Section 10.2: Infinite Series - Worksheet Solutions

#60. Each of the series below is either geometric or telescoping. Determine if each series converges or diverges, and compute its sum if it converges.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-\pi)^n}{8}$

Solution: This is a geometric series with common ratio $r = -\pi$. Since $|r| > 1$, we conclude that $\boxed{\displaystyle\sum_{n=0}^{\infty} \frac{(-\pi)^n}{8} \text{ diverges}}$.

(b) $\displaystyle\sum_{n=4}^{\infty} 2^n 3^{-n}$

Solution: Rewriting the series as

$$\sum_{n=4}^{\infty} 2^n 3^{-n} = \sum_{n=4}^{\infty} \left(\frac{2}{3}\right)^n,$$

we see that the series is geometric with common ratio $r = \frac{2}{3}$. Since $|r| < 1$, we conclude that $\boxed{\text{the series converges}}$. Using the formula for the sum of a convergent geometric series, we have

$$\sum_{n=4}^{\infty} 2^n 3^{-n} = \frac{\text{first term}}{1-r} = \frac{\frac{16}{81}}{1-\frac{2}{3}} = \boxed{\frac{16}{27}}.$$

(c) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{4}{2n+1} - \frac{4}{2n+5}\right)$

Solution: This is a telescoping series. We have

$$\begin{aligned} S_N &= \left(\frac{4}{1} - \frac{4}{5}\right) + \left(\frac{4}{3} - \frac{4}{7}\right) + \left(\frac{4}{5} - \frac{4}{9}\right) + \cdots + \left(\frac{4}{2N-1} - \frac{4}{2N+3}\right) + \left(\frac{4}{2N+1} - \frac{4}{2N+5}\right) \\ &= \frac{4}{1} + \frac{4}{3} - \frac{4}{2N+3} - \frac{4}{2N+5} \\ &= \frac{16}{3} - \frac{4}{2N+3} - \frac{4}{2N+5}. \end{aligned}$$

So

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\frac{16}{3} - \frac{4}{2N+3} - \frac{4}{2N+5}\right) = \frac{16}{3} - 0 - 0 = \frac{16}{3}.$$

Therefore, $\boxed{\text{the series converges}}$ and

$$\boxed{\sum_{n=0}^{\infty}\left(\frac{4}{2n+1}-\frac{4}{2n+5}\right)=\frac{16}{3}}.$$

(d) $\displaystyle\sum_{n=0}^{\infty}\frac{1-3\cdot 4^{2n}}{5^{n-1}}$

Solution: We have

$$\frac{1-3\cdot 4^{2n}}{5^{n-1}}=5\left(\frac{1}{5}\right)^n-15\left(\frac{16}{5}\right)^n.$$

So we have a combination of two geometric series, the first with common ratio $\frac{1}{5}$ and the second with common ratio $\frac{16}{5}$. Since $\frac{16}{5}>1$, we conclude that

$$\boxed{\sum_{n=0}^{\infty}\frac{1-3\cdot 4^{2n}}{5^{n-1}}\ \text{diverges}}.$$

(e) $\displaystyle\sum_{n=3}^{\infty}\ln\left(\frac{3n+1}{3n+4}\right)$

Solution: After rewriting the general term as

$$\ln\left(\frac{3n+1}{3n+4}\right)=\ln(3n+1)-\ln(3n+4)$$

we see that this a telescoping series. We have

$$\begin{aligned}S_N&=\sum_{n=3}^{N}(\ln(3n+1)-\ln(3n+4))\\&=(\ln(10)-\ln(13))+(\ln(13)-\ln(16))+\cdots+(\ln(3N+1)-\ln(3N+4))\\&=\ln(10)-\ln(3N+4).\end{aligned}$$

Since $\ln(3N+4)\to\infty$ when $N\to\infty$, we deduce that $S_N\to-\infty$ when $N\to\infty$. Thus

$$\boxed{\sum_{n=3}^{\infty}\ln\left(\frac{3n+1}{3n+4}\right)\ \text{diverges}}.$$

