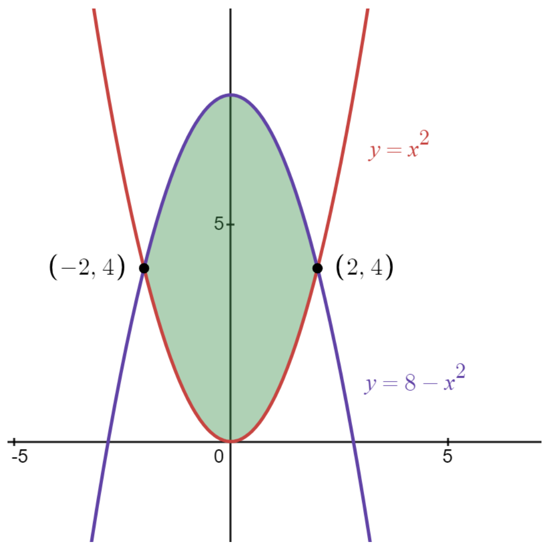
Section 6.1: Volumes by Cross-Sections - Worksheet Solutions

#11. Calculate the volume of each solid described below.

(a) The base of the solid is the region bounded by the curves $y = x^2$ and $y = 8 - x^2$ and the cross-sections perpendicular to the x-axis are rectangles of height 5.

Solution: The base of the solid is sketched below.



The cross-sections are perpendicular to the x-axis, so we look at vertical strips in the region. The length of the vertical strip at x is

$$\ell(x) = y_{\text{top}}(x) - y_{\text{bot}}(x) = (8 - x^2) - x^2 = 8 - 2x^2.$$

A rectangle of height 5 and base ℓ has area 5ℓ. With the length of the strip $\ell(x)$ that we found above, we get that the area of the cross-section at x is

$$A(x) = 5\ell(x) = 5(8 - 2x^2) = 10(4 - x^2).$$

So the volume is

$$\begin{aligned} V &= \int_{-2}^{2} A(x)dx \\ &= \int_{-2}^{2} 10(4 - x^2)dx \\ &= 2\int_{0}^{2} 10(4 - x^2)dx \quad \text{(even function)} \\ &= 20\left[4x - \frac{x^3}{3}\right]_0^2 \end{aligned}$$

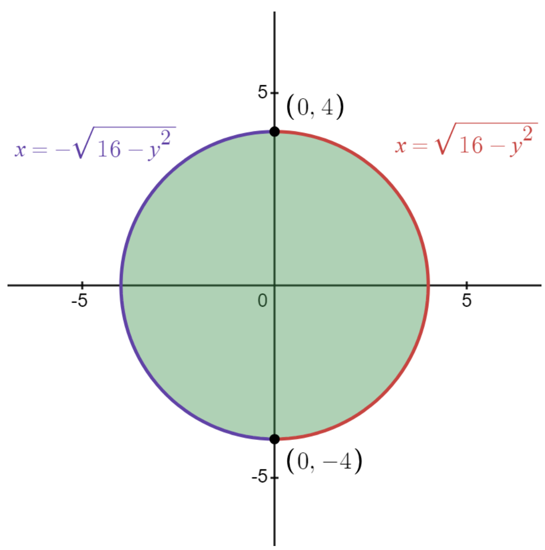
$$= 20\left(8 - \frac{8}{3} - 0\right)$$
$$= \boxed{\frac{320}{3} \text{ cubic units}}.$$

(b) The base of the solid is the region inside the circle of equation $x^2 + y^2 = 16$ and the cross-sections perpendicular to the y-axis are squares.

Solution: Since the cross-sections are perpendicular to the y-axis, we will need to use a dy-integral. So we need to express the boundaries of the region as functions of y by solving $x^2 + y^2 = 16$ for x. This gives

$$x^2 = 16 - y^2 \;\Rightarrow\; \sqrt{x^2} = \sqrt{16 - y^2} \;\Rightarrow\; |x| = \sqrt{16 - y^2} \;\Rightarrow\; x = \pm\sqrt{16 - y^2}.$$

The solution with the positive sign corresponds to the right semi-circle and the solution with the negative sign corresponds to the left semi-circle, see figure below.



So the length of the horizontal strip at y is

$$\ell(y) = x_{\text{right}}(y) - x_{\text{left}}(y) = \sqrt{16 - y^2} - (-\sqrt{16 - y^2}) = 2\sqrt{16 - y^2}.$$

Since the area of a square of base ℓ is $A = \ell^2$, the area of the cross-section at y is

$$A(y) = \ell(y)^2 = (2\sqrt{16 - y^2})^2 = 4(16 - y^2).$$

So the volume is

$$V = \int_{-4}^{4} A(y)dy$$

$$= \int_{-4}^{4} 4(16 - y^2)dy$$

$$= 2\int_{0}^{4} 4(16 - y^2)dy \quad \text{(even function)}$$

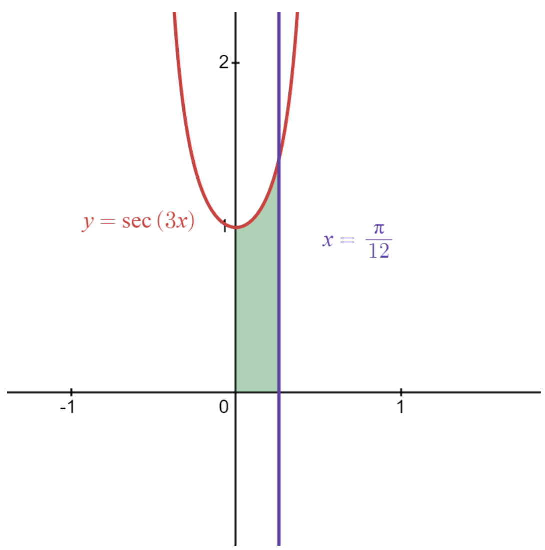
$$= 8\left[16y - \frac{y^3}{3}\right]_0^4$$

$$= 8\left(64 - \frac{64}{3} - 0\right)$$

$$= \boxed{\frac{1024}{3} \text{ cubic units}}.$$

(c) The base of the solid is the region between the graph of $y = \sec(3x)$ and the x-axis for $0 \leqslant 0 \leqslant \frac{\pi}{12}$ and the cross sections perpendicular to the x-axis are semi-circles with diameter in the base.

