Rutgers University
Math 151

Chapter 1: Review of Algebra & Precalculus - Worksheet Solutions

1. **Composite functions:** recall that given two functions f and g, the function $f \circ g$ (called f *composed with* g) is

$$(f \circ g)(x) = f(g(x)).$$

(a) Given $f(x) = \sqrt{x}$ and $g(x) = (x-3)^2$, find and simplify the following.

i. $(f \circ g)(x)$

Solution.

$$(f \circ g)(x) = f(g(x)) = f((x-3)^2) = \sqrt{(x-3)^2} = \boxed{|x-3|}.$$

ii. $(g \circ f)(x)$

Solution.

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \boxed{(\sqrt{x}-3)^2}.$$

iii. $(f \circ f \circ f)(x)$

Solution.

$$(f \circ f \circ f)(x) = f(f(f(x))) = \sqrt{\sqrt{\sqrt{x}}} = \left(\left(x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = x^{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}} = \boxed{x^{\frac{1}{8}}}.$$

(b) Let $H(x) = \cos(3x^2) + 1$. Complete the table below to find pairs of functions $f(x)$ and $g(x)$ such that $H(x) = f(g(x))$.

	$f(x) =$	$g(x) =$
i.	$\cos(x) + 1$	
ii.		x^2
iii.		$\cos(3x^2)$
iv.	x	
v.		$\cos(3x^2) + 7$

Solution.

	$f(x) =$	$g(x) =$
i.	$\cos(x) + 1$	$3x^2$
ii.	$\cos(3x) + 1$	x^2
iii.	$x + 1$	$\cos(3x^2)$
iv.	x	$\cos(3x^2) + 1$
v.	$x - 6$	$\cos(3x^2) + 7$

2. **Trigonometry:**

(a) Suppose that $\cos(a) = \frac{2}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$. Evaluate the following.

i. $\tan(a)$

Solution. We start by using the Pythagorean identity $\cos(a)^2 + \sin(a)^2$ to find $\sin(a)$. We get

$$\begin{aligned}
\left(\frac{2}{5}\right)^2 + \sin(a)^2 &= 1 \\
\sin(a)^2 &= 1 - \frac{4}{25} = \frac{21}{25} \\
\sqrt{\sin(a)^2} &= \sqrt{\frac{21}{25}} \\
|\sin(a)| &= \frac{\sqrt{21}}{5} \\
\sin(a) &= \pm\frac{\sqrt{21}}{5}.
\end{aligned}$$

To determine which sign is appropriate, we use the fact that $\frac{3\pi}{2} < \theta < 2\pi$ (quadrant IV), which implies $\sin(a) < 0$. Therefore, $\sin(a) = -\frac{\sqrt{21}}{5}$. Now

$$\tan(a) = \frac{\sin(a)}{\cos(a)} = \frac{-\frac{\sqrt{21}}{5}}{\frac{2}{5}} = \boxed{-\frac{\sqrt{21}}{2}}.$$

ii. $\sin(2a)$

Solution. We use a double-angle identity to get

$$\sin(2a) = 2\sin(a)\cos(a) = 2\left(-\frac{\sqrt{21}}{5}\right)\frac{2}{5} = \boxed{-\frac{4\sqrt{21}}{25}}.$$

iii. $\cos(2a)$

Solution. We use a double-angle identity to get

$$\cos(2a) = 2\cos(a)^2 - 1 = 2\left(\frac{2}{5}\right)^2 - 1 = \frac{8}{25} - 1 = \boxed{-\frac{17}{25}}.$$

(b) Evaluate the following.

i. $\sec\left(\frac{4\pi}{3}\right)$

ii. $\tan^{-1}(1)$

iii. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

iv. $\csc^{-1}(2)$

v. $\sin\left(\sin^{-1}(0.8)\right)$

vi. $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$

Solution.