(f) $\sum_{n=1}^{\infty} 5 \cdot 3^{1-2n}$

Solution: We can rewrite the general term as

$$5 \cdot 3^{1-2n} = \frac{15}{9^n}.$$

So this is a geometric with common ratio $r = \frac{1}{9}$. Since $|r| < 1$, $\boxed{\text{the series converges}}$. Using the formula for the sum of a convergent geometric series, we get

$$\boxed{\sum_{n=1}^{\infty} 5 \cdot 3^{1-2n} = \frac{5}{24}}.$$

(g) $\sum_{n=1}^{\infty} \frac{5^n + 2^n}{6^n}$

Solution: We have a sum of two geometric series:

$$\sum_{n=1}^{\infty} \frac{5^n + 2^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n.$$

Both geometric series converge since their common ratios ($r = \frac{5}{6}$ and $r = \frac{1}{3}$) satisfy $|r| < 1$. So $\boxed{\text{the series converges}}$. Using the formula for the sum of a convergent geometric series, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{5^n + 2^n}{6^n} &= \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n \\
&= \frac{\frac{5}{6}}{1 - \frac{5}{6}} + \frac{\frac{1}{3}}{1 - \frac{1}{3}} \\
&= \boxed{\frac{11}{2}}.
\end{aligned}$$

(h) $\sum_{n=1}^{\infty} \big(\tan^{-1}(n+1) - \tan^{-1}(n)\big)$

Solution: This is a telescoping series. We have

$$\begin{aligned}
S_N &= (\tan^{-1}(2) - \tan^{-1}(1)) + (\tan^{-1}(3) - \tan^{-1}(2)) + \cdots + (\tan^{-1}(N+1) - \tan^{-1}(N)) \\
&= -\tan^{-1}(1) + \tan^{-1}(N+1)
\end{aligned}$$

$$= -\frac{\pi}{4} + \tan^{-1}(N+1)$$

Since $\lim_{N\to\infty} \tan^{-1}(N+1) = \frac{\pi}{2}$, we have

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(-\frac{\pi}{4} + \tan^{-1}(N+1)\right) = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}.$$

So $\boxed{\text{the series converges}}$ and

$$\boxed{\sum_{n=1}^{\infty} \left(\tan^{-1}(n+1) - \tan^{-1}(n)\right) = \frac{\pi}{4}}.$$

(i) $\sum_{n=1}^{\infty} \left(5^{1/n} - 5^{1/(n+1)}\right)$

Solution: This series is telescoping. We have

$$\begin{aligned} S_N &= \left(5^1 - 5^{1/2}\right) + \left(5^{1/2} - 5^{1/3}\right) + \cdots + \left(5^{1/N} - 5^{1/(N+1)}\right) \\ &= 5 - 5^{1/(N+1)}. \end{aligned}$$

Since $\lim_{N\to\infty} 5^{1/(N+1)} = 5^0 = 1$, we have

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(5 - 5^{1/(N+1)}\right) = 5 - 1 = 4.$$

So $\boxed{\text{the series converges}}$ and

$$\boxed{\sum_{n=1}^{\infty} \left(5^{1/n} - 5^{1/(n+1)}\right) = 4}.$$

#61. Use geometric series to express the repeating decimals below as a fraction of two integers.

(a) $1.5222\cdots = 1.5\overline{2}$

Solution: We have

$$\begin{aligned} 1.5\overline{2} &= 1.5 + 0.02 + 0.002 + \cdots \\ &= \frac{15}{10} + \frac{2}{100} + \frac{2}{1000} + \cdots \end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{2} + \sum_{n=2}^{\infty} \frac{2}{10^n} \\
&= \frac{3}{2} + \frac{\frac{2}{100}}{1 - \frac{1}{10}} \\
&= \frac{3}{2} + \frac{10}{9} \cdot \frac{1}{50} \\
&= \boxed{\frac{137}{90}}.
\end{aligned}
$$

(b) $0.126126\cdots = 0.\overline{126}$

Solution: We have

$$
\begin{aligned}
0.\overline{126} &= 0.126 + 0.000126 + \cdots \\
&= \frac{126}{1000} + \frac{126}{1000000} + \cdots \\
&= \sum_{n=1}^{\infty} \frac{126}{1000^n} \\
&= \frac{\frac{126}{1000}}{1 - \frac{1}{1000}} \\
&= \frac{1000}{999} \cdot \frac{126}{1000} \\
&= \boxed{\frac{14}{111}}.
\end{aligned}
$$

#62. For each sequence $\{a_n\}_{n=n_0}^{\infty}$ given below, determine

(i) whether the **sequence** $\{a_n\}_{n=n_0}^{\infty}$ converges or diverges. If the sequence converges, find its limit.

