Section 10.6: Alternating Series and Conditional Convergence - Worksheet Solutions

#73. Determine if the series below converge absolutely, converge conditionally or diverge. Make sure to clearly label and justify the use of any convergence test used. **Note:** this problem is a comprehensive review of series - many of these problems require convergence tests from previous sections.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[7]{n^3+1}}$

Solution: This series has both positive and negative terms, so we'll start by looking into absolute convergence. We investigate the series

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt[7]{n^3+1}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt[7]{n^3+1}}.$$

For this series, we use the LCT with the reference series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[7]{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/7}}$, which diverges as a p-series with $p = \frac{3}{7} \leqslant 1$. We have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt[7]{n^3+1}}}{\frac{1}{\sqrt[7]{n^3}}} = \lim_{n\to\infty} \frac{1}{\sqrt[7]{1+\frac{1}{n^3}}} = 1.$$

Since $0 < L < \infty$, both series have the same behavior so we conclude that $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt[7]{n^3+1}}\right|$ diverges. Therefore, the given series does NOT converge absolutely.

The series could still converge conditionally. To check if it does, we can try the AST since the series is alternating. The sequence $u_n = \frac{1}{\sqrt[7]{n^3+1}}$ is decreasing (as $\sqrt[7]{n^3+1}$ is positive and increasing) and $\lim_{n\to\infty} \frac{1}{\sqrt[7]{n^3+1}} = 0$. Therefore, the AST applies and tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[7]{n^3+1}}$ converges.

In conclusion, $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[7]{n^3+1}} \text{ converges conditionally}}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{9n^2+2}}{2n^4}$

Solution: We use the LCT with the reference series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2}}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^3}$, which converges as a p-series with $p = 3 > 1$. The limit for the LCT is

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{\frac{\sqrt{9n^2+2}}{2n^4}}{\frac{\sqrt{n^2}}{n^4}} \\ &= \lim_{n\to\infty} \frac{\sqrt{9+\frac{2}{n^2}}}{2} \\ &= \frac{3}{2}. \end{aligned}$$

Since $0 < L < \infty$, both series have the same behavior, so

$$\boxed{\sum_{n=1}^{\infty} \frac{\sqrt{9n^2+2}}{2n^4} \text{ converges absolutely}}.$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n+1}{3^{2n+1}}$

Solution: We can write this series as the sum of two convergent geometric series:

$$\sum_{n=1}^{\infty} \frac{(-5)^n+1}{3^{2n+1}} = \sum_{n=1}^{\infty} \frac{1}{3}\left(-\frac{5}{9}\right)^n + \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{9}\right)^n.$$

The first series is geometric with common ratio $r = -\frac{5}{9}$ so it converges absolutely since $|r| < 1$. The second series is geometric with common ratio $r = \frac{1}{9}$ so it converges absolutely since $|r| < 1$. Therefore,

$$\boxed{\sum_{n=1}^{\infty} \frac{(-5)^n+1}{3^{2n+1}} \text{ converges absolutely}}.$$

Although this wasn't asked in this problem, we can evaluate the sum of the series as follows:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-5)^n+1}{3^{2n+1}} &= \sum_{n=1}^{\infty} \frac{1}{3}\left(-\frac{5}{9}\right)^n + \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{9}\right)^n \\ &= \frac{-\frac{5}{27}}{1+\frac{5}{9}} + \frac{\frac{1}{27}}{1-\frac{1}{9}} \\ &= \boxed{-\frac{13}{168}}. \end{aligned}$$

(d) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\log_2(n)}$

Solution: This series has both positive and negative terms, so we'll start by looking into absolute convergence. We investigate the series

$$\sum_{n=3}^{\infty} \left|\frac{(-1)^n}{n\log_2(n)}\right| = \sum_{n=3}^{\infty} \frac{1}{n\log_2(n)}.$$

For this series, we can use the Integral Test. The function $f(x) = \dfrac{1}{x\log_2(x)}$ is continuous, positive and decreasing (because $x\log_2(x)$ is positive and increasing) on $[3,\infty)$. Therefore, the Integral Test applies and we can test for convergence of the series by testing for convergence of the corresponding improper integral.