i. $\sec\left(\frac{4\pi}{3}\right) = \boxed{-2}$

ii. $\tan^{-1}(1) = \boxed{\dfrac{\pi}{4}}$

iii. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{\dfrac{5\pi}{6}}$

iv. $\csc^{-1}(2) = \boxed{\dfrac{\pi}{6}}$

v. $\sin\left(\sin^{-1}(0.8)\right) = \boxed{0.8}$

vi. $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right) = \boxed{-\dfrac{\pi}{4}}$

(c) Simplify the following. Your answers should be algebraic expressions of x (not involving any trigonometric or inverse trigonometric functions).

i. $\cos\left(\cos^{-1}(x)\right)$

Solution. By definition of the inverse function, $\boxed{\cos\left(\cos^{-1}(x)\right) = x}$.

ii. $\cos\left(\sin^{-1}(x)\right)$

Solution. We use the Pythagorean identity $\cos(\theta)^2+\sin(\theta)^2 = 1$ with $\theta = \sin^{-1}(x)$. By definition of $\sin^{-1}$, we know that $\sin(\theta) = x$ and θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We get

$$\begin{aligned}
&\cos(\theta)^2 + x^2 = 1\\
&\cos(\theta)^2 = 1 - x^2\\
&\sqrt{\cos(\theta)^2} = \sqrt{1-x^2}\\
&|\cos(\theta)| = \sqrt{1-x^2}\\
&\cos(\theta) = \pm\sqrt{1-x^2}
\end{aligned}$$

To determine which sign is appropriate, recall that $\theta = \sin^{-1}(x)$ is an angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\cos(\theta) \geqslant 0$. Hence

$$\boxed{\cos(\sin^{-1}(x)) = \sqrt{1-x^2}}.$$

iii. $\sin\left(\cos^{-1}(x)\right)$

Solution. We use the Pythagorean identity $\cos(\theta)^2+\sin(\theta)^2 = 1$ with $\theta = \cos^{-1}(x)$. By definition of $\cos^{-1}$, we know that $\cos(\theta) = x$ and θ is in $[0, \pi]$. We get

$$\begin{aligned}
&x^2 + \sin(\theta)^2 = 1\\
&\sin(\theta)^2 = 1 - x^2\\
&\sqrt{\sin(\theta)^2} = \sqrt{1-x^2}\\
&|\sin(\theta)| = \sqrt{1-x^2}\\
&\sin(\theta) = \pm\sqrt{1-x^2}
\end{aligned}$$

To determine which sign is appropriate, recall that $\theta = \cos^{-1}(x)$ is an angle in $[0, \pi]$, so $\sin(\theta) \geqslant 0$. Hence

$$\boxed{\sin(\cos^{-1}(x)) = \sqrt{1-x^2}}.$$

iv. $\sec\left(\tan^{-1}(4x)\right)$

Solution. We use the Pythagorean identity $1+\tan(\theta)^2 = \sec(\theta)^2$ with $\theta = \tan^{-1}(4x)$. By definition of $\tan^{-1}$, we know that $\tan(\theta) = 4x$ and θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We get

$$\begin{aligned}
\sec(\theta)^2 &= 1+(4x)^2 = 1+16x^2 \\
\sqrt{\sec(\theta)^2} &= \sqrt{1+16x^2} \\
|\sec(\theta)| &= \sqrt{1+16x^2} \\
\sec(\theta) &= \pm\sqrt{1+16x^2}
\end{aligned}$$

To determine which sign is appropriate, recall that $\theta = \tan^{-1}(x)$ is an angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\sec(\theta) > 0$. Hence

$$\boxed{\sec(\tan^{-1}(4x)) = \sqrt{1+16x^2}}.$$

v. $\tan\left(\cos^{-1}\left(\frac{x}{2}\right)\right)$

Solution. We start by using the Pythagorean identity $\cos(\theta)^2+\sin(\theta)^2 = 1$ with $\theta = \cos^{-1}\left(\frac{x}{2}\right)$ to find $\sin(\theta)$. By definition of $\cos^{-1}$, we know that $\cos(\theta) = \frac{x}{2}$ and θ is in $[0,\pi]$. We get

$$\begin{aligned}
\left(\frac{x}{2}\right)^2 + \sin(\theta)^2 &= 1 \\
\sin(\theta)^2 &= 1-\frac{x^2}{4} = \frac{4-x^2}{4} \\
\sqrt{\sin(\theta)^2} &= \sqrt{\frac{4-x^2}{4}} \\
|\sin(\theta)| &= \frac{\sqrt{4-x^2}}{2} \\
\sin(\theta) &= \pm\frac{\sqrt{4-x^2}}{2}
\end{aligned}$$