(ii) whether the **series** $\sum\limits_{n=n_0}^{\infty} a_n$ converges or diverges. If the series converges, find its sum if possible.

(a) $\left\{ \left(1 + \frac{4}{n} \right)^n \right\}_{n=1}^{\infty}$

Solution: (i) The limit of this sequence is an indeterminate power 1^∞. We can write it in exponential form

$$\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{4}{n}\right)}.$$

We can now compute the limit of the exponent using L'Hôpital's Rule as follows:

$$\begin{aligned}\lim_{n\to\infty} n\ln\left(1+\frac{4}{n}\right) &= \lim_{x\to\infty}\frac{\ln\left(1+\frac{4}{x}\right)}{\frac{1}{x}}\\ &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty}\frac{-\frac{4}{x^2}\cdot\frac{1}{1+\frac{4}{x}}}{-\frac{1}{x^2}}\\ &= \lim_{x\to\infty}\frac{4}{1+\frac{4}{x}}\\ &= 4.\end{aligned}$$

Therefore

$$\lim_{n\to\infty} e^{n\ln\left(1+\frac{4}{n}\right)} = e^4,$$

so $\boxed{\text{the sequence } \left\{\left(1+\frac{4}{n}\right)^n\right\}_n \text{ converges to the limit } e^4}$.

(ii) Since the limit of the general term $\left(1+\frac{4}{n}\right)^n$ is not zero, the Term Divergence Test tells us that

$$\boxed{\sum_{n=1}^{\infty}\left(1+\frac{4}{n}\right)^n \text{ diverges}}.$$

(b) $\left\{\sqrt{n+1}-\sqrt{n}\right\}_{n=0}^{\infty}$

Solution: (i) The limit of this sequence is an indeterminate form $\infty-\infty$. We can resolve the indeterminate by multiplying by the conjugate in the numerator and denominator:

$$\begin{aligned}\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) &= \lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\\ &= \lim_{n\to\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\\ &= \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}\\ &= 0.\end{aligned}$$

So $\boxed{\text{the sequence } \left\{\sqrt{n+1}-\sqrt{n}\right\}_n \text{ converges to the limit } 0}$.

(ii) To determine if the series converges or diverges, we can use the fact that this series is telescoping. The partial sums can be expressed as follows

$$\begin{aligned} S_N &= (\sqrt{1}-\sqrt{0}) + (\sqrt{2}-\sqrt{1}) + \cdots + \left(\sqrt{N+1}-\sqrt{N}\right) \\ &= \sqrt{N+1}. \end{aligned}$$

Therefore, $S_N \to \infty$ as $N \to \infty$, and

$$\boxed{\sum_{n=0}^{\infty} \left(\sqrt{n+1}-\sqrt{n}\right) \text{ diverges}}.$$

(c) $\{e^{-n}\}_{n=0}^{\infty}$

Solution: (i) This is a geometric sequence of common ratio $r = e^{-1}$, which satisfies $|r| < 1$. So

$$\lim_{n\to\infty} e^{-n} = 0,$$

and $\boxed{\text{the sequence } \left\{e^{-n}\right\}_n \text{ converges to the limit } 0}$.

(ii) Since $|r| = e^{-1} < 1$, the geometric series $\boxed{\sum_{n=0}^{\infty} e^{-n} \text{ converges}}$ and we can evaluate the sum as

$$\boxed{\sum_{n=0}^{\infty} e^{-n} = \frac{1}{1-e^{-1}}}.$$

(d) $\left\{\dfrac{e^{5n}}{n^{3/2}}\right\}_{n=1}^{\infty}$

Solution: (i) The limit of this sequence is an indeterminate form $\frac{\infty}{\infty}$. We can use L'Hôpital's Rule to compute the limit:

$$\begin{aligned} \lim_{n\to\infty} \frac{e^{5n}}{n^{3/2}} &= \lim_{x\to\infty} \frac{e^{5x}}{x^{3/2}} \\ &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{5e^{5x}}{\frac{3}{2}x^{1/2}} \\ &\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{25e^{5x}}{\frac{3}{4}x^{-1/2}} \\ &= \lim_{x\to\infty} \frac{100}{3} x^{1/2} e^{5x} \end{aligned}$$

$$= \infty.$$

So $\boxed{\text{the sequence } \left\{ \frac{e^{5n}}{n^{3/2}} \right\}_n \text{ diverges}}$.