To compute the integral, we use the substitution $u = \log_2(x)$, which gives $du = \frac{dx}{\ln(2)x}$. This gives

$$\begin{aligned}
\int_3^{\infty} \frac{dx}{x\log_2(x)} &= \lim_{t\to\infty} \int_3^t \frac{dx}{x\log_2(x)} \\
&= \lim_{t\to\infty} \int_{\log_2(3)}^{\log_2(t)} \frac{\ln(2)du}{u} && (u = \log_2(x)) \\
&= \ln(2)\int_{\log_2(3)}^{\infty} \frac{du}{u}.
\end{aligned}$$

This last integral is a type I p-integral with $p = 1$, so it diverges. Therefore, $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\log_2(n)}$ diverges. So the given series does NOT converge absolutely.

The series could still converge conditionally, and since it is alternating we can check this using the AST. The sequence $u_n = \frac{1}{n\log_2(n)}$ is decreasing (since $n\log_2(n)$ is positive and increasing) and $\lim_{n\to\infty} \frac{1}{n\log_2(n)} = 0$. So the AST applies and $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\log_2(n)}$ converges.

In conclusion, $\boxed{\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\log_2(n)} \text{ converges conditionally}}$.

(e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$

Solution: We use the Ratio Test. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n} \\ &= \lim_{n\to\infty}\frac{2}{n+1} \\ &= 0.\end{aligned}$$

Since $\rho < 1$, we conclude that $\boxed{\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n!} \text{ converges absolutely}}$.

(f) $\displaystyle\sum_{n=1}^{\infty}\left(5+\frac{3}{n}\right)^2$

Solution: Since

$$\lim_{n\to\infty}\left(5+\frac{3}{n}\right)^2 = (5+0)^2 = 25 \neq 0,$$

the Term Divergence Test guarantees that $\boxed{\sum_{n=1}^{\infty}\left(5+\frac{3}{n}\right)^2 \text{ diverges}}$.

(g) $\displaystyle\sum_{n=0}^{\infty}\frac{n\arctan(n)}{\sqrt[3]{8n^6+1}}$

Solution: Note that this series has only non-negative terms. We use the LCT with $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{n}{\sqrt[3]{n^6}} = \sum_{n=1}^{\infty}\frac{1}{n}$, which diverges as a p-series with $p = 1$. We have

$$\begin{aligned}L &= \lim_{n\to\infty}\frac{a_n}{b_n} \\ &= \lim_{n\to\infty}\frac{\frac{n\arctan(n)}{\sqrt[3]{8n^6+1}}}{\frac{1}{n}} \\ &= \lim_{n\to\infty}\frac{n^2\arctan(n)}{\sqrt[3]{8n^6+1}}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\ &= \lim_{n\to\infty}\frac{\arctan(n)}{\sqrt[3]{8+\frac{1}{n^6}}}\end{aligned}$$

$$= \frac{\frac{\pi}{2}}{\sqrt[3]{8}}$$
$$= \frac{\pi}{4}.$$

Since $0 < L < \infty$, both series have the same behavior, so we deduce that

$$\boxed{\sum_{n=0}^{\infty} \frac{n \arctan(n)}{\sqrt[3]{8n^6+1}} \text{ diverges}}.$$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^3+5\sqrt{n}}$

Solution: This series is alternating. We'll start by checking if it converges absolutely or not by looking at the series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{2n^3+5\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{2n^3+5\sqrt{n}}.$$

For this series, we use the LCT with the reference series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$, which converges as a p-series with $p = 1$. We have

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{\frac{1}{2n^3+5\sqrt{n}}}{\frac{1}{n^3}} \\ &= \lim_{n\to\infty} \frac{1}{2+\frac{5}{n^{5/2}}} \\ &= \frac{1}{2}. \end{aligned}$$

Since $0 < L < \infty$, both series have the same behavior. So $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{2n^3+5\sqrt{n}} \right|$ converges. This means

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+5\sqrt{n}} \text{ converges absolutely}}.$$

Remark: the AST would apply to this series, but it would only tell us that the series converges. The AST would not tell us whether the series converges absolutely or converges conditionally.

