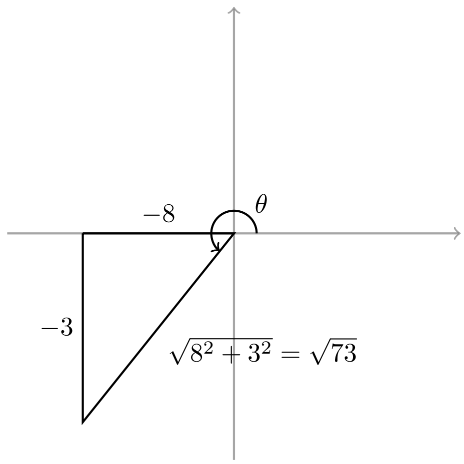
Rutgers University
Math 151

Midterm 1 Practice Session - Solutions

1. Suppose that θ is an angle such that $\cot(\theta) = \dfrac{8}{3}$ and $\csc(\theta) < 0$. Find $\sec(\theta)$, $\sin(2\theta)$ and $\cos(2\theta)$.

Solution. We use a right triangle placed in the correct quadrant. Since $\cot(\theta) > 0$ but $\csc(\theta) < 0$, we know that the terminal ray of the angle θ will be in quadrant III.



Using this triangle, we get $\boxed{\sec(\theta) = -\dfrac{\sqrt{73}}{8}}$. For the trigonometric functions of 2θ, we use double angle identities and we obtain

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\left(-\frac{3}{\sqrt{73}}\right)\left(-\frac{8}{\sqrt{73}}\right) = \boxed{\frac{48}{73}},$$

$$\cos(2\theta) = 2\cos^2(\theta) - 1 = 2\left(-\frac{8}{\sqrt{73}}\right)^2 - 1 = \boxed{\frac{55}{73}}.$$

2. Evaluate the following limits.

(a) $\displaystyle\lim_{x\to 1} \frac{\sqrt{4x^2+7} - \sqrt{x+10}}{x-1}$

Solution. Direct substitution gives $\frac{0}{0}$, so we rewrite the expression by rationalizing the numerator and canceling out common factors.

$$\begin{aligned}\lim_{x\to 1} \frac{\sqrt{4x^2+7} - \sqrt{x+10}}{x-1} &= \lim_{x\to 1} \frac{\sqrt{4x^2+7} - \sqrt{x+10}}{x-1} \cdot \frac{\sqrt{4x^2+7} + \sqrt{x+10}}{\sqrt{4x^2+7} + \sqrt{x+10}} \\ &= \lim_{x\to 1} \frac{4x^2+7-(x+10)}{(x-1)(\sqrt{4x^2+7} + \sqrt{x+10})}\end{aligned}$$

$$\begin{aligned}
&= \lim_{x\to 1} \frac{4x^2 - x - 3}{(x-1)(\sqrt{4x^2+7} + \sqrt{x+10})} \\
&= \lim_{x\to 1} \frac{(x-1)(4x+3)}{(x-1)(\sqrt{4x^2+7} + \sqrt{x+10})} \\
&= \lim_{x\to 1} \frac{4x+3}{\sqrt{4x^2+7} + \sqrt{x+10}} \\
&= \boxed{\frac{7}{2\sqrt{11}}}.
\end{aligned}$$

(b) $\displaystyle \lim_{h\to 0} \frac{\dfrac{6}{3+7h} - 2}{h}$

Solution. Direct substitution gives $\frac{0}{0}$, so we rewrite the expression as a simple fraction and cancel the common factors.

$$\begin{aligned}
\lim_{h\to 0} \frac{\dfrac{6}{3+7h} - 2}{h} &= \lim_{h\to 0} \frac{\dfrac{6-2(3+7h)}{3+7h}}{h} \\
&= \lim_{h\to 0} \frac{-14h}{h(3+7h)} \\
&= \lim_{h\to 0} \frac{-14}{3+7h} \\
&= \boxed{-\frac{14}{3}}.
\end{aligned}$$

(c) $\displaystyle \lim_{x\to 1^+} \frac{x^2-6x}{x^2-5x+4}$

Solution. Substitution gives $\frac{-5}{0}$, so the limit is infinite. To determine if the limit is ∞ or $-\infty$, we use a sign analysis. The function can be written as $\dfrac{x(x-6)}{(x-4)(x-1)}$. When $x \to 1^+$, the signs in this expression are $\dfrac{(+)(-)}{(-)(+)}$, so the expression is positive. Therefore,

$$\boxed{\lim_{x\to 1^+} \frac{x^2-6x}{x^2-5x+4} = \infty}.$$

(d) $\displaystyle \lim_{x\to -\infty} \frac{4x+3}{\sqrt{9x^2+5x+2}}$

Solution.

