Section 10.4: Comparison Tests - Worksheet Solutions

#70. Determine if the series below converge or diverge. Make sure to clearly label and justify the use of any convergence test used. **Note:** some of these problems require convergence tests from previous sections.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{25n^6 - 1}}$

Solution: We use the LCT with $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^6}} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges as a p-series with $p = 2 > 1$. We have

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{\frac{n}{\sqrt{25n^6-1}}}{\frac{n}{\sqrt{n^6}}} \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{25 - \frac{1}{n^6}}} \\ &= \frac{1}{5}. \end{aligned}$$

Since $0 < L < \infty$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we conclude that

$$\boxed{\sum_{n=1}^{\infty} \frac{n}{\sqrt{25n^6 - 1}} \text{ converges}}.$$

(b) $\displaystyle\sum_{n=0}^{\infty} (2n+3)^{1/n}$

Solution: Observe that

$$\begin{aligned} \lim_{n\to\infty} (2n+3)^{1/n} &= \lim_{n\to\infty} \left(n\left(2 + \frac{3}{n}\right)\right)^{1/n} \\ &= \lim_{n\to\infty} n^{1/n} \cdot \left(2 + \frac{3}{n}\right)^{1/n} \\ &= 1 \cdot 2^0 \\ &= 1. \end{aligned}$$

Since the general term of the series does not approach 0, the Term Divergence Test tells us that

$$\boxed{\sum_{n=0}^{\infty} (2n+3)^{1/n} \text{ diverges}}.$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n!)}{2^n}$

Solution: We use the DCT. Since the range of cos is $[-1, 1]$, we have $0 \leqslant \cos^2(n!) \leqslant 1$. Then

$$0 \leqslant \frac{\cos^2(n!)}{2^n} \leqslant \frac{1}{2^n}.$$

Furthermore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with common ratio $r = \frac{1}{2}$, so it converges since $|r| < 1$. We can conclude that

$$\boxed{\sum_{n=1}^{\infty} \frac{\cos^2(n!)}{2^n} \text{ converges}}.$$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{(5\sqrt{n}-2)^3}{3n^2-2n+4}$

Solution: We use the LCT with $\displaystyle\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{n^{3/2}}{n^2} = \sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$, which diverges as a p-series with $p = \frac{1}{2} \leqslant 1$. We have

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\
&= \lim_{n\to\infty} \frac{\frac{(5\sqrt{n}-2)^3}{3n^2-2n+4}}{\frac{n^{3/2}}{n^2}} \\
&= \lim_{n\to\infty} \frac{\left(5 - \frac{2}{n^{1/2}}\right)^3}{3 - \frac{2}{n} + \frac{4}{n^2}} \\
&= \frac{125}{3}.
\end{aligned}$$

Since $0 < L < \infty$ and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$ diverges, we conclude that

$$\boxed{\sum_{n=2}^{\infty} \frac{(5\sqrt{n}-2)^3}{3n^2-2n+4} \text{ diverges}}.$$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n5^n}$

Solution: We use the DCT. Observe that for any $n \geqslant 1$, we have

$$0 < \frac{1}{n} \leqslant 1 \;\Rightarrow\; 0 < \frac{3^n}{n5^n} \leqslant \frac{3^n}{5^n}.$$

Furthermore, $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$ converges since it is a geometric series with common ratio $r = \frac{3}{5}$ satisfying $|r| < 1$. Therefore, $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n5^n} \text{ converges}}$.

Remark. The LCT with $b_n = \frac{3^n}{5^n}$ would have worked equally well here since

- $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3^n}{n5^n}}{\frac{3^n}{5^n}} = \lim_{n\to\infty} \frac{1}{n} = 0$, and
- $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^n}$ converges.

Since $L = 0$ and $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^n}$ converges, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n5^n}$ converges.