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n+\cos(n)}$

Solution: Note that this series has non-negative terms. We use the LCT with $\sum_{n=1}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{1}{3^n}$, which converges as a geometric series with common ratio $r = \frac{1}{3}$, $|r| < 1$. We have

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{\frac{1}{3^n+\cos(n)}}{\frac{1}{3^n}} \\ &= \lim_{n\to\infty} \frac{3^n}{3^n+\cos(n)} \cdot \frac{\frac{1}{3^n}}{\frac{1}{3^n}} \\ &= \lim_{n\to\infty} \frac{1}{1+\frac{\cos(n)}{3^n}}. \end{aligned}$$

Because $-1 \leqslant \cos(n) \leqslant 1$, we have

$$-\frac{1}{3^n} \leqslant \frac{\cos(n)}{3^n} \leqslant \frac{1}{3^n}$$

and $\lim_{n\to\infty} -\frac{1}{3^n} = \lim_{n\to\infty} \frac{1}{3^n} = 0$. By the Sandwich Theorem, we obtain $\lim_{n\to\infty} \frac{\cos(n)}{3^n} = 0$ and

$$L = \lim_{n\to\infty} \frac{1}{1+\frac{\cos(n)}{3^n}} = \frac{1}{1+0} = 1.$$

Since $0 < L < \infty$, both series have the same behavior, so we deduce that

$$\boxed{\sum_{n=0}^{\infty} \frac{1}{3^n+\cos(n)} \text{ converges absolutely}}.$$

(j) $\sum_{n=2}^{\infty} \frac{\sec(\pi n)}{\sqrt{n}}$

Solution: Note that $\sec(\pi n) = (-1)^n$ for any integer n. So

$$\sum_{n=2}^{\infty} \frac{\sec(\pi n)}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}},$$

and the series is alternating. Let us use the AST. The sequence $u_n = \frac{1}{\sqrt{n}}$ is decreasing (since $\sqrt{n}$ is positive and increasing) and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. Therefore, the AST applies and $\sum_{n=2}^{\infty} \frac{\sec(\pi n)}{\sqrt{n}}$ converges.

We need to determine if the convergence is absolute or conditional, so we consider the series

$$\sum_{n=2}^{\infty} \left| \frac{\sec(\pi n)}{\sqrt{n}} \right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}.$$

This series is a p-series with $p = \frac{1}{2} \leqslant 1$, so it diverges. In conclusion,

$$\boxed{\sum_{n=2}^{\infty} \frac{\sec(\pi n)}{\sqrt{n}} \text{ converges conditionally}}.$$

(k) $\displaystyle\sum_{n=1}^{\infty} \frac{5\cos(n) - 2}{n^2}$

Solution: We test for absolute convergence with the DCT. We have $-1 \leqslant \cos(n) \leqslant 1$, so $-7 \leqslant 5\cos(n) - 2 \leqslant 3$. Therefore, $0 \leqslant |5\cos(n) - 2| \leqslant 7$. So we have

$$0 \leqslant \left| \frac{5\cos(n) - 2}{n^2} \right| \leqslant \frac{7}{n^2},$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{7}{n^2}$ converges as a p-series with $p = 2 > 1$. It follows that

$$\boxed{\sum_{n=1}^{\infty} \frac{5\cos(n) - 2}{n^2} \text{ converges absolutely}}.$$

(l) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \ln\left(\frac{n+1}{n}\right)$

Solution: This series is alternating. Let us start by checking if it converges absolutely. We consider the series

$$\sum_{n=2}^{\infty} \left| (-1)^n \ln\left(\frac{n+1}{n}\right) \right| = \sum_{n=2}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=2}^{\infty} \left(\ln(n+1) - \ln(n)\right).$$