$$\begin{aligned}
\lim_{x\to -\infty} \frac{4x+3}{\sqrt{9x^2+5x+2}} &= \lim_{x\to -\infty} \frac{4x+3}{|x|\sqrt{9+\frac{5}{x}+\frac{2}{x^2}}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to -\infty} \frac{4x+3}{-x\sqrt{9+\frac{5}{x}+\frac{2}{x^2}}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}
\end{aligned}$$

$$= \lim_{x\to-\infty} \frac{4+\frac{3}{x}}{-\sqrt{9+\frac{5}{x}+\frac{2}{x^2}}}$$
$$= \frac{4+0}{-\sqrt{9+0+0}}$$
$$= \boxed{-\frac{4}{3}}.$$

(e) $\lim_{x\to 8} \frac{|3x-24|}{x^2-64}$.

Solution. Substitution gives $\frac{0}{0}$, so we need to do more work to determine this limit. Given that $x = 8$ is the transition point of the absolute value in the numerator, we'll investigate one-sided limits.

$$\lim_{x\to 8^-} \frac{|3x-24|}{x^2-64} = \lim_{x\to 8^-} \frac{-3(x-8)}{(x-8)(x+8)} = \lim_{x\to 8^-} -\frac{3}{x+8} = -\frac{3}{16},$$
$$\lim_{x\to 8^+} \frac{|3x-24|}{x^2-64} = \lim_{x\to 8^+} \frac{3(x-8)}{(x-8)(x+8)} = \lim_{x\to 8^+} \frac{3}{x+8} = \frac{3}{16}.$$

Since left and right limits are different, we conclude that $\boxed{\lim_{x\to 8} \frac{|3x-24|}{x^2-64} \text{ does not exist}}$.

(f) $\lim_{\theta\to\frac{\pi}{3}^+} \frac{1}{2\cos(\theta)-1}$

Solution. Subsitution gives $\frac{1}{0}$, so this limit is infinite. To determine if the limit is ∞ or $-\infty$, we investigate the sign of the expression. When $\theta \to \frac{\pi}{3}^+$, we have $\cos(\theta) < \frac{1}{2}$ as $\cos(\theta)$ decreases from $\theta = 0$ to $\theta = \frac{\pi}{2}$. Therefore, the denominator is negative as $\theta \to \frac{\pi}{3}^+$, and

$$\boxed{\lim_{\theta\to\frac{\pi}{3}^+} \frac{1}{2\cos(\theta)-1} = -\infty}.$$

(g) $\lim_{x\to 0} \frac{x\sin(5x)}{\tan^2(3x)}$

Solution.

$$\lim_{x\to 0} \frac{x\sin(5x)}{\tan^2(3x)} = \lim_{x\to 0} \frac{x\sin(5x)\cos^2(3x)}{\sin^2(3x)} \cdot \frac{5x}{5x} \cdot \frac{(3x)^2}{(3x)^2}$$
$$= \lim_{x\to 0} \frac{\sin(5x)}{5x} \cdot \left(\frac{3x}{\sin(3x)}\right)^2 \cdot \frac{x(5x)\cos^2(3x)}{(3x)^2}$$
$$= \left(\lim_{x\to 0} \frac{\sin(5x)}{5x}\right) \cdot \left(\lim_{x\to 0} \frac{3x}{\sin(3x)}\right)^2 \cdot \left(\lim_{x\to 0} \frac{5\cos^2(3x)}{9}\right)$$
$$= 1 \cdot 1^2 \cdot \frac{5\cos^2(0)}{9}$$
$$= \boxed{\frac{5}{9}}.$$

(h) $\lim_{x\to\infty} \dfrac{4e^{2x}+e^{3x}}{5e^{3x}+2e^{x}}$

Solution.

$$\begin{aligned}\lim_{x\to\infty} \frac{4e^{2x}+e^{3x}}{5e^{3x}+2e^{x}} &= \lim_{x\to\infty} \frac{4e^{2x}+e^{3x}}{5e^{3x}+2e^{x}} \cdot \frac{e^{-3x}}{e^{-3x}} \\ &= \lim_{x\to\infty} \frac{4e^{-x}+1}{5+2e^{-2x}} \\ &= \boxed{\frac{1}{5}}.\end{aligned}$$

(i) $\lim_{x\to -2} \dfrac{3x}{x^2+4x+4}$

Solution. The limit has the form $\frac{-6}{0}$, so the limit is infinite. Observe that the expression is equal to $\dfrac{3x}{(x+2)^2}$. As $x \to -2$, the signs in the expression are $\dfrac{(-)}{(+)} = (-)$, so

$$\boxed{\lim_{x\to -2} \frac{3x}{x^2+4x+4} = -\infty}.$$

3. Suppose that $f(x)$ is a function such that $\cos(\pi x) \leqslant f(x) \leqslant x^4 - 8x^2 + 17$ for $-5 < x < 6$. Find the following limits, if possible. If there is not enough information to find a limit, explain why.

