Rutgers University
Math 151

Midterm 2 Practice Session Solutions

1. Find $\frac{dy}{dx}$ for the following equations.

(a) $y = \cos(7) + 4e^{3x} - \frac{5}{\sqrt[8]{x^3}}$

Solution. $\boxed{\frac{dy}{dx} = 0 + 12e^{3x} - 5\left(-\frac{3}{8}\right)x^{-11/3}}$

(b) $y = \arctan(7\ln(x))$

Solution. $\boxed{\frac{dy}{dx} = \frac{1}{1+(7\ln(x))^2} \cdot \frac{7}{x}}$

(c) $y = x^2 \sin^{-1}(2x)e^{-x}$

Solution. $\boxed{\frac{dy}{dx} = 2x\sin^{-1}(2x)e^{-x} + x^2\frac{2}{\sqrt{1-(2x)^2}}e^{-x} + x^2\sin^{-1}(2x)e^{-x}(-1)}$

(d) $y = \sec^3\left(\frac{2}{x} - e^{-4x}\right)$

Solution. $\boxed{\frac{dy}{dx} = 3\sec^2\left(\frac{2}{x} - e^{-4x}\right)\sec\left(\frac{2}{x} - e^{-4x}\right)\tan\left(\frac{2}{x} - e^{-4x}\right)\left(-\frac{2}{x^2} + 4e^{-4x}\right)}$

(e) $y = (1-3x)^{8\cot(5x^2)}$

Solution. We use logarithmic differentiation because the base and the exponent both depend on x.

$$\begin{aligned}
\ln(y) &= \ln\left((1-3x)^{8\cot(5x^2)}\right) = 8\cot(5x^2)\ln(1-3x) \\
\Rightarrow \frac{y'}{y} &= 8(-\csc^2(5x^2))(10x)\ln(1-3x) + 8\cot(5x^2)\frac{-3}{1-3x} \\
&= -8\left(10x\csc^2(5x^2) + \frac{3\cot(5x^2)}{1-3x}\right) \\
\Rightarrow y' &= -8y\left(10x\csc^2(5x^2) + \frac{3\cot(5x^2)}{1-3x}\right) \\
&= \boxed{-8(1-3x)^{8\cot(5x^2)}\left(10x\csc^2(5x^2) + \frac{3\cot(5x^2)}{1-3x}\right)}
\end{aligned}$$

(f) $y = \sqrt{4-9x^2} - \sec^{-1}(3x)$

Solution. $\boxed{\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{4-9x^2}}(-18x) - \dfrac{3}{|3x|\sqrt{(3x)^2-1}}}$

(g) $y = \dfrac{\sqrt[7]{x^2}(x^2+6x+1)^{32}}{(2x+1)^{10}(x+2)^3}$

Solution. We use logarithmic differentiation to make this derivative calculation less of a pain. We have

$$\ln(y) = \frac{2}{7}\ln(x) + 32\ln(x^2+6x+1) - 10\ln(2x+1) - 3\ln(x+2).$$

Differentiating both sides with respect to x gives

$$\begin{aligned} \frac{y'}{y} &= \frac{2}{7x} + \frac{32(2x+6)}{x^2+6x+1} - \frac{20}{2x+1} - \frac{3}{x+2} \\ \Rightarrow y' &= y\left(\frac{2}{7x} + \frac{32(2x+6)}{x^2+6x+1} - \frac{20}{2x+1} - \frac{3}{x+2}\right) \\ &= \boxed{\frac{\sqrt[7]{x^2}(x^2+6x+1)^{32}}{(2x+1)^{10}(x+2)^3}\left(\frac{2}{7x} + \frac{32(2x+6)}{x^2+6x+1} - \frac{20}{2x+1} - \frac{3}{x+2}\right)} \end{aligned}$$

(h) $y = \dfrac{5^x}{\cos(2x)+3x}$

Solution. $\boxed{\dfrac{dy}{dx} = \dfrac{\ln(5)5^x(\cos(2x)+3x) - 5^x(-2\sin(2x)+3)}{(\cos(2x)+3x)^2}}$

(i) $y = \sin(3x)^{x^2}$

Solution. We use logarithmic differentiation because the base and the exponent both depend on x.