Solution: The base of the solid is sketched below.



The length of the vertical strip at x is

$$\ell(x) = y_{\text{top}}(x) - y_{\text{bot}}(x) = \sec(3x) - 0 = \sec(3x).$$

The area of a semi-circle with diameter ℓ is $A = \frac{1}{2}\pi\left(\frac{\ell}{2}\right)^2 = \frac{\pi}{8}\ell^2$, so the cross-section at x has area

$$A(x) = \frac{\pi}{8}\ell(x)^2 = \frac{\pi}{8}\sec^2(3x).$$

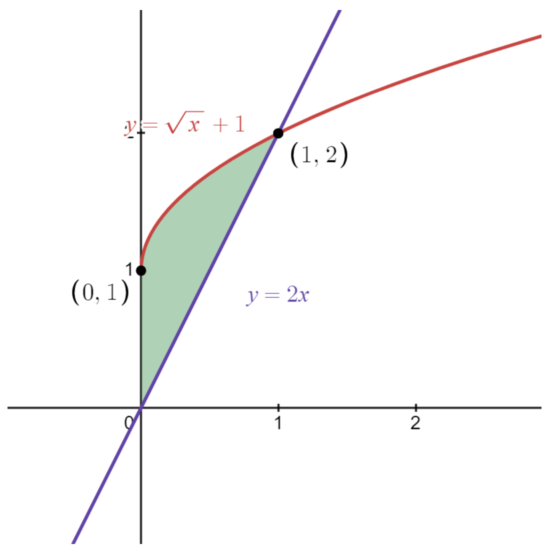
Therefore the volume is

$$V = \int_0^{\pi/12} A(x)dx$$

$$= \int_0^{\pi/12} \frac{\pi}{8} \sec^2(3x)dx$$
$$= \frac{\pi}{8}\left[\frac{1}{3}\tan(3x)\right]_0^{\pi/12}$$
$$= \boxed{\frac{\pi}{24}\text{ cubic units}}.$$

(d) The base of the solid is the region bounded by the curves $y = 2x$, $y = \sqrt{x} + 1$ and the y-axis, and the cross-sections perpendicular to the y-axis are isosceles triangles of height 3.

Solution: The base of the solid is sketched below.



Since the cross-sections are perpendicular to the y-axis, we consider horizontal strips in the region. We will need to have both equations as functions of y by solving for x, which gives

$$y = 2x \;\Rightarrow\; x = \frac{y}{2},$$
$$y = \sqrt{x} + 1 \;\Rightarrow\; \sqrt{x} = y - 1 \;\Rightarrow\; x = (y-1)^2.$$

The right boundary of the region is $x_{\text{right}}(y) = \frac{y}{2}$. The left boundary is $x_{\text{left}} = 0$ (the y-axis) for $0 \leqslant y \leqslant 1$ and $x_{\text{left}}(y) = (y-1)^2$ for $1 \leqslant y \leqslant 2$. So the length of the horizontal strip at y is

$$\ell(y) = x_{\text{right}}(y) - x_{\text{left}}(y) = \begin{cases} \frac{y}{2} & \text{if } 0 \leqslant y \leqslant 1, \\ \frac{y}{2} - (y-1)^2 & \text{if } 1 \leqslant y \leqslant 2. \end{cases}$$

The area of an isosceles triangle with base ℓ and height 3 is $A = \frac{3}{2}\ell$, so the area of the cross-section at y is

$$A(y) = \frac{3}{2}\ell(y) = \begin{cases} \frac{3y}{4} & \text{if } 0 \leqslant y \leqslant 1, \\ \frac{3}{2}\left(\frac{y}{2} - (y-1)^2\right) & \text{if } 1 \leqslant y \leqslant 2. \end{cases}$$

Hence the volume is

$$\begin{aligned} V &= \int_0^2 A(y)dy \\ &= \int_0^1 \frac{3y}{4}dy + \int_1^2 \frac{3}{2}\left(\frac{y}{2} - (y-1)^2\right)dy \\ &= \left[\frac{3y^2}{8}\right]_0^1 + \frac{3}{2}\left[\frac{y^2}{4} - \frac{(y-1)^3}{3}\right]_1^2 \\ &= \left(\frac{3}{8} - 0\right) + \frac{3}{2}\left(1 - \frac{1}{3} - \frac{1}{4}\right) \\ &= \boxed{1 \text{ cubic unit}}. \end{aligned}$$

#12. Consider the region bounded by the curve $y = \dfrac{4}{x^3+1}$, the y-axis and the line $y = 2$.

This region is revolved around an axis to form a solid of revolution. Use the disk/washer method to set-up an integral that calculates the volume of the solid for each of the axes of revolution given below. You do not need to evaluate the integrals.