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n}}{3^n + 11n^2}$

Solution: We use the LCT with $\displaystyle\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{2^{2n}}{3^n} = \sum_{n=2}^{\infty} \left(\frac{4}{3}\right)^n$, which diverges as a geometric series with common ratio $r = \frac{4}{3}$ satisfying $|r| \geqslant 1$. We have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

$$= \lim_{n\to\infty} \frac{\frac{2^{2n}}{3^n+11n^2}}{\frac{2^{2n}}{3^n}}$$
$$= \lim_{n\to\infty} \frac{1}{1-\frac{11n^2}{3^n}}.$$

Using L'Hôpital's Rule twice, we see that

$$\lim_{x\to\infty} \frac{11x^2}{3^x} \overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{22x}{\ln(3)3^x} \overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{22}{\ln(3)^2 3^x} = 0.$$

Thus,

$$L = \lim_{n\to\infty} \frac{1}{1-\frac{11n^2}{3^n}} = \frac{1}{1-0} = 1.$$

Since $0 < L < \infty$ and $\sum_{n=0}^{\infty} \frac{2^{2n}}{3^n}$ diverges, we conclude that $\boxed{\sum_{n=0}^{\infty} \frac{2^{2n}}{3^n+11n^2} \text{ diverges}}$.

Remark. We cannot use the DCT with $b_n = \frac{2^{2n}}{3^n}$ here since

$$0 < 3^n < 3^n + 11n^2 \;\Rightarrow\; 0 < \frac{2^{2n}}{3^n+11n^2} < \frac{2^{2n}}{3^n}.$$

And knowing that the series of the "bigger terms" diverges does not tell us anything about the series of the "smaller terms".

(g) $\sum_{n=3}^{\infty} \frac{\ln(n)^2}{\sqrt{n}}$

Solution: We use the DCT. Observe that for $n \geqslant 3$, we have

$$0 < \ln(3)^2 < \ln(n)^2 \;\Rightarrow\; 0 < \frac{\ln(3)^2}{\sqrt{n}} < \frac{\ln(n)^2}{\sqrt{n}}.$$

Furthermore, $\sum_{n=3}^{\infty} \frac{\ln(3)^2}{\sqrt{n}} = \ln(3)^2 \sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$ diverges as a p-series with $p = \frac{1}{2} \leqslant 1$. Therefore, $\boxed{\sum_{n=3}^{\infty} \frac{\ln(n)^2}{\sqrt{n}} \text{ diverges}}$.

Remark. We could have also used the LCT with $b_n = \frac{1}{\sqrt{n}}$ here, observing that

- $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\ln(n)^2}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \ln(n)^2 = \infty$, and

- $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

Since $L = \infty$ and $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$ diverges, we can conclude that $\displaystyle\sum_{n=3}^{\infty} \frac{\ln(n)^2}{\sqrt{n}}$ diverges.

(h) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln(n)^2}$

Solution: Intuitively, we expect this series to diverge because $\ln(n)$ grows slower than any power of n, so the $\sqrt{n}$ in the denominator dictates the behavior. But because the $\ln(n)^2$ in the denominator is making the fraction smaller, any attempt to compare this series with $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ will be inconclusive. So instead, we will try to compare with another p-series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p}$. Because we expect divergence, we will need to pick $p \leqslant 1$. Because we need the denominator n^p to grow faster than $\sqrt{n}\ln(n)^2$, we will pick $p > \frac{1}{2}$. A suitable value of p would therefore be $p = \frac{3}{4}$ (but you can repeat the reasoning below with any value of p between $\frac{1}{2}$ and 1).

We use the LCT with $\displaystyle\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^{3/4}}$, which diverges as a p-series with $p = \frac{3}{4} \leqslant 1$.. We have

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\
&= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}\ln(n)^2}}{\frac{1}{n^{3/4}}} \\
&= \lim_{x\to\infty} \frac{x^{1/4}}{\ln(x)^2} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{x^{-3/4}}{\frac{8\ln(x)}{x}} \\
&= \lim_{x\to\infty} \frac{x^{1/4}}{8\ln(x)} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{x^{-3/4}}{\frac{32}{x}} \\
&= \lim_{x\to\infty} \frac{x^{1/4}}{32} \\
&= \infty
\end{aligned}$$

Since $L = \infty$ and $\sum_{n=2}^{\infty} \frac{1}{n^{3/4}}$ diverges, we conclude that $\boxed{\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln(n)^2} \text{ diverges}}$.

(i) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)^2}$

Solution: If we try to use a comparison with a p-series smartly chosen as in the previous problem, we cannot find a suitable exponent for a conclusive test because the exponent of n is exactly 1. Instead, the Integral Test will come to the rescue here.

The function $f(x) = \frac{1}{x\ln(x)^2}$ is continuous, positive and decreasing (because $x\ln(x)^2$ is increasing) on $[2, \infty)$. Therefore, the Integral Test applies and we can test for convergence of the series by testing for convergence of the corresponding improper integral.

