We can now use this to compute the area. We obtain

$$\begin{aligned}
A &= 2\int_0^1 \left(\frac{4}{3+x^2} - 1\right) dx \\
&= 2\left[\frac{4\sqrt{3}}{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - x\right]_0^1 \\
&= 2\left(\frac{4\sqrt{3}}{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - 1\right) \\
&= 2\left(\frac{4\sqrt{3}}{3}\cdot\frac{\pi}{6} - 1\right) \\
&= \boxed{2\left(\frac{2\sqrt{3}\pi}{9} - 1\right) \text{ square units}}.
\end{aligned}$$

(c) The region bounded by $y = 2\ln(x+1)$, the x-axis and the line $x = 4$.

Solution. A sketch of the region is included below.



The region is both vertically and horizontally simple. Calculating the area using an x-integral would require finding an antiderivative of ln, which we do not know how to do (yet! we will learn how to do this in section 8.2). So we will prefer a y-integral here. We can express the curve $y = 2\ln(x+1)$ as a function of y as follows

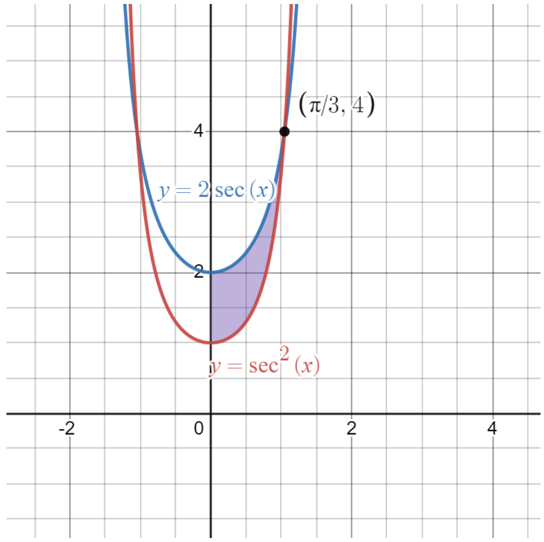
$$y = 2\ln(x+1) \;\Rightarrow\; \frac{y}{2} = \ln(x+1) \;\Rightarrow\; x+1 = e^{y/2} \;\Rightarrow x = e^{y/2} - 1.$$

The horizontal strip at y is bounded by the line $x = 4$ on the right and the curve $x = e^{y/2} - 1$ on the left. Therefore the area is

$$\begin{aligned} A &= \int_0^{2\ln(5)} \left(4 - \left(e^{y/2} - 1\right)\right) dy \\ &= \int_0^{2\ln(5)} \left(5 - e^{y/2}\right) dy \\ &= \left[5y - 2e^{y/2}\right]_0^{2\ln(5)} \\ &= \left(10\ln(5) - 2e^{\ln(5)}\right) - (-2) \\ &= \boxed{10\ln(5) - 8 \text{ square units}}. \end{aligned}$$

(d) The region to the right of the y-axis, above the graph of $y = \sec(x)^2$ and below the graph of $y = 2\sec(x)$.

Solution. A sketch of the region is included below.



The region is vertically simple. The vertical strip at x is bounded on the top by $y = 2\sec(x)$ and on the bottom by $y = \sec(x)^2$. Therefore the area is given by

$$\begin{aligned}
A &= \int_0^{\pi/3} \left(2\sec(x) - \sec(x)^2\right) dx \\
&= \left[2\ln|\sec(x) + \tan(x)| - \tan(x)\right]_0^{\pi/3} \\
&= \left(2\ln\left|\sec\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right)\right| - \tan\left(\frac{\pi}{3}\right)\right) - (2\ln|\sec(0) + \tan(0)| - \tan(0)) \\
&= \left(2\ln\left|2 + \sqrt{3}\right| - \sqrt{3}\right) - (2\ln|1 + 0| - 0) \\
&= \boxed{2\ln(2 + \sqrt{3}) - \sqrt{3} \text{ square units}}.
\end{aligned}$$