(a) the x-axis

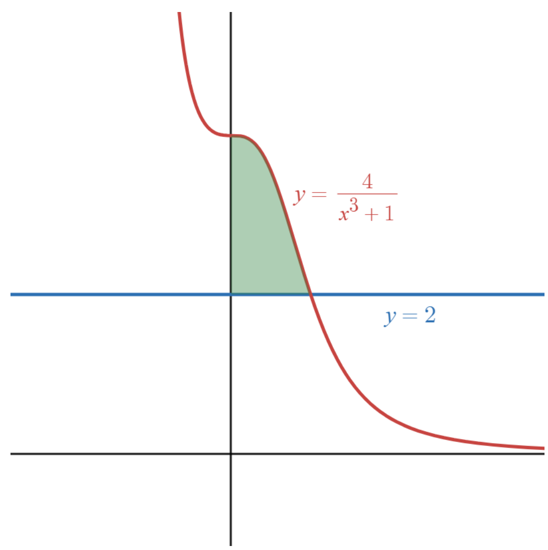
(b) the y-axis

(c) $y = -5$

(d) $y = 2$

(e) $x = 3$

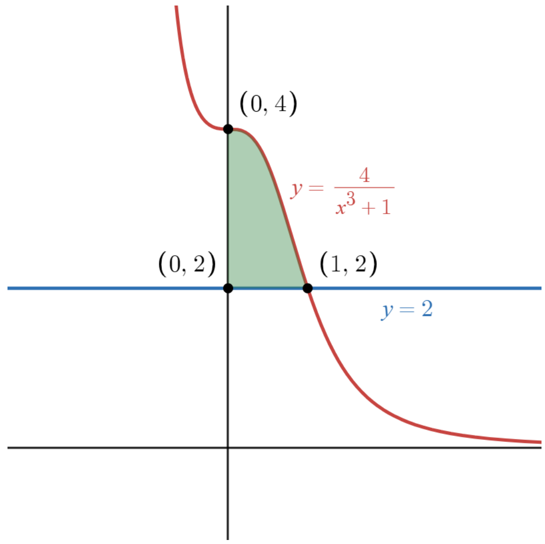
(f) $x = -2$



Solution: First, find the intersection points of the bounding curves:

$$\frac{4}{x^3+1} = 2 \Rightarrow x^3 + 1 = 2 \Rightarrow x^3 = 1 \Rightarrow x = 1.$$

See the figure below for all intersection points.



(a) To use the method of disk/washers, we need to revolve a strip perpendicular to the axis of revolution. Since the axis is horizontal here, we revolve vertical strips and use integration with respect to x. Revolving the vertical strip at x creates a washer with inner radius $r_{\text{in}}(x) = 2 - 0 = 2$ and outer radius $r_{\text{out}}(x) = \frac{4}{x^3+1} - 0 = \frac{4}{x^3+1}$. Therefore, the volume is

$$V = \int_0^1 \pi\left(r_{\text{out}}(x)^2 - r_{\text{in}}(x)^2\right) dx = \boxed{\int_0^1 \pi\left(\left(\frac{4}{x^3+1}\right)^2 - 2^2\right) dx}.$$

Although not necessary, we can simplify further to the answer

$$\boxed{V = \int_0^1 \pi\left(\frac{16}{(x^3+1)^2} - 4\right) dx}.$$

(b) To use the method of disk/washers, we need to revolve a strip perpendicular to the axis of revolution. Since the axis is vertical here, we revolve horizontal strips and use integration with respect to y. We need to express the bounding curve as a function of y by solving for x in its equation:

$$y = \frac{4}{x^3+1} \;\Rightarrow\; x^3 + 1 = \frac{4}{y} \;\Rightarrow\; x^3 = \frac{4}{y} - 1 \;\Rightarrow\; x = \sqrt[3]{\frac{4}{y} - 1}.$$

Revolving the horizontal strip at y around the y-axis creates a disk of radius $r(y) = \sqrt[3]{\frac{4}{y} - 1}$. So the volume is given by

$$V = \int_2^4 \pi r(y)^2 dy = \boxed{\int_2^4 \pi\left(\sqrt[3]{\frac{4}{y} - 1}\right)^2 dy}.$$

Although not necessary, we can simplify the answer to

$$\boxed{V = \int_2^4 \pi \left(\frac{4}{y} - 1\right)^{2/3} dy}.$$

(c) This is similar to part (a), only the axis of revolution $y = -5$ is shifted 5 units down and further away from the region. Thus, the inner and outer radii are 5 units greater and become $r_{\text{in}}(x) = 2 - (-5) = 7$, $r_{\text{out}}(x) = \frac{4}{x^3+1} - (-5) = \frac{4}{x^3+1} + 5$. Therefore, the volume is

$$V = \int_0^1 \pi \left(r_{\text{out}}(x)^2 - r_{\text{in}}(x)^2\right) dx = \boxed{\int_0^1 \pi \left(\left(\frac{4}{x^3+1} + 5\right)^2 - 7^2\right) dx}.$$

(d) This is again similar to part (a), only the axis of revolution $y = 2$ is shifted 2 units up and closer to the region. Because $y = 2$ is also the bottom boundary of the region, the inner radius is now 0, meaning we are using the disk method. Disks have radius $r(x) = \frac{4}{x^3+1} - 2$, so the volume is given by

$$V = \int_0^1 \pi r(x)^2 dx = \boxed{\int_0^1 \pi \left(\frac{4}{x^3+1} - 2\right)^2 dx}.$$

(e) This time, the axis is vertical and to the right of the region. So we revolve horizontal strips and use integration with respect to y. The inner radius of the washer at y is $r_{\text{in}}(y) = 3 - \sqrt[3]{\frac{4}{y} - 1}$ and the outer radius is $r_{\text{out}}(y) = 3 - 0 = 3$. Thus, the volume is

$$V = \int_2^4 \pi \left(r_{\text{out}}(y)^2 - r_{\text{in}}(y)^2\right) dy = \boxed{\int_2^4 \pi \left(3^2 - \left(3 - \sqrt[3]{\frac{4}{y} - 1}\right)^2\right) dy}.$$

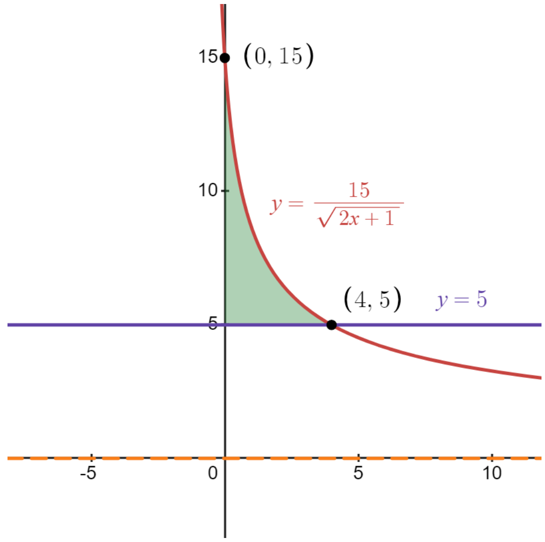
(f) This is similar to (b), only the axis $x = -2$ is shifted 2 units to the left and further away from the region. So the inner and outer radii increase by 2 units and become $r_{\text{in}}(y) = 0 - (-2) = 2$ and $r_{\text{out}}(y) = \sqrt[3]{\frac{4}{y} - 1} - (-2) = \sqrt[3]{\frac{4}{y} - 1} + 2$. Thus, the volume is

$$V = \int_2^4 \pi \left(r_{\text{out}}(y)^2 - r_{\text{in}}(y)^2\right) dy = \boxed{\int_2^4 \pi \left(\left(\sqrt[3]{\frac{4}{y} - 1} + 2\right)^2 - 2^2\right) dy}.$$

#13. Use the method of disks/washers to calculate the volume of the solids of revolutions obtained by revolving the regions described below about the given axis.