$$\begin{aligned} \ln(y) &= \ln\left(\sin(3x)^{x^2}\right) \\ &= x^2\ln(\sin(3x)) \\ \Rightarrow \frac{y'}{y} &= 2x\ln(\sin(3x)) + x^2\frac{3\cos(3x)}{\sin(3x)} \\ &= 2x\ln(\sin(3x)) + 3x^2\cot(3x) \\ \Rightarrow y' &= y\left(2x\ln(\sin(3x)) + 3x^2\cot(3x)\right) \\ &= \boxed{\sin(3x)^{x^2}\left(2x\ln(\sin(3x)) + 3x^2\cot(3x)\right)} \end{aligned}$$

2. Find the values of the constants A, B for which the following function is differentiable at $x = 1$.

$$f(x) = \begin{cases} 10 - 3x & \text{if } x < 1, \\ x^A + Bx + 3 & \text{if } x \geqslant 1. \end{cases}$$

Solution. For the function to be continuous at $x = 1$, we will need $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$. This produces the condition $10 - 3 = 1^A + B + 3$, so $7 = B + 4$ and $\boxed{B = 3}$.

To have differentiability at $x = 1$, the slope from the left and right at $x = 1$ must be equal. We have

$$f'(x) = \begin{cases} -3 & \text{if } x < 1, \\ Ax^{A-1} + B & \text{if } x > 1. \end{cases}$$

So we get the condition $-3 = A1^{A-1} + B = A + B$. Since $B = 3$, we obtain $\boxed{A = -6}$.

3. Suppose that f is a one-to-one differentiable function. The following table of values is given for f and f'.

x	-1	0	1	2
$f(x)$	2	3	6	11
$f'(x)$	7	2	8	5

(a) Find an equation of the tangent line to the graph of $y = f(x)$ at the point $x = -1$.

Solution. $\boxed{y - 2 = 7(x + 1)}$

(b) Find an equation of the tangent line to the graph of $y = f^{-1}(x)$ at the point $x = 2$.

Solution. $\boxed{y + 1 = \frac{1}{7}(x - 2)}$

(c) Let $G(x) = 2^{7x} f(3x)$. Calculate $G'(0)$.

Solution. The derivative of G is

$$G'(x) = 7\ln(2)2^{7x} f(3x) + 2^{7x}3f'(3x).$$

So $G'(0) = 7\ln(2)f(0) + 3f'(0) = \boxed{21\ln(2) + 6}$.

(d) Let $H(x) = \tan^{-1}\left(f(x^2)\right)$. Calculate $H'(-1)$.

Solution. The derivative of H is

$$H'(x) = \frac{1}{1 + f(x^2)^2} f'(x^2)(2x).$$

So $H'(-1) = \frac{1}{1 + f(1)^2} f'(1)(-2) = \boxed{-\frac{16}{37}}$.

(e) Let $K(x) = \sqrt{f(2 - x)^2 + e^{16x}}$. Calculate $K'(0)$.

Solution. The derivative of K is

$$K'(x) = \frac{1}{2\sqrt{f(2 - x) + e^{16x}}}(f'(2 - x)(-1) + 16e^{16x}).$$

So $K'(0) = \dfrac{1}{2\sqrt{f(2)+1}}(-f'(2)+16) = \boxed{\dfrac{11}{2\sqrt{12}}}$.

(f) Let $M(x) = \cos(\pi x) f(2x)$. Calculate $M''\left(\frac{1}{2}\right)$.

Solution. The first derivative of M is

$$M'(x) = -\pi \sin(\pi x) f(2x) + 2\cos(\pi x) f'(2x).$$

The second derivative of M is

$$M''(x) = -\pi^2 \cos(\pi x) f(2x) - 2\pi \sin(\pi x) f'(2x) - 2\pi \sin(\pi x) f'(2x) + 4\cos(\pi x) f''(2x) = -\pi^2 \cos(\pi x) f(2x) - 4\pi \sin(\pi x) f'$$

So $M''\left(\frac{1}{2}\right) = -\pi^2 \cos\left(\frac{\pi}{2}\right) f(1) - 4\pi \sin\left(\frac{\pi}{2}\right) f'(1) + 4\cos\left(\frac{\pi}{2}\right) f''(1) = 0 - 4\pi f'(1) + 0 = \boxed{-32\pi}$.

