Rutgers University
Math 151

Midterm 3 Practice Session Solutions

1. The two parts of this problem are independent.

(a) Suppose $f(-2) = 7$ and $f(1) = -4$. Fill in the blanks below. Your answer to the last blank must be a real number.

Solution.
If f is $\boxed{\textbf{continuous}}$ on the interval $[-2, 1]$ and $\boxed{\textbf{differentiable}}$ on the interval $(-2, 1)$, then the Mean Value Theorem guarantees the existence of a number c in the interval $\boxed{(-2,1)}$ such that the the slope of the tangent line to the graph of f at $x = c$ is equal to $\dfrac{f(1) - f(-2)}{1 - (-2)} = \boxed{-\dfrac{11}{3}}$.

(b) Suppose that f is a differentiable function such that $f'(x) \geqslant -2$ and $f(3) = 4$. Find the maximum possible value of $f(-1)$ and the minimum possible value of $f(5)$.

Solution. Using the MVT, we have $\dfrac{f(3) - f(-1)}{3 - (-1)} = f'(c)$ for some c in $(-1, 3)$. So

$$f(3) - f(-1) = 4f'(c) \ \Rightarrow\ f(-1) = f(3) - 4f'(c) = 4 - 4f'(c).$$

Since $f'(c) \geqslant -2$, we have $4f'(c) \geqslant -8$, so $f(-1) \leqslant 4 + 8 = \boxed{12}$.

Likewise, we have $\dfrac{f(5) - f(3)}{5 - 3} = f'(d)$ for some d in $(3, 5)$. So

$$f(5) - f(3) = 2f'(d) \ \Rightarrow\ f(5) = f(3) + 2f'(d) = 4 + 2f'(d).$$

Since $f'(d) \geqslant -2$, we have $2f'(c) \geqslant -4$, so $f(5) \geqslant 4 - 4 = \boxed{0}$.

2. Let $f(x) = \sqrt[3]{4\cos^2(x) - 1}$. Find the absolute extrema and where they occur for $f(x)$ on the interval $\left[-\frac{\pi}{4}, \frac{\pi}{2}\right]$.

Solution. First, we find the critical points of $f(x)$ in $\left(-\frac{\pi}{4}, \frac{\pi}{2}\right)$. We have

$$f'(x) = \frac{1}{3}(4\cos^2(x) - 1)^{-2/3}(-8\cos(x)\sin(x)) = -\frac{8\cos(x)\sin(x)}{3(4\cos^2(x) - 1)^{2/3}}.$$

- $f'(x) = 0$ gives $-8\cos(x)\sin(x) = 0$. The only solution in $\left(-\frac{\pi}{4}, \frac{\pi}{2}\right)$ is $x = 0$.
- $f'(x)$ undefined gives $3\left(4\cos^2(x) - 1\right)^{2/3} = 0$, that is $\cos(x) = \pm\frac{1}{2}$. The only solution in $\left(-\frac{\pi}{4}, \frac{\pi}{2}\right)$ is $x = \frac{\pi}{3}$.

We now evaluate $f(x)$ at the critical points in $\left(-\frac{\pi}{4}, \frac{\pi}{2}\right)$ and at the endpoints.

- $f\left(-\frac{\pi}{4}\right) = \sqrt[3]{4\left(\frac{\sqrt{2}}{2}\right)^2 - 1} = 1$
- $f(0) = \sqrt[3]{4-1} = \sqrt[3]{3}$
- $f\left(\frac{\pi}{3}\right) = \sqrt[3]{4\left(\frac{1}{2}\right)^2 - 1} = 0$
- $f\left(\frac{\pi}{2}\right) = \sqrt[3]{0-1} = -1$

Therefore, the absolute maximum of f on $\left(-\frac{\pi}{4}, \frac{\pi}{2}\right)$ is $\boxed{\sqrt[3]{3}}$ and it occurs at $\boxed{x=0}$. The absolute minimum of f on $\left(-\frac{\pi}{4}, \frac{\pi}{2}\right)$ is -1 and it occurs at $\boxed{x = \frac{\pi}{2}}$.

3. Let $f(x) = \ln(x^2+4)$. Find:

   (a) Find:

   (i) the critical points of f.
   (ii) the open intervals where f is increasing and decreasing.
   (iii) the open intervals where f is concave up and concave down.
   (iv) the x-coordinates of the local maxima and local minima of f.
   (v) the x-coordinates of the inflection points of f.