(a) $\lim_{x\to -2} f(x)$

Solution. We use the Squeeze Theorem. Since

$$\lim_{x\to -2} \cos(\pi x) = \cos(-2\pi) = 1, \qquad \lim_{x\to -2} x^4 - 8x^2 + 17 = 16 - 32 + 17 = 1,$$

we conclude that $\boxed{\lim_{x\to -2} f(x) = 1}$.

(b) $\lim_{x\to 1} f(x)$

Solution. The Squeeze Theorem does not apply since

$$\lim_{x\to 1} \cos(\pi x) = \cos(\pi) = -1, \qquad \lim_{x\to 1} x^4 - 8x^2 + 17 = 1 - 8 + 17 = 10.$$

Therefore, there is not enough information to find $\lim_{x\to 1} f(x)$.

(c) $\lim_{x\to 2} f(x)$

Solution. We use the Squeeze Theorem. Since

$$\lim_{x\to 2} \cos(\pi x) = \cos(2\pi) = 1, \qquad \lim_{x\to 2} x^4 - 8x^2 + 17 = 16 - 32 + 17 = 1,$$

we conclude that $\boxed{\lim_{x\to 2} f(x) = 1}$.

4. Find horizontal asymptotes of $f(x) = \dfrac{|6x+1|+7x}{3x+8}$.

Solution. To find the horizontal asymptotes of f, we need to compute the limits at ∞ and $-\infty$. We have

$$\lim_{x\to\infty} \frac{|6x+1|+7x}{3x+8} = \lim_{x\to\infty} \frac{6x+1+7x}{3x+8} = \lim_{x\to\infty} \frac{13x+1}{3x+8} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{13+\frac{1}{x}}{3+\frac{8}{x}} = \frac{13}{3},$$
$$\lim_{x\to-\infty} \frac{|6x+1|+7x}{3x+8} = \lim_{x\to-\infty} \frac{-(6x+1)+7x}{3x+8} = \lim_{x\to-\infty} \frac{x-1}{3x+8} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to-\infty} \frac{1-\frac{1}{x}}{3+\frac{8}{x}} = \frac{1}{3}.$$

So the horizontal asymptotes of f are $\boxed{y = \frac{13}{3}, y = \frac{1}{3}}$.

5. Find the vertical asymptotes of $f(x) = \dfrac{\sqrt{x^2+8}-3}{x^2-3x-4}$. Also find the limits to the left and right of any vertical asymptote.

Solution. The discontinuities of f occur when $x^2-3x-4=0$, that is $x=-1$, $x=4$. We now test if the limits at these points are finite or infinite.

- At $x=4$, substitution gives $\frac{\sqrt{24}-3}{0}$, so $\boxed{x=4}$ is a vertical asymptote. For the one sided limits, we have

$$\lim_{x\to4^-} \frac{\sqrt{x^2+8}-3}{x^2-3x-4} = \lim_{x\to4^-} \frac{\sqrt{x^2+8}-3}{(x+1)(x-4)} = \frac{(+)}{(+)(-)}\infty = \boxed{-\infty},$$
$$\lim_{x\to4^+} \frac{\sqrt{x^2+8}-3}{x^2-3x-4} = \lim_{x\to4^+} \frac{\sqrt{x^2+8}-3}{(x+1)(x-4)} = \frac{(+)}{(+)(+)}\infty = \boxed{\infty}.$$

- At $x=-1$, substitution gives $\frac{0}{0}$, so we need more analysis. We have

$$\begin{aligned}
\lim_{x\to-1} \frac{\sqrt{x^2+8}-3}{x^2-3x-4} &= \lim_{x\to-1} \frac{\sqrt{x^2+8}-3}{(x+1)(x-4)} \cdot \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3} \\
&= \lim_{x\to-1} \frac{x^2+8-9}{(x+1)(x-4)\left(\sqrt{x^2+8}+3\right)} \\
&= \lim_{x\to-1} \frac{x^2-1}{(x+1)(x-4)\left(\sqrt{x^2+8}+3\right)} \\
&= \lim_{x\to-1} \frac{(x-1)(x+1)}{(x+1)(x-4)\left(\sqrt{x^2+8}+3\right)} \\
&= \lim_{x\to-1} \frac{x-1}{(x-4)\left(\sqrt{x^2+8}+3\right)} \\
&= \frac{-2}{-5\,(3+3)}.
\end{aligned}$$

Since this limit is finite, $x=-1$ is not a vertical asymptote. (There is a removable discontinuity at $x=-1$.)