This series looks telescoping, and inspecting the partial sums, we see that

$$\begin{aligned} S_N &= \sum_{n=2}^{N} \left(\ln(n+1) - \ln(n)\right) \\ &= (\ln(3) - \ln(2)) + (\ln(4) - \ln(3)) + \cdots + (\ln(N) - \ln(N-1)) + (\ln(N+1) - \ln(N)) \end{aligned}$$

$$= \ln(N+1) - \ln(2).$$

Therefore,

$$\sum_{n=2}^{\infty} (\ln(n+1) - \ln(n)) = \lim_{N\to\infty} S_N = \lim_{N\to\infty} (\ln(N+1) - \ln(2)) = \infty.$$

So $\displaystyle\sum_{n=2}^{\infty} (\ln(n+1) - \ln(n))$ diverges. Hence, the original series does NOT converge absolutely.

The series could still converge conditionally, and we use the AST with $u_n = \ln\left(\frac{n+1}{n}\right)$ to check this. The sequence u_n is decreasing since

$$\frac{d}{dx} \ln\left(\frac{x+1}{x}\right) = \frac{d}{dx} (\ln(x+1) - \ln(x)) = \frac{1}{x+1} - \frac{1}{x} = -\frac{1}{x(x+1)} < 0 \quad \text{for} \quad x > 0.$$

Also, observe that

$$\lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right) = \ln(1+0) = \ln(1) = 0.$$

Therefore, the AST applies and $\displaystyle\sum_{n=2}^{\infty} (-1)^n \ln\left(\frac{n+1}{n}\right)$ converges.

In conclusion, $\boxed{\displaystyle\sum_{n=2}^{\infty} (-1)^n \ln\left(\frac{n+1}{n}\right) \text{ converges conditionally}}$.

(m) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^{\sqrt{n}}}$

Solution: This series is a bit tricky because it is not geometric. Indeed, the exponent of e is $\sqrt{n}$, and not n. The Root Test is also inconclusive since

$$\lim_{n\to\infty} \left(\frac{1}{e^{\sqrt{n}}}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{e^{\sqrt{n}/n}} = \lim_{n\to\infty} \frac{1}{e^{1/\sqrt{n}}} = \frac{1}{e^0} = 1.$$

Still, we expect the series to converge because $e^{\sqrt{n}}$ grows faster than any power of n (even though it grows slower than the exponential e^n). This hints that we might be able to prove convergence by comparing with a convergent p-series.

So let us use the LCT with $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which converges as a p-series with $p = \frac{3}{2} > 1$. We have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{\frac{1}{e^{\sqrt{n}}}}{\frac{1}{n^{3/2}}} \\
&= \lim_{x\to\infty} \frac{x^{3/2}}{e^{\sqrt{x}}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \frac{\frac{3}{2}x^{1/2}}{\frac{1}{2}x^{-1/2}e^{\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{3x}{e^{\sqrt{x}}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \frac{3}{\frac{1}{2}x^{-1/2}e^{\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{6\sqrt{x}}{e^{\sqrt{x}}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \frac{3x^{-1/2}}{\frac{1}{2}x^{-1/2}e^{\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{6}{e^{\sqrt{x}}} \\
&= 0.
\end{aligned}
$$

Since $L = 0$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, we conclude that

$$\boxed{\sum_{n=0}^{\infty} \frac{1}{e^{\sqrt{n}}} \text{ converges absolutely}}.$$

(n) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n}{2n+1}$

Solution: This series is alternating, but the AST does not apply. Indeed, we have

$$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2}.$$

It follows that $\lim_{n\to\infty} (-1)^n \frac{n}{2n+1}$ does not exists. Therefore, the Term Divergence Test tells us that $\boxed{\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n}{2n+1} \text{ diverges}}$.

(o) $\displaystyle\sum_{n=3}^{\infty} \left(\frac{n+3}{n}\right)^{n^2}$

Solution: We use the Root Test. We have

$$\rho = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{n+3}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{3}{n}\right)}.$$

We now compute the limit of the exponent using L'Hôpital's Rule:

$$\begin{aligned}
\lim_{n\to\infty} n\ln\left(1+\frac{3}{n}\right) &= \lim_{x\to\infty} \frac{\ln(1+3/x)}{\frac{1}{x}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \frac{-\frac{3}{x^2}\cdot\frac{1}{1+3/x}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{3}{1+3/x} \\
&= 3.
\end{aligned}$$

So

$$\rho = \lim_{n\to\infty} e^{n\ln\left(1+\frac{3}{n}\right)} = e^3.$$

Since $\rho > 1$, we conclude that $\boxed{\sum_{n=3}^{\infty} \left(\frac{n+3}{n}\right)^{n^2} \text{ diverges}}$.