(ii) Since the limit of the general term $\frac{e^{5n}}{n^{3/2}}$ is not zero, the Term Divergence Test tells us that

$$\boxed{\sum_{n=1}^{\infty} \frac{e^{5n}}{n^{3/2}} \text{ diverges}}.$$

(e) $\left\{ \frac{3n + 2\cos(n)}{5n} \right\}_{n=1}^{\infty}$

Solution: (i) To compute the limit of this sequence, observe that $-1 \leqslant \cos(n) \leqslant 1$, so

$$\frac{3n-2}{5n} \leqslant \frac{3n + 2\cos(n)}{5n} \leqslant \frac{3n+2}{5n}.$$

We have

$$\lim_{n\to\infty} \frac{3n-2}{5n} = \lim_{n\to\infty} \frac{3}{5} - \frac{2}{5n} = \frac{3}{5},$$
$$\lim_{n\to\infty} \frac{3n+2}{5n} = \lim_{n\to\infty} \frac{3}{5} + \frac{2}{5n} = \frac{3}{5}.$$

By the Squeeze Theorem, it follows that

$$\boxed{\text{the sequence } \left\{ \frac{3n + 2\cos(n)}{5n} \right\}_n \text{ converges to the limit } \frac{3}{5}}.$$

(ii) Since the limit of the general term $\frac{3n + 2\cos(n)}{5n}$ is not zero, the Term Divergence Term tells us that

$$\boxed{\sum_{n=1}^{\infty} \frac{3n + 2\cos(n)}{5n} \text{ diverges}}.$$

(f) $\left\{ \cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+2}\right) \right\}_{n=3}^{\infty}$

Solution: (i) We have

$$\lim_{n\to\infty} \left(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+2}\right) \right) = \cos(0) - \cos(0) = 0.$$

So $\boxed{\text{the sequence } \left\{\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+2}\right)\right\} \text{ converges to the limit } 0}$.

(ii) The series is telescoping, so we can determine if it converges or diverges by explicitly computing its partial sums S_N. We have

$$\begin{aligned}
S_N &= \left(\cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{5}\right)\right) + \left(\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{6}\right)\right) + \left(\cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{\pi}{7}\right)\right) + \cdots \\
&\quad \cdots + \left(\cos\left(\frac{\pi}{N-1}\right) - \cos\left(\frac{\pi}{N+1}\right)\right) + \left(\cos\left(\frac{\pi}{N}\right) - \cos\left(\frac{\pi}{N+2}\right)\right) \\
&= \cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{N+1}\right) - \cos\left(\frac{\pi}{N+2}\right) \\
&= \frac{1}{2} + \frac{\sqrt{2}}{2} - \cos\left(\frac{\pi}{N+1}\right) - \cos\left(\frac{\pi}{N+2}\right).
\end{aligned}$$

So

$$\begin{aligned}
\lim_{N\to\infty} S_N &= \lim_{N\to\infty}\left(\frac{1}{2} + \frac{\sqrt{2}}{2} - \cos\left(\frac{\pi}{N+1}\right) - \cos\left(\frac{\pi}{N+2}\right)\right) \\
&= \frac{1}{2} + \frac{\sqrt{2}}{2} - \cos(0) - \cos(0) \\
&= \frac{1+\sqrt{2}}{2} - 2.
\end{aligned}$$

Since this limit is finite, we conclude that

$$\boxed{\sum_{n=3}^{\infty}\left(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+2}\right)\right) \text{ converges}}$$

and the sum is $\boxed{\sum_{n=3}^{\infty}\left(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+2}\right)\right) = \frac{1+\sqrt{2}}{2} - 2}$.

#63. Consider the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\sin(n)}{n} - \frac{\sin(n+1)}{n+1}\right)$.

(a) Find an explicit formula for the partial sum $S_N = \displaystyle\sum_{n=1}^{N}\left(\frac{\sin(n)}{n} - \frac{\sin(n+1)}{n+1}\right)$ of the series.

Solution: This is a telescoping series; the partial sum is given by

$$S_N = \left(\frac{\sin(1)}{1} - \frac{\sin(2)}{2}\right) + \left(\frac{\sin(2)}{2} - \frac{\sin(3)}{3}\right) + \cdots \left(\frac{\sin(N)}{N} - \frac{\sin(N+1)}{N+1}\right)$$
$$= \boxed{\sin(1) - \frac{\sin(N+1)}{N+1}}$$

(b) Does the series $\sum_{n=1}^{\infty} \left(\frac{\sin(n)}{n} - \frac{\sin(n+1)}{n+1}\right)$ converge or diverge? If it converges, find its sum. If it diverges, explain why.