6. Find the values of the constants A and B making the following function continuous on $\mathbb{R}$.

$$f(x) = \begin{cases} \arctan(Ax) & \text{if } x > 3, \\ \dfrac{\pi(x+2)}{20} & \text{if } 0 \leqslant x \leqslant 3, \\ \dfrac{x^2 + Bx}{|x|} & \text{if } x < 0. \end{cases}$$

Solution. Each piece of f is a continuous function, so it suffices to test for continuity at the transition points. At $x = 3$, we want $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = f(3)$. We have

$$\begin{aligned} \lim_{x\to 3^+} f(x) &= \lim_{x\to 3^+} \arctan(Ax) = \arctan(3A), \\ \lim_{x\to 3^-} f(x) &= \lim_{x\to 3^-} \frac{\pi(x+2)}{20} = \frac{\pi(3+2)}{20} = \frac{\pi}{4}, \\ f(3) &= \frac{\pi(3+2)}{20} = \frac{\pi}{4}. \end{aligned}$$

So we obtain the condition $\arctan(3A) = \frac{\pi}{4}$, which gives $3A = \tan\left(\frac{\pi}{4}\right) = 1$, so $\boxed{A = \dfrac{1}{3}}$.

At $x = 0$, we want $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$. We have

$$\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \frac{\pi(x+2)}{20} = \frac{\pi(0+2)}{20} = \frac{\pi}{10}, \\ \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} \frac{x^2+Bx}{|x|} = \lim_{x\to 0^-} \frac{x(x+B)}{-x} = \lim_{x\to 0^-} -(x+B) = -B, \\ f(0) &= \frac{\pi(0+2)}{20} = \frac{\pi}{10}. \end{aligned}$$

So we obtain $\boxed{B = -\dfrac{\pi}{10}}$.

7. Use the Intermediate Value Theorem to show that the equation $\sec^{-1}(x) = 3 - x$ has a solution in the interval $[1, 2]$.

Solution. The equation is equivalent to

$$\sec^{-1}(x) + x = 3.$$

Put $f(x) = \sec^{-1}(x) + x$ and $y_0 = 3$. The function f is continuous on $[1, 2]$. Furthermore, we have

- $f(1) = \sec^{-1}(1) + 1 = 0 + 1 = 1 < 3$,
- $f(2) = \sec^{-1}(2) + 2 = \dfrac{\pi}{3} + 2 > 3$ since $\pi > 3$, so $\dfrac{\pi}{3} > 1$.

Thus $y_0 = 3$ is an intermediate value between $f(1)$ and $f(2)$. By the IVT, there exists x_0 in $[1, 2]$ such that $f(x_0) = y_0$, that is $\sec^{-1}(x_0) + x_0 = 3$. Thus x_0 provides a solution to the given equation.

8. Let $f(x) = \dfrac{3x}{\sqrt{x}+5}$. Use the limit definition of derivatives to compute $f'(1)$, then find an equation of the tangent line to the graph of f at $x = 1$.

Solution. We have $f(1) = \dfrac{3}{6} = \dfrac{1}{2}$ and

$$\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{3(1+h)}{\sqrt{1+h}+5} - \dfrac{1}{2}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{6(1+h) - \sqrt{1+h} - 5}{2(\sqrt{1+h}+5)}}{h} \\
&= \lim_{h\to 0} \frac{1+6h-\sqrt{1+h}}{2h(\sqrt{1+h}+5)} \cdot \frac{1+6h+\sqrt{1+h}}{1+6h+\sqrt{1+h}} \\
&= \lim_{h\to 0} \frac{(1+6h)^2 - (\sqrt{1+h})^2}{2h(\sqrt{1+h}+5)\left(1+6h+\sqrt{1+h}\right)} \\
&= \lim_{h\to 0} \frac{1+12h+36h^2-1-h}{2h(\sqrt{1+h}+5)\left(1+6h+\sqrt{1+h}\right)} \\
&= \lim_{h\to 0} \frac{11h+36h^2}{2h(\sqrt{1+h}+5)\left(1+6h+\sqrt{1+h}\right)} \\
&= \lim_{h\to 0} \frac{11+36h}{2(\sqrt{1+h}+5)\left(1+6h+\sqrt{1+h}\right)} \\
&= \frac{11+0}{2(\sqrt{1+0}+5)\left(1+0+\sqrt{1+0}\right)} \\
&= \boxed{\frac{11}{24}}.
\end{aligned}$$

Therefore, the tangent line at $x = 1$ passes through the point $\left(1, \frac{1}{2}\right)$ and has slope $\frac{11}{24}$. Hence, it has equation

$$\boxed{y - \frac{1}{2} = \frac{11}{24}(x-1)}.$$