(a) The region bounded by the curve $y = \frac{15}{\sqrt{2x+1}}$, the line $y = 5$ and the y-axis revolved about the x-axis.

Solution: The region and axis of revolution are sketched below.

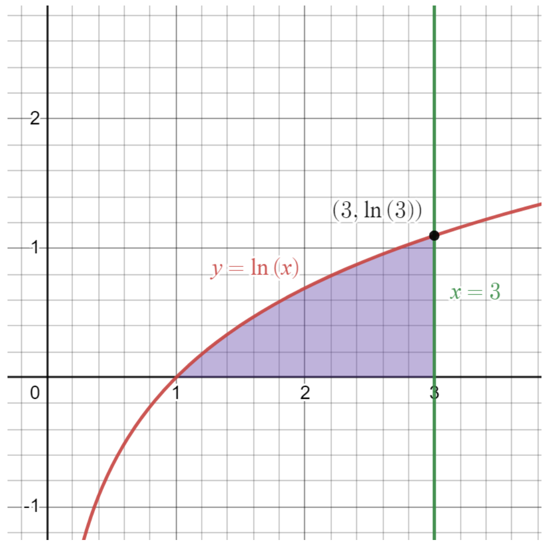


To use the method of disk/washers, we need to revolve a strip perpendicular to the axis of revolution. Since the axis is horizontal here, we revolve vertical strips and use integration with respect to x. Revolving the vertical strip at x creates a washer with inner radius $r_{\text{in}}(x) = 5 - 0 = 5$ and outer radius $r_{\text{out}}(x) = \frac{15}{\sqrt{2x+1}} - 0 = \frac{15}{\sqrt{2x+1}}$. Therefore, the volume is

$$\begin{aligned} V &= \int_0^4 \pi \left(r_{\text{out}}(x)^2 - r_{\text{in}}(x)^2\right) dx \\ &= \int_0^4 \pi \left(\frac{225}{2x+1} - 25\right) dx \\ &= 25\pi \int_0^4 \left(\frac{9}{2x+1} - 1\right) dx \\ &= 25\pi \left[\frac{9}{2} \ln|2x+1| - x\right]_0^4 \\ &= 25\pi \left(\frac{9}{2}\ln(9) - 4\right) \\ &= \boxed{\frac{25\pi}{2}\left(9\ln(9) - 8\right) \text{ cubic units}}. \end{aligned}$$

(b) The region below the graph of $y = \ln(x)$ on $1 \leqslant x \leqslant 3$ revolved about the line $x = 3$.

Solution: The region and axis of revolution are sketched below.



To use the method of disks/washers, we need to revolve strips perpendicular to the axis of revolution. Since the axis is vertical here, we revolve horizontal strips and use integration with respect to y. We need to express the curve as a function of y:

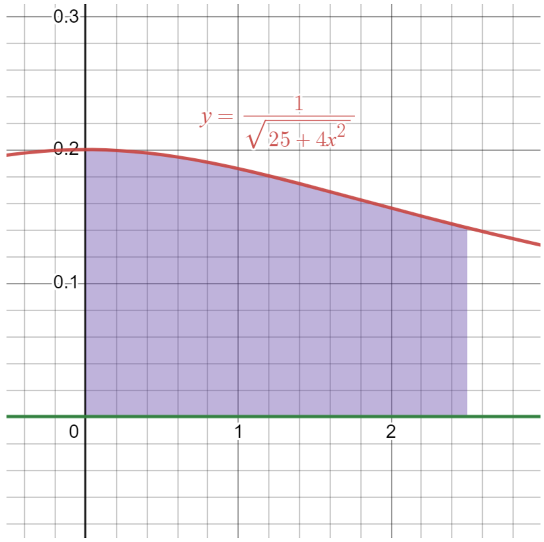
$$y = \ln(x) \;\Rightarrow\; x = e^y.$$

Revolving the horizontal strip at y in the region around $x = 3$ creates a disk with radius $r(y) = 3 - e^y$. So the volume is given by

$$\begin{aligned}
V &= \int_0^{\ln(3)} \pi r(y)^2 dy \\
&= \pi \int_0^{\ln(3)} (3 - e^y)^2 dy \\
&= \pi \int_0^{\ln(3)} (9 - 6e^y + e^{2y}) dy \\
&= \pi \left[9y - 6e^y + \frac{1}{2} e^{2y} \right]_0^{\ln(3)} \\
&= \pi \left(9\ln(3) - 6e^{\ln(3)} + \frac{1}{2} e^{2\ln(3)} + 6e^0 - \frac{1}{2} e^0 \right) \\
&= \pi \left(9\ln(3) - 6 \cdot 3 + \frac{1}{2} 3^2 + 6 - \frac{1}{2} \right) \\
&= \boxed{\pi \left(9\ln(3) - 8 \right) \text{ cubic units}}.
\end{aligned}$$

(c) The region below the graph of $y = \dfrac{1}{\sqrt{25 + 4x^2}}$ on $0 \leqslant x \leqslant \dfrac{5}{2}$ revolved about the x-axis.

Solution: The region and axis of revolution are sketched below.