Solution: We need to compute the limit of the partial sums $S_N = \sin(1) - \frac{\sin(N+1)}{N+1}$. For this, observe that $-1 \leqslant \sin(N+1) \leqslant 1$, so

$$-\frac{1}{N+1} \leqslant \frac{\sin(N+1)}{N+1} \leqslant \frac{1}{N+1}.$$

Since $\lim_{N\to\infty} -\frac{1}{N+1} = \lim_{N\to\infty} \frac{1}{N+1} = 0$, the Squeeze Theorem tells us that

$$\lim_{N\to\infty} \frac{\sin(N+1)}{N+1} = 0.$$

So we have

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\sin(1) - \frac{\sin(N+1)}{N+1}\right) = \sin(1) - 0 = \sin(1).$$

Since the limit of the partial sums exists and is finite, we conclude that

$$\boxed{\sum_{n=1}^{\infty} \left(\frac{\sin(n)}{n} - \frac{\sin(n+1)}{n+1}\right) \text{ converges}}$$

and its sum is equal to

$$\boxed{\sum_{n=1}^{\infty} \left(\frac{\sin(n)}{n} - \frac{\sin(n+1)}{n+1}\right) = \sin(1)}.$$

#64. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence such that for all $n \geqslant 1$, we have

$$\frac{4n}{2n+1} \leqslant a_n \leqslant 1 + 5^{1/n}.$$

(a) Determine whether the **sequence** $\{a_n\}_{n=1}^{\infty}$ converges or diverges. If it converges, find $\lim_{n\to\infty} a_n$. If it diverges, explain why.

Solution: We use the Squeeze Theorem. We have

$$\lim_{n\to\infty} \frac{4n}{2n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{4}{2+\frac{1}{n}} = \frac{4}{2+0} = 2,$$
$$\lim_{n\to\infty} 1 + 5^{1/n} = 1 + 5^0 = 2.$$

Since the smaller and bigger sequence converge to the same limit, we conclude that

$$\boxed{\text{the sequence } \{a_n\}_{n=1}^{\infty} \text{ converges to the limit 2}}.$$

(b) Determine whether the **series** $\sum_{n=1}^{\infty} a_n$ converges or diverges. Justify your answer and name any test used.

Solution: Since the general term a_n does not converge to 0, the Term Divergence Test tells us that

$$\boxed{\sum_{n=1}^{\infty} a_n \text{ diverges}}.$$

#65. Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{2 \cdot 5^{n+1}}$. Find the values of x for which the series converges and find the sum of the series when it converges.

Solution: Observe that $f(x)$ is a geometric series of common ratio $r = \frac{x}{5}$. So it will converge when

$$|r| < 1 \;\Rightarrow\; \left|\frac{x}{5}\right| < 1 \;\Rightarrow\; |x| < 5 \;\Rightarrow\; \boxed{-5 < x < 5}.$$

When $-5 < x < 5$, the sum of the series is

$$\begin{aligned} f(x) &= \frac{\text{first term}}{1 - (\text{common ratio})} \\ &= \frac{\frac{1}{10}}{1 - \frac{x}{5}} \\ &= \boxed{\frac{1}{10 - 2x}}. \end{aligned}$$

#66. Consider the series $\sum_{n=0}^{\infty}\left(\frac{5}{A-2}\right)^n$, where A is an unspecified positive constant.

(a) Find the values of the positive constant A for which the series converges.

Solution: The series is geometric with common ratio $r = \frac{5}{A-2}$. By the Geometric Series Test, it converges when $|r| < 1$, so we get the condition

$$\left|\frac{5}{A-2}\right| < 1 \;\Rightarrow\; 5 < |A-2|.$$

This last inequality gives $A-2 > 5$ or $A-2 < -5$, that is $A > 7$ or $A < -3$. Since A is assumed to be positive, the possible values of A are $\boxed{A > 7}$.

(b) For the values of A you found in part (a), evaluate the sum of the series.

Solution: Using the formula for the sum of a convergent geometric series, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{5}{A-2}\right)^n &= \frac{\text{first term}}{1-r} \\ &= \frac{1}{1-\frac{5}{A-2}} \\ &= \boxed{\frac{A-2}{A-7}}.\end{aligned}$$