To determine which sign is appropriate, recall that $\theta = \cos^{-1}\left(\frac{x}{2}\right)$ is an angle in $[0,\pi]$, so $\sin(\theta) \geqslant 0$. Hence $\sin(\theta) = \frac{\sqrt{4-x^2}}{2}$ and

$$\tan\left(\cos^{-1}\left(\frac{x}{2}\right)\right) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{\sqrt{4-x^2}}{2}}{\frac{x}{2}} \boxed{\frac{\sqrt{4-x^2}}{x}}.$$

vi. $\csc\left(\cot^{-1}\left(\frac{3x}{5}\right)\right)$

Solution. We use the Pythagorean identity $1+\cot(\theta)^2 = \csc(\theta)^2$ with $\theta = \cot^{-1}\left(\frac{3x}{5}\right)$. By definition of $\cot^{-1}$, we know that $\cot(\theta) = \frac{3x}{5}$ and θ is in $(0,\pi)$. We get

$$\begin{aligned}
\csc(\theta)^2 &= 1+\left(\frac{3x}{5}\right)^2 = 1+\frac{9x^2}{25} = \frac{25+9x^2}{25} \\
\sqrt{\csc(\theta)^2} &= \sqrt{\frac{25+9x^2}{25}} \\
|\csc(\theta)| &= \frac{\sqrt{25+9x^2}}{5}
\end{aligned}$$

$$\csc(\theta) = \pm\frac{\sqrt{25+9x^2}}{5}$$

To determine which sign is appropriate, recall that $\theta = \cot^{-1}\left(\frac{3x}{5}\right)$ is an angle in $(0,\pi)$, so $\csc(\theta) > 0$. Hence

$$\boxed{\csc\left(\cot^{-1}\left(\frac{3x}{5}\right)\right) = \frac{\sqrt{25+9x^2}}{5}}.$$

3. **Exponential and Logarithmic Functions:**

(a) Evaluate the following.

i. $e^{\ln(75)-2\ln(5)}$

Solution.

$$e^{\ln(75)-2\ln(5)} = e^{\ln\left(\frac{75}{5^2}\right)} = \frac{75}{5^2} = \boxed{3}.$$

ii. $\log_{\frac{1}{2}}(32)$

Solution. Recall that by definition of logarithms, $\log_{\frac{1}{2}}(32)$ is the exponent to which the base $\frac{1}{2}$ must be raised to obtain 32. Therefore $\boxed{\log_{\frac{1}{2}}(32) = -5}$.

iii. $\ln(9e^2) + \ln(\sqrt{9e}) - \ln(27e^{1/3})$

Solution.

$$\ln(9e^2)+\ln(\sqrt{9e})-\ln(27e^{1/3}) = \ln\left(\frac{9e^2\sqrt{9e}}{27e^{1/3}}\right) = \ln\left(\frac{e^2e^{1/2}}{e^{1/3}}\right) = \ln\left(e^{2+1/2-1/3}\right) = \ln\left(e^{13/6}\right) = \boxed{\frac{13}{6}}.$$

(b) Solve the following equations.

i. $2^{5x-1} = 4^{-3x}$

Solution.

$$\begin{aligned}
&2^{5x-1} = 4^{-3x}\\
&2^{5x-1} = \left(2^2\right)^{-3x}\\
&2^{5x-1} = 2^{-6x}\\
&5x - 1 = -6x\\
&11x = 1\\
&\boxed{x = \frac{1}{11}}.
\end{aligned}$$

ii. $\log_4(x+5) - \log_4(x) = 2$

Solution.