To use the method of disks/washers, we need to revolve strips perpendicular to the axis of revolution. Since the axis is horizontal here, we revolve vertical strips and use integration with respect to x. Revolving the vertical strip at x in the region around the x-axis creates a disk with radius $r(x) = \dfrac{1}{\sqrt{25+4x^2}}$. So the volume is given by

$$V = \int_0^{5/2} \pi r(x)^2 dx = \pi \int_0^{5/2} \frac{dx}{25+4x^2}.$$

This integral can be calculated using the reference antiderivative

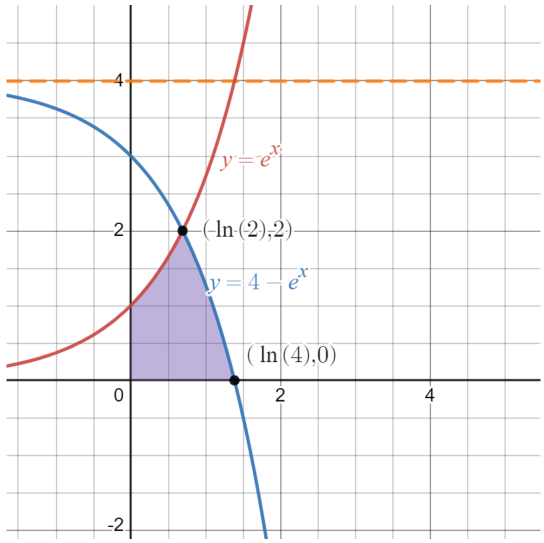
$$\int \frac{dx}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$$

after substituting $u = 2x$, $du = 2dx$. This gives

$$\begin{aligned} V &= \pi \int_0^5 \frac{du}{2(25+u^2)} \\ &= \frac{\pi}{2}\left[\frac{1}{5}\tan^{-1}\left(\frac{u}{5}\right)\right]_0^5 \\ &= \frac{\pi}{10}\left(\tan^{-1}(1) - \tan^{-1}(0)\right) \\ &= \frac{\pi}{10}\left(\frac{\pi}{4} - 0\right) \\ &= \boxed{\frac{\pi^2}{40} \text{ cubic units}}. \end{aligned}$$

(d) The region bounded by $y = e^x$, $y = 4 - e^x$ and the coordinate axes revolved about the line $y = 4$.

Solution: The region and axis of revolution are sketched below.



To use the method of disks/washers, we need to revolve strips perpendicular to the axis of revolution. Since the axis is horizontal here, we revolve vertical strips and use integration with respect to x. Revolving the vertical strip at x in the region creates a washer. The outer radius of the washer is $r_{\text{out}}(x) = 4 - 0 = 4$. the inner radius of the washer is

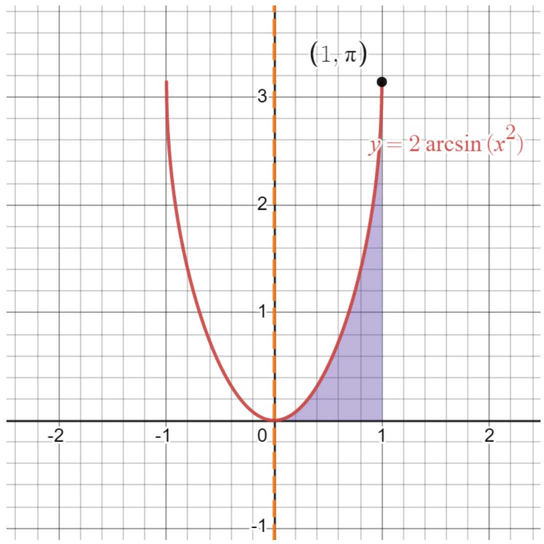
$$r_{\text{in}}(x) = \begin{cases} 4 - e^x & \text{if } 0 \leqslant x \leqslant \ln(2), \\ 4 - (4 - e^x) = e^x & \text{if } \ln(2) \leqslant x \leqslant \ln(4). \end{cases}$$

So the volume is

$$\begin{aligned} V &= \int_0^{\ln(4)} \pi \left(r_{\text{out}}(x)^2 - r_{\text{in}}(x)^2 \right) dx \\ &= \int_0^{\ln(2)} \pi \left(4^2 - (4 - e^x)^2 \right) dx + \int_{\ln(2)}^{\ln(4)} \pi \left(4^2 - (e^x)^2 \right) dx \\ &= \pi \int_0^{\ln(2)} \left(8e^x - e^{2x} \right) dx + \pi \int_{\ln(2)}^{\ln(4)} \left(16 - e^{2x} \right) dx \\ &= \pi \left(\left[8e^x - \frac{1}{2} e^{2x} \right]_0^{\ln(2)} + \left[16x - \frac{1}{2} e^{2x} \right]_{\ln(2)}^{\ln(4)} \right) \\ &= \pi \left(\left(8e^{\ln(2)} - \frac{e^{2\ln(2)}}{2} \right) - \left(8 - \frac{1}{2} \right) + \left(16\ln(4) - \frac{e^{2\ln(4)}}{2} \right) - \left(16\ln(2) - \frac{e^{2\ln(2)}}{2} \right) \right) \\ &= \boxed{\pi \left(16\ln(2) + \frac{1}{2} \right) \text{ cubic units}}. \end{aligned}$$

(e) The region below the graph of $y = 2\sin^{-1}\left(x^2\right)$ on $0 \leqslant x \leqslant 1$ revolved about the y-axis.

Solution: The region and axis of revolution are sketched below.