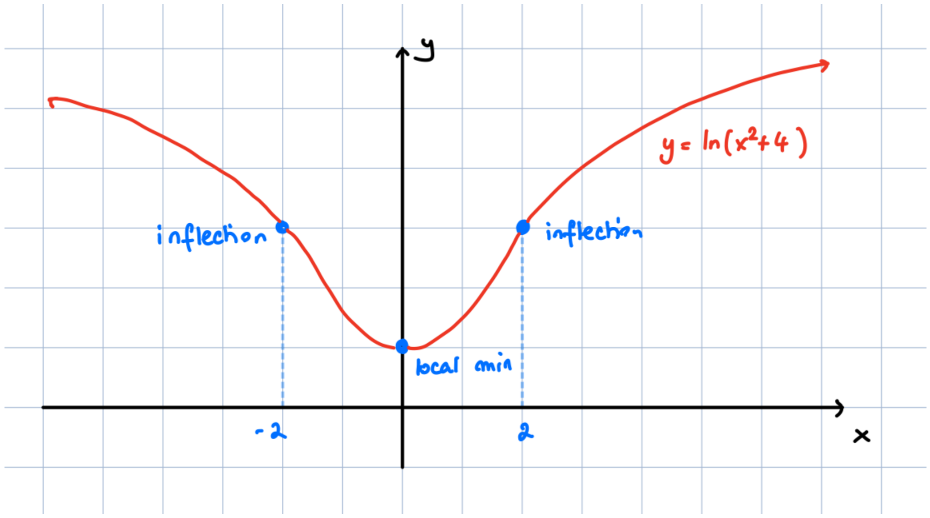
   Solution. We have

$$f'(x) = \frac{2x}{x^2+4}, \quad f''(x) = \frac{2(x^2+4)-(2x)(2x)}{(x^2+4)^2} = \frac{8-2x^2}{(x^2+4)^2} = \frac{2(2-x)(2+x)}{(x^2+4)^2}.$$

   (i) the critical points of f: $\boxed{x=0}$.
   (ii) the open intervals where f is increasing and decreasing: increasing on $\boxed{(0,\infty)}$ and decreasing on $\boxed{(-\infty,0)}$.
   (iii) the open intervals where f is concave up and concave down: concave up on $\boxed{(-2,2)}$ and concave down on $\boxed{(\infty,-2),(2,\infty)}$.
   (iv) the x-coordinates of the local maxima and local minima of f: no local maximum, local minimum at $\boxed{x=0}$.
   (v) the x-coordinates of the inflection points of f: $\boxed{x=-2,2}$.

   (b) Sketch the graph of f.

   Solution.



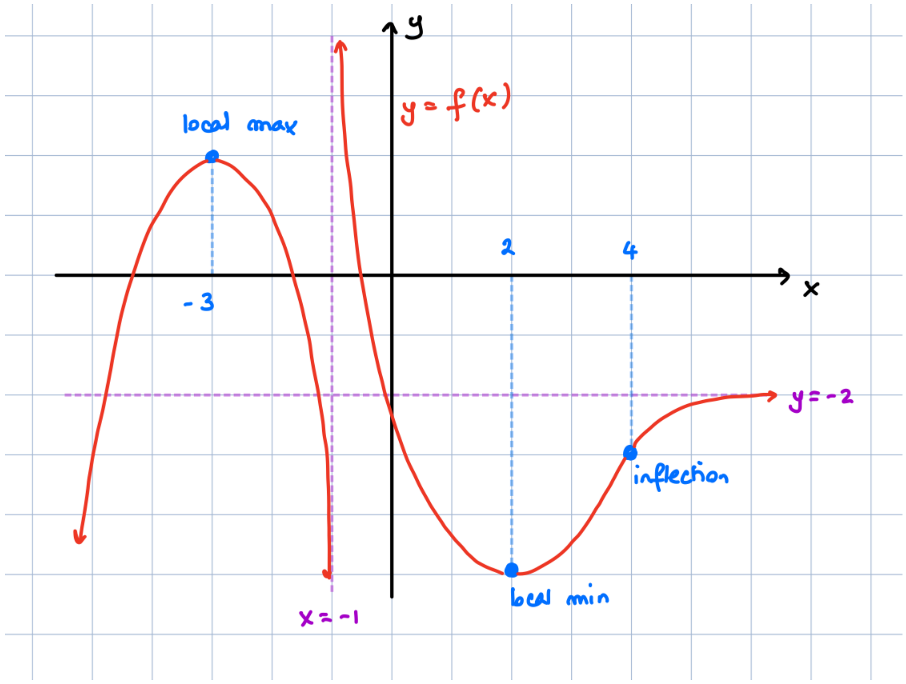
4. Sketch the graph of a function f with the given properties.

- $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = -2$.
- $\lim_{x\to-1^-} f(x) = -\infty$ and $\lim_{x\to-1^+} f(x) = \infty$.
- $f(-3) = 2$, $f(2) = -5$, $f(4) = -3$.
- The first two derivatives of f have the following sign chart.

x	$(-\infty, -3)$	$(-3, -1)$	$(-1, 2)$	$(2, 4)$	$(4, \infty)$
$f'(x)$	$+$	$-$	$-$	$+$	$+$
$f''(x)$	$-$	$-$	$+$	$+$	$-$

Label all asymptotes, local extrema and inflection points.