(p) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+3)!}{(-5)^n}$

Solution: The presence of factorials strongly suggests using the Ratio Test. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{(n+4)!}{(-5)^{n+1}} \cdot \frac{(-5)^n}{(n+3)!}\right| \\
&= \lim_{n\to\infty} \frac{n+4}{5} \\
&= \infty.
\end{aligned}$$

Since $\rho > 1$, we conclude that $\boxed{\sum_{n=1}^{\infty} \frac{(n+3)!}{(-5)^n} \text{ diverges}}$.

(q) $\displaystyle\sum_{n=3}^{\infty} \cos\left(\frac{\pi}{n}\right)^{n^2}$

Solution: Given the exponent n^2, the Root Test is tempting here, but it would turn out to be inconclusive (try it). Let us try to directly compute the limit of the general term to see if the Term Divergence Test would apply. The limit $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right)^{n^2}$ is an indeterminate power 1^∞. Let us write it as

$$\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right)^{n^2} = \lim_{n\to\infty} e^{n^2 \ln\left(\cos\left(\frac{\pi}{n}\right)\right)}$$

and compute the limit of the exponent using L'Hôpital's Rule. We have

$$\begin{aligned}
\lim_{n\to\infty} n^2 \ln\left(\cos\left(\frac{\pi}{n}\right)\right) &= \lim_{x\to\infty} \frac{\ln\left(\cos\left(\frac{\pi}{x}\right)\right)}{\frac{1}{x^2}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} \frac{\frac{\pi}{x^2}\tan\left(\frac{\pi}{x}\right)}{-\frac{2}{x^3}} \\
&= \lim_{x\to\infty} -\frac{\pi\tan\left(\frac{\pi}{x}\right)}{\frac{2}{x}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} -\frac{-\frac{\pi^2}{x^2}\sec\left(\frac{\pi}{x}\right)^2}{-\frac{2}{x^2}} \\
&= \lim_{x\to\infty} -\frac{\pi^2}{2}\sec\left(\frac{\pi}{x}\right)^2 \\
&= -\frac{\pi^2}{2}\sec(0)^2 \\
&= -\frac{\pi^2}{2}.
\end{aligned}$$

So

$$\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right)^{n^2} = \lim_{n\to\infty} e^{n^2 \ln\left(\cos\left(\frac{\pi}{n}\right)\right)} = e^{-\pi^2/2}.$$

Since this limit is not equal to zero, the Term Divergence Test tells us that

$$\boxed{\sum_{n=3}^{\infty} \cos\left(\frac{\pi}{n}\right)^{n^2} \text{ diverges}}.$$

(r) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n+2}{7\sqrt{25^n+1}}$

Solution: We will use the LCT with the reference series $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{\sqrt{25^n}} = \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$, which converges as a geometric series with common ratio $r = \frac{3}{5}$, $|r| < 1$. We have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

$$= \lim_{n\to\infty} \frac{\frac{3^n+2}{7\sqrt{25^n+1}}}{\frac{3^n}{\sqrt{25^n}}}$$

$$= \lim_{n\to\infty} \frac{1+\frac{2}{3^n}}{7\sqrt{1+\frac{1}{25^n}}}$$

$$= \frac{1}{7}.$$

Since $0 < L < \infty$, both series have the same behavior. So

$$\boxed{\sum_{n=0}^{\infty} \frac{3^n+2}{7\sqrt{25^n+1}} \text{ converges absolutely}}.$$

(s) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2 e^n}{(2n)!}$

Solution: We use the Ration Test. We have

$$\rho \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{((n+1)!)^2 e^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2 e^n}\right|$$

$$= \lim_{n\to\infty} \frac{((n+1)!)^2 e^{n+1}(2n)!}{(2n+2)!(n!)^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 e}{(2n+1)(2n+2)} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2 e}{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}$$

$$= \frac{e}{4}.$$

Since $\rho < 1$, we conclude that $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2 e^n}{(2n)!} \text{ converges absolutely}}$.