4. Consider the curve of equation $x^2 + 6xy - y^2 = 40$. Find the points on the curve, if any, where the tangent line is (a) horizontal, (b) vertical, (c) perpendicular to $y = 2x + 9$.

Solution. First, let us differentiate the relation with respect to x:

$$\begin{aligned} 2x + 6y + 6xy' - 2yy' &= 0 \\ x + 3y + 3xy' - yy' &= 0. \end{aligned}$$

(a) The tangent line is horizontal when $y' = 0$. Using this in the previous equation, we get $x + 3y = 0$, or $x = -3y$. Plugging this in the equation of the curve gives $(-3y)^2 + 6(-3y)y - y^2 = 40$, or $-10y^2 = 40$. This equation has no solution, so there are no points on the curve where the tangent line is horizontal.

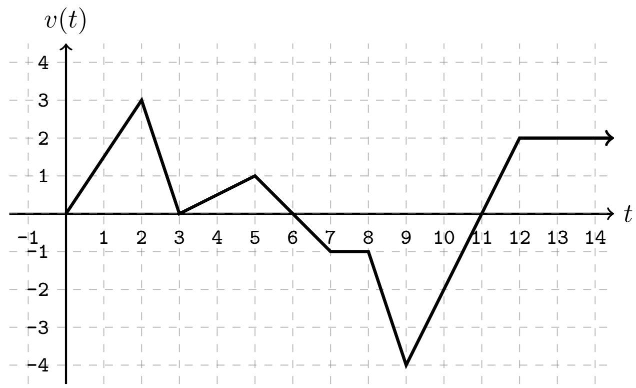
(b) Solving for y' in the previous equation gives $y' = -\dfrac{x+3y}{3x-y}$, so the tangent line is vertical when $y = 3x$. Plugging this in the equation of the curve gives $x^2 + 6x(3x) - (3x)^2 = 40$, or $10x^2 = 40$. We get $x^2 = 4$, that is $x = 2$ (which gives $y = 6$) and $x = -2$ (which gives $y = -6$). So the points where the tangent line is vertical are $\boxed{(2,6), (-2,-6)}$.

(c) The tangent line is perpendicular to $y = 2x+9$ when $y' = -\frac{1}{2}$. Plugging this in $2x+6y+6xy'-2yy' = 0$ gives $2x + 6y - 3x + y = 0$, or $x = 7y$. Substituting $x = 7y$ in the equation of the curve gives

$$\begin{aligned} (7y)^2 + 6(7y)y - y^2 &= 40 \\ 90y^2 &= 40 \\ y^2 &= \frac{4}{9} \\ y &= \pm\frac{2}{3}. \end{aligned}$$

For $y = \frac{2}{3}$, we get $x = \frac{14}{3}$ and for $y = -\frac{2}{3}$, we get $x = -\frac{14}{3}$. Therefore, the points on the curve where the tangent line is perpendicular to $y = 2x + 9$ are $\boxed{\left(\frac{14}{3}, \frac{2}{3}\right), \left(-\frac{14}{3}, -\frac{2}{3}\right)}$.

5. The graph below shows the velocity v of an object moving along an axis.



(a) When is the object moving forward? backward? standing still?

Solution. The object moves forward when $v(t) > 0$, that is $\boxed{0 < t < 3, 3 < t < 6, 11 < t}$. The object moves backward when $v(t) < 0$, that is $\boxed{6 < t < 11}$. The object is standing still when $v(t) = 0$, that is $\boxed{t = 0, 3, 6, 11}$.

(b) When does the object reverse direction?

Solution. The object reverses direction when $v(t)$ changes sign, which happens at $\boxed{t = 6, 11}$.

(c) When does the object move at greatest speed?

Solution. The greatest speed reached by the object is $|-4| = 4$ at $\boxed{t = 9}$.

(d) When is the acceleration positive?

Solution. The acceleration is positive when $v(t)$ is increasing, that is $\boxed{0 < t < 2, 3 < t < 5, 9 < t < 12}$.