To use the method of disks/washers, we need to revolve strips perpendicular to the axis of revolution. Since the axis is vertical here, we revolve horizontal strips and use integration with respect to y. We need to express the curve as a function of y:

$$y = 2\sin^{-1}(x^2) \;\Rightarrow\; x^2 = \sin\left(\frac{y}{2}\right) \;\Rightarrow\; |x| = \sqrt{\sin\left(\frac{y}{2}\right)} \;\Rightarrow\; x = \sqrt{\sin\left(\frac{y}{2}\right)}$$

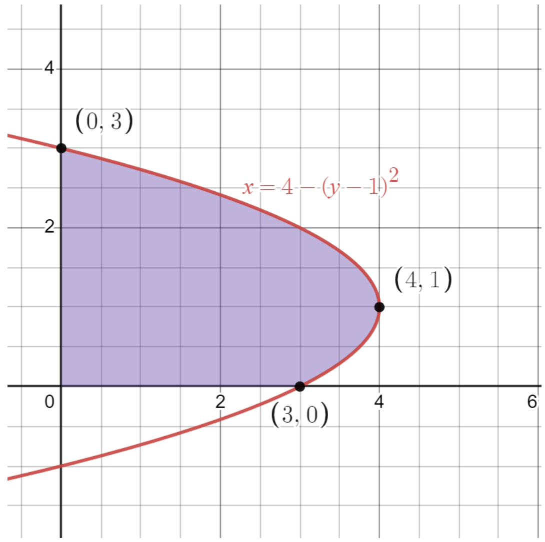
where the simplification $|x| = x$ is because $x \geqslant 0$ for the portion of the curve considered. Revolving the horizontal strip at y in the region around the y-axis creates a washer with outer radius $r_{\text{out}}(y) = 1$ and inner radius $r_{\text{in}}(y) = \sqrt{\sin\left(\frac{y}{2}\right)}$. So the volume is

$$\begin{aligned} V &= \int_0^{\pi} \pi\left(r_{\text{out}}(y)^2 - r_{\text{in}}(y)^2\right) dy \\ &= \int_0^{\pi} \pi\left(1 - \sin\left(\frac{y}{2}\right)\right) dy \\ &= \pi\left[y + 2\cos\left(\frac{y}{2}\right)\right]_0^{\pi} \\ &= \pi\left(\pi + 2\cos\left(\frac{\pi}{2}\right) - 2\cos(0)\right) \\ &= \boxed{\pi(\pi - 2)\ \text{cubic units}}. \end{aligned}$$

#14. Consider the region $\mathcal{R}$ in the first quadrant bounded by the curve $x = 4 - (y-1)^2$.

(a) Sketch the region. Make sure to clearly label the curve and its intercepts.

Solution:



(b) A solid has base $\mathcal{R}$ and cross-sections perpendicular to the y-axis. Calculate the volume of the solid if the cross-sections are (i) semi-circles with diameter in the base and (ii) equilateral triangles with a side in the base.

Solution: The horizontal strip at y in the region is bounded on the right by $x = 4 - (y-1)^2$ and on the left by $x = 0$. Therefore it has length

$$\ell(y) = 4 - (y-1)^2 - 0 = 4 - (y^2 - 2y + 1) = 3 + 2y - y^2.$$

(i) A semi-circle of diameter ℓ has area $\frac{1}{2}\pi\left(\frac{\ell}{2}\right)^2 = \frac{\pi}{8}\ell^2$. With the length of the strip $\ell(y)$ we found above, we can express the area of the cross-section at y as

$$A(y) = \frac{\pi}{8}\ell(y)^2 = \frac{\pi}{8}\left(3 + 2y - y^2\right)^2 = \frac{\pi}{8}\left(9 + 12y - 2y^2 - 4y^3 + y^4\right).$$

So the volume of the solid is

$$\begin{aligned} V &= \int_0^3 A(y)dy \\ &= \int_0^3 \frac{\pi}{8}\left(9 + 12y - 2y^2 - 4y^3 + y^4\right)dy \\ &= \frac{\pi}{8}\left[9y + 6y^2 - \frac{2}{3}y^3 - y^4 + \frac{1}{5}y^5\right]_0^3 \\ &= \frac{\pi}{8}\left(9 \cdot 3 + 6(3)^2 - \frac{2}{3}(3)^3 - 3^4 + \frac{1}{5}(3)^5\right) \\ &= \boxed{\frac{63\pi}{40} \text{ cubic units}}. \end{aligned}$$

(ii) An equilateral triangle with side length ℓ has area $\frac{\sqrt{3}}{4}\ell^2$. With the length of the strip $\ell(y)$ we found above, we can express the area of the cross-section at y as

$$A(y) = \frac{\sqrt{3}}{4}\ell(y)^2 = \frac{\sqrt{3}}{4}\left(3 + 2y - y^2\right)^2 = \frac{\sqrt{3}}{4}\left(9 + 12y - 2y^2 - 4y^3 + y^4\right).$$

So the volume of the solid is

$$\begin{aligned} V &= \int_0^3 A(y)dy \\ &= \int_0^3 \frac{\sqrt{3}}{4}\left(9 + 12y - 2y^2 - 4y^3 + y^4\right) dy \\ &= \frac{\sqrt{3}}{4}\left[9y + 6y^2 - \frac{2}{3}y^3 - y^4 + \frac{1}{5}y^5\right]_0^3 \\ &= \frac{\sqrt{3}}{4}\left(9 \cdot 3 + 6(3)^2 - \frac{2}{3}(3)^3 - 3^4 + \frac{1}{5}(3)^5\right) \\ &= \boxed{\frac{63\sqrt{3}}{20} \text{ cubic units}}. \end{aligned}$$

Note: we could have used the integral that we had already compute in part (i) to minimize computations:

$$\int_0^3 \left(9 + 12y - 2y^2 - 4y^3 + y^4\right) dy = \frac{63}{5}.$$

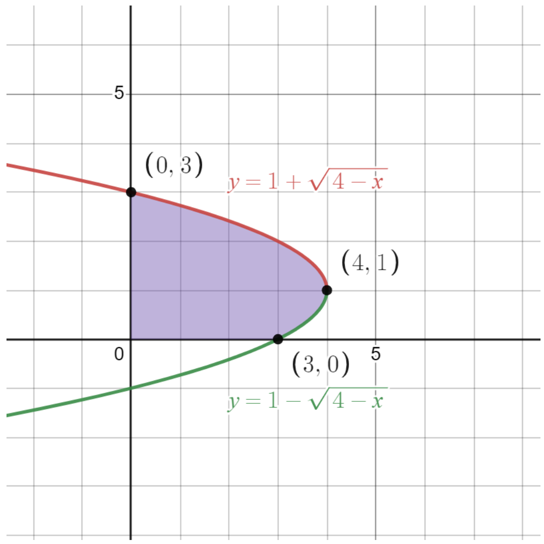
Only the factor in front of the integral changed for part (ii).