(t) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3\cos(n)}{n^4+1}$

Solution: We use the LCT with the reference series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges as a

p-series with $p = 2 > 1$. We have

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{\frac{2n^2+3\cos(n)}{n^4+1}}{\frac{n^2}{n^4}} \\ &= \lim_{n\to\infty} \frac{2 + \frac{3\cos(n)}{n^2}}{1 + \frac{1}{n^4}}. \end{aligned}$$

To finish computing this limit, we must use the Squeeze Theorem. Since $-1 \leqslant \cos(n) \leqslant 1$, we have

$$-\frac{3}{n^2} \leqslant \frac{3\cos(n)}{n^2} \leqslant \frac{3}{n^2}.$$

Since $\lim_{n\to\infty} -\frac{3}{n^2} = \lim_{n\to\infty} \frac{3}{n^2} = 0$, we conclude that $\lim_{n\to\infty} \frac{3\cos(n)}{n^2} = 0$. Therefore, our limit for the LCT becomes

$$L = \lim_{n\to\infty} \frac{2 + \frac{3\cos(n)}{n^2}}{1 + \frac{1}{n^4}} = \frac{2+0}{1+0} = 2.$$

Since $0 < L < 0$, the two series have the same behavior. So

$$\boxed{\sum_{n=1}^{\infty} \frac{2n^2 + 3\cos(n)}{n^4+1} \text{ converges absolutely}}.$$

(u) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)^2}{n}$

Solution: Note that the first term of the series is 0. We use the DCT. Since ln is an increasing function, we have

$$\frac{\ln(n)^2}{n} \geqslant \frac{\ln(2)^2}{n} \geqslant 0$$

for all $n \geqslant 2$. The series $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(2)^2}{n} = \ln(2)^2 \sum_{n=1}^{\infty} \frac{1}{n}$ diverges as a p-series with $p = 1$. Therefore, $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)^2}{n} \text{ diverges}}$.

#74. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{7n+4}}$.

(a) Show that this series meets the conditions of the Alternating Series Estimation Theorem.

Solution: The sequence $u_n = \frac{1}{\sqrt[3]{7n+4}}$ satisfies the following conditions.

- The sequence $u_n = \frac{1}{\sqrt[3]{7n+4}}$ is decreasing since $\sqrt[3]{7n+4}$ is positive and increasing.
- $\lim\limits_{n\to\infty} \frac{1}{\sqrt[3]{7n+4}} = 0$.

Therefore, $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{7n+4}} \text{ meets the conditions of the ASET}}$.

(b) Find the smallest integer N for which the partial sum $S_N = \sum_{n=1}^{N} \frac{(-1)^n}{\sqrt[3]{7n+4}}$ approximates the sum of the series with an error of at most 0.1.

Solution: The Alternating Series Estimation Theorem tells us that the best estimate for the error is $|S - S_N| \leqslant u_{N+1}$. Therefore, we will want $u_{N+1} \leqslant 0.1$. This gives

$$\begin{aligned}
&\frac{1}{\sqrt[3]{7(N+1)+4}} \leqslant 0.1 \\
&\Rightarrow \quad \sqrt[3]{7N+11} \geqslant 10 \\
&\Rightarrow 7N + 11 \geqslant 1000 \\
&\Rightarrow 7N \geqslant 989 \\
&\Rightarrow N \geqslant \frac{989}{7} \simeq 141.3.
\end{aligned}$$

Therefore, the smallest value of N giving us the desired error is $\boxed{N = 142}$.

#75. Consider the series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{3n-7}+9}$.

(a) Show that this series meets the conditions of the Alternating Series Estimation Theorem.

Solution: The sequence $u_n = \frac{1}{2^{3n-7}+9}$ satisfies the following conditions.