(e) What the particle average acceleration on the interval $5 \leqslant t \leqslant 8$?

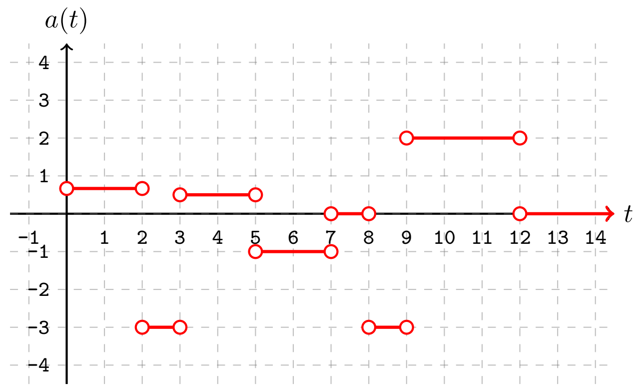
Solution. The average acceleration on $5 \leqslant t \leqslant 8$ is

$$\frac{v(8) - v(5)}{8 - 5} = \frac{-1 - 1}{3} = \boxed{-\frac{2}{3}}.$$

(f) What the exact value of the acceleration at $t = 1$?

Solution. The acceleration at $t = 1$ is the slope of $v(t)$ at $t = 1$. Inspecting the graph, we see that the graph of $v(t)$ is a line of slope $\frac{3}{2}$ on $0 < t < 2$. So $\boxed{a(1) = \frac{3}{2}}$.

(g) Sketch the graph of the acceleration of the object.
Solution. Recall that the value of the acceleration is the slope of the graph of the velocity.



6. A snow ball in the shape of a perfect sphere melts at a rate of 4 cm^3/min. How fast is the surface area changing when the radius of the sphere is 7 cm? *[Hint: the volume and surface area of a sphere of radius R are given by the formulas $V = \frac{4}{3}\pi R^3$, $S = 4\pi R^2$]*

   Solution. We'll start with the relations relating the variables and differentiate them with respect to the time t to get equations relating the rates of change.

$$\begin{cases} V = \frac{4}{3}\pi R^3 \\ \\ S = 4\pi R^2 \end{cases} \xrightarrow{\frac{d}{dt}} \begin{cases} \dfrac{dV}{dt} = \frac{4}{3}\pi 3R^2 \dfrac{dR}{dt} = 4\pi R^2 \dfrac{dR}{dt} \\ \\ \dfrac{dS}{dt} = 8\pi R \dfrac{dR}{dt} \end{cases}$$

   We know the values $\dfrac{dV}{dt} = -4$ (negative because the ball is melting, so volume decreases) and $R = 7$, and we want to solve for $\dfrac{dS}{dt}$. Plugging the values in the equations relating the rates gives

$$\begin{cases} -4 = 4\pi \cdot 7^2 \cdot \dfrac{dR}{dt} \\ \\ \dfrac{dS}{dt} = 8\pi \cdot 7 \cdot \dfrac{dR}{dt} \end{cases}$$

   Using the first equation, we can solve for $\dfrac{dR}{dt}$ to get $\dfrac{dR}{dt} = -\dfrac{1}{49\pi}$. Using this in the second equation gives

$$\frac{dS}{dt} = 8\pi \cdot 7 \cdot \left(-\frac{1}{49\pi}\right) = \boxed{-\frac{8}{7}\ \text{cm}^2/\text{min}}.$$

7. Find the value of the constant A such that the tangent line to $y = 2e^{Ax} + \tan^{-1}(7x)$ at $x = 0$ passes through the point $(-3, 11)$.

   Solution. First, let us find the equation of the tangent line to the graph at $x = 0$. We have $f(0) = 2e^0 + \tan^{-1}(0) = 2$ and $f'(x) = 2Ae^{Ax} + \frac{7}{1+(7x)^2}$, so $f'(0) = 2A + 7$. Therefore, the equation of the tangent line at $x = 0$ is

$$y - 2 = (2A + 7)x.$$

   Since we want the tangent line to pass through $(-3, 11)$, we must have $11 - 2 = (2A + 7)(-3)$. This gives $2A + 7 = -3$, so $\boxed{A = -5}$.