(c) A solid has base $\mathcal{R}$ and its cross-sections perpendicular to the x-axis are isosceles right triangles with hypotenuse in the base. Calculate the volume of the solid.

Solution: We need to express the curve as functions of x:

$$x = 4 - (y-1)^2 \;\Rightarrow\; (y-1)^2 = 4 - x \;\Rightarrow |y - 1| = \sqrt{4-x} \;\Rightarrow\; y = 1 \pm \sqrt{4-x}.$$

The solution with the positive sign corresponds to the upper branch of the parabola and the negative sign corresponds to the lower branch of the parabola, see figure below.



The vertical strip at x is bounded on the top by $y = 1 + \sqrt{4-x}$. On the bottom, the vertical strip at x is bounded by $y = 0$ for $0 \leqslant x \leqslant 3$ and $y = 1 - \sqrt{4-x}$ for $3 \leqslant x \leqslant 4$. Therefore, the length of the strip is

$$\ell(x) = \begin{cases} 1 + \sqrt{4-x} - 0 = 1 + \sqrt{4-x} & \text{if } 0 \leqslant x \leqslant 3, \\ (1 + \sqrt{4-x}) - (1 - \sqrt{4-x}) = 2\sqrt{4-x} & \text{if } 3 \leqslant x \leqslant 4. \end{cases}$$

The area of an isosceles right triangle with hypotenuse ℓ is $\frac{1}{4}\ell^2$, so the area of the cross-section at x is

$$A(x) = \frac{1}{4}\ell(x)^2 = \begin{cases} \frac{1}{4}\left(1 + \sqrt{4-x}\right)^2 = \frac{1}{4}\left(5 - x + 2\sqrt{4-x}\right) & \text{if } 0 \leqslant x \leqslant 3, \\ \frac{1}{4}\left(2\sqrt{4-x}\right)^2 = 4 - x & \text{if } 3 \leqslant x \leqslant 4. \end{cases}$$

So the volume of the solid is given by

$$\begin{aligned} V &= \int_0^4 A(x)dx \\ &= \int_0^3 \frac{1}{4}\left(5 - x + 2\sqrt{4-x}\right)dx + \int_3^4 (4-x)dx \\ &= \frac{1}{4}\left[5x - \frac{1}{2}x^2 - \frac{4}{3}(4-x)^{3/2}\right]_0^3 + \left[4x - \frac{1}{2}x^2\right]_3^4 \\ &= \frac{1}{4}\left(15 - \frac{9}{2} - \frac{4}{3} + \frac{4}{3}4^{3/2}\right) + \left(16 - \frac{1}{2}16 - 12 + \frac{9}{2}\right) \\ &= \boxed{\frac{131}{24} \text{ cubic units}}. \end{aligned}$$

(d) Calculate the volume of the solid of revolution obtained by revolving $\mathcal{R}$ about (i) the y-axis and (ii) the line $y = -2$.

Solution: (i) Revolving the horizontal strip at y about the y-axis creates a disk of radius $r(y) = 4 - (y-1)^2 = 3 + 2y - y^2$. So the volume is given by

$$\begin{aligned} V &= \int_0^3 \pi r(y)^2 dy \\ &= \pi \int_0^3 (3 + 2y - y^2)^2 dy \\ &= \boxed{\frac{63\pi}{5} \text{ cubic units}}. \end{aligned}$$

(We have used the integral already computed in part (b)(i).)

(ii) Revolving the vertical strip at x about the line $y = -2$ creates a washer. The outer radius of the washer is

$$r_{\text{out}}(x) = 1 + \sqrt{4-x} - (-2) = 3 + \sqrt{4-x}.$$

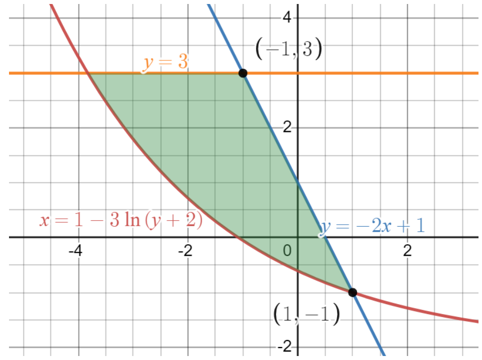
The inner radius is

$$r_{\text{in}}(x) = \begin{cases} 0 - (-2) = 2 & \text{if } 0 \leqslant x \leqslant 3, \\ 1 - \sqrt{4-x} - (-2) = 3 - \sqrt{4-x} & \text{if } 3 \leqslant x \leqslant 4. \end{cases}$$

So the volume is given by

$$\begin{aligned} V &= \int_0^4 \pi \left(r_{\text{out}}(x)^2 - r_{\text{in}}(x)^2\right) dx \\ &= \int_0^3 \pi \left((3 + \sqrt{4-x})^2 - 2^2\right) dx + \int_3^4 \pi \left((3 + \sqrt{4-x})^2 - (3 - \sqrt{4-x})^2\right) dx \\ &= \pi \int_0^3 \left(9 + 6\sqrt{4-x} - x\right) dx + \pi \int_3^4 12\sqrt{4-x} dx \\ &= \pi \left(\left[9x - 4(4-x)^{3/2} - \frac{1}{2}x^2 \right]_0^3 + \left[-8\sqrt{4-x} \right]_3^4 \right) \\ &= \boxed{\frac{117\pi}{2} \text{ cubic units}}. \end{aligned}$$

#15. Consider the region $\mathcal{R}$ bounded by the curves $x = 1 - 3\ln(y+2)$, $y = 3$ and $y = -2x + 1$. The region $\mathcal{R}$ is shaded in the figure to the right. Use the disk/washer method to set-up integrals that compute the volume of the solid obtained by revolving $\mathcal{R}$ about each of the lines given below.