- The sequence $u_n = \frac{1}{2^{3n-7}+9}$ is decreasing since $2^{3n-7} + 11$ is positive and increasing.
- $\lim\limits_{n\to\infty} \frac{1}{2^{3n-7}+9} = 0$.

Therefore, $\boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{3n-7}+9} \text{ meets the conditions of the ASET}}$.

(b) Find the smallest integer N for which the partial sum $S_N = \sum_{n=0}^{N} \frac{(-1)^{n+1}}{2^{3n-7}+9}$ approximates the sum of the series with an error of at most 10^{-3}.

Solution: The Alternating Series Estimation Theorem tells us that the best estimate for the error is $|S - S_N| \leqslant u_{N+1}$. Therefore, we will want $u_{N+1} \leqslant 10^{-3}$. This gives

$$\begin{aligned}
&\frac{1}{2^{3(N+1)-7}+9} \leqslant 10^{-3} \\
&\Rightarrow\ 2^{3N-4}+9 \geqslant 1000 \\
&\Rightarrow 2^{3N-4} \geqslant 991 \\
&\Rightarrow 3N-4 \geqslant 10 \quad (2^9 = 512, 2^{10} = 1024) \\
&\Rightarrow N \geqslant \frac{14}{3} \simeq 4.7.
\end{aligned}$$

Therefore, the smallest value of N giving us the desired error is $\boxed{N = 5}$.

#76. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+2}$.

(a) Show that this series meets the conditions of the Alternating Series Estimation Theorem.

Solution: The sequence $u_n = \frac{1}{\sqrt{n}+2}$ satisfies the following conditions.

- $\frac{1}{\sqrt{n+1}+2} < \frac{1}{\sqrt{n}+2}$ as $\sqrt{n+1} > \sqrt{n}$.
- $\lim_{n\to\infty} \frac{1}{\sqrt{n}+2} = 0$.

Therefore, $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+2} \text{ meets the conditions of the ASET}}$.

(b) Find the smallest integer N for which the partial sum $S_N = \sum_{n=1}^{N} \frac{(-1)^n}{\sqrt{n}+2}$ approximates the sum of the series with an error of at most $\frac{3}{20}$.

Solution: The Alternating Series Estimation Theorem tells us that the best estimate for the error is $|S - S_N| \leqslant u_{N+1}$. Therefore, we will want $u_{N+1} \leqslant \frac{3}{20}$. This gives

$$\frac{1}{\sqrt{N+1}+2} \leqslant \frac{3}{20}$$

$$\begin{aligned}
&\Rightarrow\ \sqrt{N+1}+2 \geqslant \frac{20}{3}\\
&\Rightarrow\ \sqrt{N+1} \geqslant \frac{14}{3}\\
&\Rightarrow\ N+1 \geqslant \frac{196}{9}\\
&\Rightarrow\ N \geqslant \frac{187}{9} \simeq 20.8.
\end{aligned}$$

Therefore, the smallest value of N giving us the desired error is $\boxed{N = 21}$.

#77. Consider the series $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{(3n+11)^2}$.

(a) Show that this series meets the conditions of the Alternating Series Estimation Theorem.

Solution: We have

- $\left\{ \frac{1}{(3n+11)^2} \right\}$ is decreasing since $\frac{1}{(3(n+1)+11)^2} < \frac{1}{(3n+11)^2}$.
- $\lim_{n\to\infty} \frac{1}{(3n+11)^2} = 0$.

Therefore, $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{(3n+11)^2} \text{ meets the conditions of the ASET}}$.

(b) Find the smallest integer N for which the partial sum $S_N = \sum_{n=1}^{N} \frac{(-1)^n}{(3n+11)^2}$ approximates the sum of the series S with an error of at most 0.0001.

Solution: The ASET tells us that the error is bounded by u_{N+1}. Therefore, we will want

$$\begin{aligned}
&u_{N+1} < 0.0001\\
&\frac{1}{(3(N+1)+11)^2} < 0.0001\\
&(3N+14)^2 > 10000\\
&3N+14 > 100\\
&3N > 86\\
&N > \frac{86}{3} \simeq 28.66...
\end{aligned}$$

Therefore the smallest integer N giving us the desired level of accuracy is $\boxed{N = 29}$.