Solution.



5. A closed cylindrical box has total surface area 150π ft^2. Find the dimensions of the box (height and radius) that give the maximum possible volume.

Solution. The objective function is the volume $V = \pi r^2 h$. This is subject to the constraint that the total surface area is 150π, so $2\pi r^2 + 2\pi rh = 150$. This gives $h = \frac{150\pi - 2\pi r^2}{2\pi r} = \frac{75-r^2}{r}$. So the volume expressed in terms of r only is

$$V(r) = \pi r^2 \frac{75 - r^2}{r} = \pi r(75 - r^2) = \pi(75r - r^3).$$

To find the interval of interest, observe that lengths cannot be negative, so we need $r \geqslant 0$ and $\frac{75-r^2}{r}$. This last condition gives $r \neq 0$ and $r \leqslant \sqrt{75}$. So the interval of interest is $(0, \sqrt{75}]$.

We now find the critical points in the interval. We have $V'(r) = \pi(75 - 3r^2)$, so $V'(r) = 0$ when $r = 5$. To check that this does give the maximum, we can use the SDT and observe that $V''(r) = \pi(-6r)$, which is negative on $(0, \sqrt{75}]$. Hence, f is concave down on $(0, \sqrt{75}]$. So the volume is maximal when $\boxed{r = 5 \text{ ft}}$ and $\boxed{h = 10 \text{ ft}}$.

6. A particle moving along an axis has acceleration $a(t) = \dfrac{8}{t^2} + 6$. Find the position $s(t)$ of the particle if $v(-1) = 3$ and $s(-1) = 5$.

Solution. First, we find the velocity.

$$v(t) = \int \left(\frac{8}{t^2} + 6\right) dt = -\frac{8}{t} + 6t + C.$$

To find the value of C, we use the initial condition $v(-1) = 3$. This gives $8 - 6 + C = 3$, so $C = 1$ and $v(t) = -\frac{8}{t} + 6t + 1$. for the position, we get

$$s(t) = \int \left(-\frac{8}{t} + 6t + 1\right) dt = -8\ln|t| + 3t^2 + t + D.$$

To find the value of D, we use the initial condition $s(-1) = 5$. This gives $-8\ln|-1| + 3 - 1 + D = 5$, so $D = 3$. Hence, $\boxed{s(t) = -8\ln|t| + 3t^2 + t + 3}$.

7. Evaluate the following limits.

   (a) $\lim_{x \to \frac{\pi}{6}} \sec^2(3x)\ln(\sin(3x))$

   Solution. This limit is an indeterminate form $\infty \cdot 0$. We can rewrite the expression as a fraction and use L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x \to \frac{\pi}{6}} \sec^2(3x)\ln(\sin(3x)) &= \lim_{x \to \frac{\pi}{6}} \frac{\ln(\sin(3x))}{\cos^2(3x)} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x \to \frac{\pi}{6}} \frac{\frac{3\cos(3x)}{\sin(3x)}}{-6\cos(3x)\sin(3x)} \\
&= \lim_{x \to \frac{\pi}{6}} -\frac{1}{2\sin^2(3x)} \\
&= \boxed{-\frac{1}{2}}.
\end{aligned}$$

   (b) $\lim_{x \to \infty} \left(\frac{2\arctan(5x)}{\pi}\right)^x$

   Solution. This limit is an indeterminate power 1^∞. **Warning:** limits of the form 1^∞ need not be equal to 1! This is because the base is not equal to 1, it is *approaching* 1. We can resolve the indeterminate form by taking the ln of the limit L and applying L'Hôpital's Rule. This gives.

$$\begin{aligned}
\ln(L) &= \lim_{x \to \infty} x \ln\left(\frac{2\arctan(5x)}{\pi}\right) \\
&= \lim_{x \to \infty} \frac{\ln\left(\frac{2\arctan(5x)}{\pi}\right)}{\frac{1}{x}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x \to \infty} \frac{\frac{\pi}{2\arctan(5x)} \cdot \frac{2\cdot 5}{\pi(1+25x^2)}}{-\frac{1}{x^2}} \\
&= \lim_{x \to \infty} -\frac{5x^2}{\arctan(5x)(1+25x^2)} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= \lim_{x \to \infty} -\frac{5}{\arctan(5x)\left(\frac{1}{x^2} + 25\right)} \\
&= -\frac{5}{\frac{\pi}{2}(0 + 25)} \\
&= -\frac{2}{5\pi}.
\end{aligned}$$

   This is the ln of the original limit, so we now solve for L and we get $\boxed{L = e^{-2/5\pi}}$.