(a) $x = 2$

Solution: We need to express the blue line as a function of y:

$$y = -2x + 1 \;\Rightarrow\; x = \frac{1-y}{2}$$

Revolving the horizontal strip at y in the region about the line $x = 2$ creates a washer with inner radius $r_{\text{in}}(y) = 2 - \frac{1-y}{2} = \frac{3+y}{2}$ and outer radius $r_{\text{out}}(y) = 2 - (1 - 3\ln(y+2)) = 1 + 3\ln(y+2)$. So the volume of the solid is given by

$$\boxed{V = \int_{-1}^{3} \pi \left((1 + 3\ln(y+2))^2 - \left(\frac{3+y}{2}\right)^2 \right) dy}.$$

(b) $y = 3$

Solution: We need to express the red curve as a function of x:

$$x = 1 - 3\ln(y+2) \;\Rightarrow\; \ln(y+2) = \frac{1-x}{3} \;\Rightarrow\; y = e^{(1-x)/3} - 2.$$

Revolving the vertical strip at x in the region about $y = 3$ creates a washer of outer radius

$$r_{\text{out}}(x) = 3 - \left(e^{(1-x)/3} - 2\right) = 5 - e^{(1-x)/3}.$$

The inner radius is

$$r_{\text{in}}(x) = \begin{cases} 0 & \text{if } 1 - 3\ln(5) \leqslant x \leqslant -1, \\ 3 - (-2x+1) = 2 + 2x & \text{if } -1 \leqslant x \leqslant 1. \end{cases}$$

So the volume is given by

$$\boxed{V = \int_{1-3\ln(5)}^{-1} \pi \left(5 - e^{(1-x)/3}\right)^2 dx + \int_{-1}^{1} \pi \left(\left(5 - e^{(1-x)/3}\right)^2 - (2+2x)^2 \right) dx}.$$

(c) $x = -4$

Solution: Revolving the horizontal strip at y in the region about the line $x = -4$ creates a washer with outer radius $r_{\text{out}}(y) = \frac{1-y}{2} - (-4) = \frac{9-y}{2}$ and inner radius $r_{\text{in}}(y) = (1 - 3\ln(y + 2)) - (-4) = 5 - 3\ln(y+2)$. So the volume of the solid is given by

$$\boxed{V = \int_{-1}^{3} \pi \left(\left(\frac{9-y}{2} \right)^2 - (5 - 3\ln(y+2))^2 \right) dy}.$$

(d) $y = -2$

Solution: Revolving the vertical strip at x in the region creates a washer with inner radius $r_{\text{in}}(x) = e^{(1-x)/3} - 2 - (-2) = e^{(1-x)/3}$. The outer radius is given by

$$r_{\text{out}}(x) = \begin{cases} 3 - (-2) = 5 & \text{if } 1 - 3\ln(5) \leqslant x \leqslant -1, \\ -2x + 1 - (-2) = 3 - 2x & \text{if } -1 \leqslant x \leqslant 1. \end{cases}$$

So the volume is given by

$$\boxed{V = \int_{1-3\ln(5)}^{-1} \pi \left(5^2 - \left(e^{(1-x)/3} \right)^2 \right) dx + \int_{-1}^{1} \pi \left((3-2x)^2 - \left(e^{(1-x)/3} \right)^2 \right) dx}.$$

#16. Consider the region below the graph $y = \cot(5x) + \csc(5x)$ for $\frac{\pi}{20} \leqslant x \leqslant \frac{\pi}{10}$.

(a) Find the area of the region.

Solution: The area is given by

$$\begin{aligned} A &= \int_{\pi/20}^{\pi/10} (\cot(5x) + \csc(5x)) dx \\ &= \frac{1}{5} \int_{\pi/4}^{\pi/2} (\cot(u) + \csc(u))\, du \quad (u = 5x) \\ &= \frac{1}{5} \left[\ln|\sin(u)| - \ln|\csc(u) + \cot(u)| \right]_{\pi/4}^{\pi/2} \\ &= \frac{1}{5} \left(\ln(1) - \ln(1+0) - \ln\left(\frac{1}{\sqrt{2}} \right) + \ln\left(\sqrt{2} + 1 \right) \right) \\ &= \boxed{\frac{1}{5} \ln(2 + \sqrt{2}) \text{ square units}}. \end{aligned}$$

(b) Find the volume of the solid obtained by revolving the region about the x-axis.

Solution: We use the disk method. Revolving the vertical strip at x about the x-axis gives a disk of radius $r(x) = \cot(5x) + \csc(5x)$. Therefore, the volume is

$$\begin{aligned}
V &= \int_{\pi/20}^{\pi/10} \pi \left(\cot(5x) + \csc(5x)\right)^2 dx \\
&= \frac{\pi}{5} \int_{\pi/4}^{\pi/2} \left(\cot(u) + \csc(u)\right)^2 du \quad (u = 5x) \\
&= \frac{\pi}{5} \int_{\pi/4}^{\pi/2} \big(\cot^2(u) + \csc^2(u) + 2\cot(u)\csc(u)\big)\, du \\
&= \frac{\pi}{5} \int_{\pi/4}^{\pi/2} \big((\csc^2(u) - 1) + \csc^2(u) + 2\cot(u)\csc(u)\big)\, du \\
&= \frac{\pi}{5} \int_{\pi/4}^{\pi/2} \big(2\csc^2(u) + 2\cot(u)\csc(u) - 1\big)\, du \\
&= \frac{\pi}{5} \left[-2\cot(u) - 2\csc(u) - u\right]_{\pi/4}^{\pi/2} \\
&= \frac{\pi}{5} \left(-2 \cdot 0 - 2 - \frac{\pi}{2} + 2 + 2\sqrt{2} + \frac{\pi}{4}\right) \\
&= \boxed{\frac{\pi}{5} \left(2\sqrt{2} - \frac{\pi}{4}\right) \text{ cubic units}}.
\end{aligned}$$