$$\log_4(x+5) - \log_4(x) = 2$$

$$\begin{aligned}
&\log_4\left(\frac{x+5}{x}\right) = 2\\
&\frac{x+5}{x} = 4^2\\
&x+5 = 16x\\
&15x = 5\\
&\boxed{x = \frac{1}{3}}
\end{aligned}$$

iii. $e^{2x} - 3e^x - 10 = 0$

Solution.

$$\begin{aligned}
&e^{2x} - 3e^x - 10 = 0\\
&(e^x)^2 - 3e^x - 10 = 0\\
&(e^x - 5)(e^x + 2) = 0\\
&e^x = 5 \text{ or } e^x = -2\\
&\boxed{x = \ln(5)}
\end{aligned}$$

4. **Inverse Functions:** each function below is one-to-one. Find the inverse function.

(a) $f(x) = (x+8)^{7/4}$

Solution.

$$\begin{aligned}
&f(y) = x\\
&(y+8)^{7/4} = x\\
&\left((y+8)^{7/4}\right)^{4/7} = x^{4/7}\\
&y+8 = x^{4/7}\\
&y = x^{4/7} - 8\\
&\boxed{f^{-1}(x) = x^{4/7} - 8}
\end{aligned}$$

(b) $f(x) = \dfrac{3-2x}{4x+7}$

Solution.

$$\begin{aligned}
&f(y) = x\\
&\frac{3-2y}{4y+7} = x\\
&3-2y = x(4y+7)\\
&3-2y = 4xy+7x
\end{aligned}$$

$$
\begin{aligned}
4xy + 2y &= 3 - 7x \\
y(4x+2) &= 3 - 7x \\
y &= \frac{3-7x}{4x+2} \\
&\boxed{f^{-1}(x) = \frac{3-7x}{4x+2}}
\end{aligned}
$$

(c) $f(x) = 5 + 2e^{3x+1}$

Solution.

$$
\begin{aligned}
f(y) &= x \\
5 + 2e^{3y+1} &= x \\
e^{3y+1} &= \frac{x-5}{2} \\
3y + 1 &= \ln\left(\frac{x-5}{2}\right) \\
y &= \frac{1}{3}\left(\ln\left(\frac{x-5}{2}\right) - 1\right) \\
&\boxed{f^{-1}(x) = \frac{1}{3}\left(\ln\left(\frac{x-5}{2}\right) - 1\right)}
\end{aligned}
$$

(d) $f(x) = 1 - \arcsin(x^3)$

Solution.

$$
\begin{aligned}
f(y) &= x \\
1 - \arcsin(y^3) &= x \\
\arcsin(y^3) &= 1 - x \\
y^3 &= \sin(1-x) \\
y &= \sqrt[3]{\sin(1-x)} \\
&\boxed{f^{-1}(x) = \sqrt[3]{\sin(1-x)}}
\end{aligned}
$$

(e) $f(x) = \ln(x) - \ln(x-3)$

Solution.

$$
\begin{aligned}
f(y) &= x \\
\ln(y) - \ln(y-3) &= x \\
\ln\left(\frac{y}{y-3}\right) &= x \\
\frac{y}{y-3} &= e^x
\end{aligned}
$$

$$
\begin{aligned}
&y = e^x(y-3)\\
&y = e^x y - 3e^x\\
&e^x y - y = 3e^x\\
&y(e^x - 1) = 3e^x\\
&y = \frac{3e^x}{e^x - 1}\\
&\boxed{f^{-1}(x) = \frac{3e^x}{e^x - 1}}
\end{aligned}
$$

(f) $f(x) = \dfrac{2^x}{2^x + 3}$

Solution.

$$
\begin{aligned}
&f(y) = x\\
&\frac{2^y}{2^y + 3} = x\\
&2^y = x(2^y + 3)\\
&2^y = x2^y + 3x\\
&2^y - x2^y = 3x\\
&2^y(1-x) = 3x\\
&2^y = \frac{3x}{1-x}\\
&y = \log_2\left(\frac{3x}{1-x}\right)\\
&\boxed{f^{-1}(x) = \log_2\left(\frac{3x}{1-x}\right)}
\end{aligned}
$$