#78. Determine whether the series $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\sqrt{\ln(n)^2+1}}$ converges absolutely, converges conditionally or diverges.

Solution: Let us start by checking if the series converges absolutely or not. We look at the series

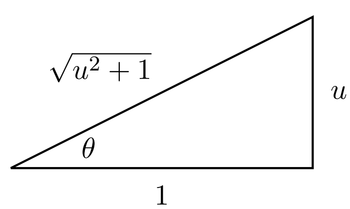
$$\sum_{n=3}^{\infty} \left| \frac{(-1)^n}{n\sqrt{\ln(n)^2+1}} \right| = \sum_{n=3}^{\infty} \frac{1}{n\sqrt{\ln(n)^2+1}}.$$

For this series, we use the Integral Test. The function $f(x) = \dfrac{1}{x\sqrt{\ln(x)^2+1}}$ is continuous and positive on $[3, \infty)$. It is also decreasing as the reciprocal of a positive increasing function. Therefore, the Integral Test applies and we can test for the convergence of the series by calculating the improper integral $\displaystyle\int_3^{\infty} \frac{dx}{x\sqrt{\ln(x)^2+1}}$.

Before we calculate the improper integral, let us start by finding an antiderivative. We first use the substitution $u = \ln(x)$, $du = \frac{dx}{x}$ to get

$$\int \frac{dx}{x\sqrt{\ln(x)^2+1}} = \int \frac{du}{\sqrt{u^2+1}}.$$

This integral can be calculated with the substitution $u = \tan(\theta)$, which gives $du = \sec^2(\theta)d\theta$ and $\sqrt{u^2+1} = \sqrt{\tan^2(\theta)+1} = \sec(\theta)$. The right triangle for this substitution has base angle θ so that $\tan(\theta) = u$, as shown below.



The integral becomes

$$\begin{aligned}\int \frac{dx}{x\sqrt{\ln(x)^2+1}} &= \int \frac{du}{\sqrt{u^2+1}} \\ &= \int \frac{\sec^2(\theta)d\theta}{\sec(\theta)} \\ &= \int \sec(\theta)d\theta\end{aligned}$$

$$
\begin{aligned}
&= \ln|\sec(\theta) + \tan(\theta)| + C \\
&= \ln\left|\sqrt{u^2+1} + u\right| + C \\
&= \ln\left|\sqrt{\ln(x)^2+1} + \ln(x)\right| + C.
\end{aligned}
$$

Now for the improper integral, we get

$$
\begin{aligned}
\int_3^\infty \frac{dx}{x\sqrt{\ln(x)^2+1}} &= \lim_{t\to\infty} \int_3^t \frac{dx}{x\sqrt{\ln(x)^2+1}} \\
&= \lim_{t\to\infty} \left[\ln\left|\sqrt{\ln(x)^2+1} + \ln(x)\right|\right]_3^t \\
&= \lim_{t\to\infty} \left(\ln\left|\sqrt{\ln(t)^2+1} + \ln(t)\right| - \ln\left|\sqrt{\ln(3)^2+1} + \ln(3)\right|\right) \\
&= \infty.
\end{aligned}
$$

So $\displaystyle\int_3^\infty \frac{dx}{x\sqrt{\ln(x)^2+1}}$ diverges, and thus $\displaystyle\sum_{n=3}^\infty \frac{(-1)^n}{n\sqrt{\ln(n)^2+1}}$ does not converge absolutely.

Next, since the series is alternating, we can use the AST with $u_n = \frac{1}{n\sqrt{\ln(n)^2+1}}$, which is decreasing and converges to 0 as the reciprocal of an increasing positive sequence going to infinity. Therefore, the series converges.

In conclusion, $\boxed{\displaystyle\sum_{n=3}^\infty \frac{(-1)^n}{n\sqrt{\ln(n)^2+1}} \text{ converges conditionally}}$.