To compute the integral, we use the substitution $u = \ln(x)$, which gives $du = \frac{dx}{x}$. This gives

$$\begin{aligned} \int_2^{\infty} \frac{dx}{x\ln(x)^2} &= \lim_{b\to\infty} \int_2^b \frac{dx}{x\ln(x)^2} \\ &= \lim_{b\to\infty} \int_{\ln(2)}^{\ln(b)} \frac{du}{u^2} \\ &= \int_{\ln(2)}^{\infty} \frac{du}{u^2}. \end{aligned}$$

This last integral is a type I p-integral with $p = 2 > 1$, so it converges. Therefore

$$\boxed{\sum_{n=2}^{\infty} \frac{1}{n\ln(n)^2} \text{ converges}}.$$

(j) $\sum_{n=0}^{\infty} \left(\frac{n}{n+3}\right)^n$

Solution: The limit of the general term is an indeterminate form 1^{∞}. If this indetermination resolves into something not equal to zero, the Term Divergence Test will immediately tell us that the series diverges. So let us try to compute the limit of the general term.

We can start by writing the power in exponential form

$$\lim_{n\to\infty} \left(\frac{n}{n+3}\right)^n = \lim_{n\to\infty} e^{n\ln\left(\frac{n}{n+3}\right)}.$$

We now compute the limit of the exponent using L'Hôpital's Rule:

$$\begin{aligned}\lim_{n\to\infty} n\ln\left(\frac{n}{n+3}\right) &= \lim_{x\to\infty}\frac{\ln(x)-\ln(x+3)}{\frac{1}{x}}\\ &\underset{\frac{0}{0}}{\overset{\text{L'H}}{=}} \frac{\frac{1}{x}-\frac{1}{x+3}}{-\frac{1}{x^2}}\\ &= \lim_{x\to\infty} -x^2\frac{(x+3)-x}{x(x+3)}\\ &= \lim_{x\to\infty} -\frac{3x}{x+3}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}\\ &= \lim_{x\to\infty} -\frac{3}{1+3/x}\\ &= -3.\end{aligned}$$

So

$$\lim_{n\to\infty} e^{n\ln\left(\frac{n}{n+3}\right)} = e^{-3} \neq 0.$$

By the Term Divergence Test, it follows that $\boxed{\sum_{n=0}^{\infty}\left(\frac{n}{n+3}\right)^n \text{ diverges}}$.

(k) $\displaystyle\sum_{n=1}^{\infty}\frac{7-3\cos\left(n^2\right)}{n^5+3}$

Solution: We use the DCT. Observe that $-1 \leqslant \cos(n^2) \leqslant 1$, so $4 \leqslant 7-3\cos\left(n^2\right) \leqslant 10$. Also, $n^5+3 < n^5$. It follows that

$$0 < \frac{7-3\cos\left(n^2\right)}{n^5+3} < \frac{10}{n^5}.$$

Furthermore, $\displaystyle\sum_{n=1}^{\infty}\frac{10}{n^5} = 10\sum_{n=1}^{\infty}\frac{1}{n^5}$ converges as a p-series with $p=5>1$. Therefore, $\boxed{\displaystyle\sum_{n=1}^{\infty}\frac{7-3\cos\left(n^2\right)}{n^5+3} \text{ converges}}$.

(l) $\displaystyle\sum_{n=2}^{\infty} n\sin\left(\frac{5}{n^3}\right)$

Solution: We use the LCT with $\displaystyle\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} n\frac{1}{n^3} = \sum_{n=2}^{\infty}\frac{1}{n^2}$, which converges as a p-series with $p = 2 > 1$. We have

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{n\sin\left(\frac{5}{n^3}\right)}{\frac{1}{n^2}} \\ &= \lim_{x\to\infty} \frac{\sin\left(\frac{5}{x^3}\right)}{\frac{1}{x^3}} \\ &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} \frac{-\frac{15}{x^4}\cos\left(\frac{5}{x^3}\right)}{-\frac{3}{x^4}} \\ &= \lim_{x\to\infty} 5\cos\left(\frac{5}{x^3}\right) \\ &= 5. \end{aligned}$$

Since $0 < L < \infty$ and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, we conclude that $\boxed{\displaystyle\sum_{n=2}^{\infty} n\sin\left(\frac{5}{n^3}\right) \text{ converges}}$.