Section 8.2: Integration by Parts - Worksheet Solutions

#33. Evaluate the following integrals. **Note:** some of these problems use integration techniques from earlier sections.

(a) $\int 5xe^{8x+1}dx$

Solution: We use IBP with

$$u = 5x \;\Rightarrow\; du = 5dx,$$
$$dv = e^{8x+1}dx \;\Rightarrow\; v = \frac{1}{8}e^{8x+1}.$$

This gives

$$\int 5xe^{8x+1}dx = \frac{5xe^{8x+1}}{8} - \int \frac{5e^{8x+1}}{8}dx = \boxed{\frac{5xe^{8x+1}}{8} - \frac{5e^{8x+1}}{64} + C}.$$

(b) $\int \arctan(7x)dx$

Solution: We use IBP with

$$u = \arctan(7x) \;\Rightarrow\; du = \frac{7dx}{1+49x^2},$$
$$dv = dx \;\Rightarrow\; v = x.$$

This gives

$$\int \arctan(7x)dx = x\arctan(7x) - \int \frac{7x}{1+49x^2}dx.$$

This last integral can be evaluated by substituting $t = 1 + 49x^2$, so $dt = 98xdx$. Then

$$\int \frac{7x}{1+49x^2}dx = \int \frac{7}{98t}dt = \frac{\ln|t|}{14} + C = \frac{\ln(1+49x^2)}{14} + C.$$

where we have dropped the absolute values since $1 + 49x^2 > 0$ for all x. Putting everything together, we have

$$\boxed{\int \arctan(7x)dx = x\arctan(7x) - \frac{\ln(1+49x^2)}{14} + C}.$$

(c) $\int x^3 \cos(5x)dx$

Solution: We will evaluate this integral with three consecutive IBPs, taking u to be the power of x each time. For the first IBP, we take

$$\begin{aligned} u &= x^3 \;\Rightarrow\; du = 3x^2 dx, \\ dv &= \cos(5x)dx \;\Rightarrow\; v = \frac{1}{5}\sin(5x). \end{aligned}$$

This gives

$$\begin{aligned} \int x^3 \cos(5x)dx &= \frac{x^3 \sin(5x)}{5} - \int 3x^2 \frac{\sin(5x)}{5} dx \\ &= \frac{x^3 \sin(5x)}{5} - \frac{3}{5}\int x^2 \sin(5x)dx. \end{aligned}$$

For the second IBP, we take

$$\begin{aligned} u &= x^2 \;\Rightarrow\; du = 2x dx, \\ dv &= \sin(5x)dx \;\Rightarrow\; v = -\frac{1}{5}\cos(5x). \end{aligned}$$

This gives

$$\begin{aligned} \int x^3 \cos(5x)dx &= \frac{x^3 \sin(5x)}{5} - \frac{3}{5}\left(-\frac{x^2\cos(5x)}{5} - \int -2x \frac{\cos(5x)}{5} dx\right) \\ &= \frac{x^3 \sin(5x)}{5} + \frac{3x^2\cos(5x)}{25} - \frac{6}{25}\int x\cos(5x)dx. \end{aligned}$$

Finally, for the third and final IBP, we take

$$\begin{aligned} u &= x \;\Rightarrow\; du = dx, \\ dv &= \cos(5x)dx \;\Rightarrow\; v = \frac{1}{5}\sin(5x), \end{aligned}$$

and we obtain

$$\begin{aligned} \int x^3 \cos(5x)dx &= \frac{x^3 \sin(5x)}{5} + \frac{3x^2\cos(5x)}{25} - \frac{6}{25}\left(\frac{x\sin(5x)}{5} - \int \frac{\sin(5x)}{5}dx\right) \\ &= \frac{x^3 \sin(5x)}{5} + \frac{3x^2\cos(5x)}{25} - \frac{6x\sin(5x)}{125} + \frac{6}{125}\int \sin(5x)dx \\ &= \boxed{\frac{x^3 \sin(5x)}{5} + \frac{3x^2\cos(5x)}{25} - \frac{6x\sin(5x)}{125} - \frac{6\cos(5x)}{625} + C}. \end{aligned}$$

(d) $\int x^3 \cos(5x^4)dx$

Solution: This integral will use a substitution rather than an IBP. This is because of the form of the integrand: we have an "inside function", $5x^4$, whose derivative, $20x^3$, appears as a factor of the integrand up to a constant factor. So we choose to substitute $u = 5x^4$, which gives $du = 20x^3 dx$. Then we have

$$\begin{aligned}\int x^3 \cos(5x^4) dx &= \int \frac{\cos(u)}{20} du \\ &= \frac{\sin(u)}{20} + C \\ &= \boxed{\frac{\sin(5x^4)}{20} + C}.\end{aligned}$$

(e) $\displaystyle\int_1^e (\ln(x))^2 \, dx$

Solution: We use an IBP with

$$\begin{aligned}u &= (\ln(x))^2 \;\Rightarrow\; du = \frac{2\ln(x)}{x} dx, \\ dv &= dx \;\Rightarrow\; v = x.\end{aligned}$$

We get

$$\begin{aligned}\int_1^e (\ln(x))^2 \, dx &= \left[x (\ln(x))^2\right]_1^e - \int_1^e x \frac{2\ln(x)}{x} dx \\ &= e - 2\int_1^e \ln(x) dx.\end{aligned}$$

For this last integral, we again use an IBP with

$$\begin{aligned}u &= \ln(x) \;\Rightarrow\; du = \frac{dx}{x}, \\ dv &= dx \;\Rightarrow\; v = x.\end{aligned}$$

We get

$$\begin{aligned}\int_1^e (\ln(x))^2 \, dx &= e - 2\left([x\ln(x)]_1^e - \int_1^e x\frac{1}{x} dx\right) \\ &= e - 2\left(e - \int_1^e dx\right) \\ &= e - 2\,(e - (e-1)) \\ &= \boxed{e - 2}.\end{aligned}$$

(f) $\int x^2 \sin^{-1}(x)dx$

Solution: We start with an IBP taking

$$u = \sin^{-1}(x) \;\Rightarrow\; du = \frac{dx}{\sqrt{1-x^2}},$$
$$dv = x^2 dx \;\Rightarrow\; v = \frac{x^3}{3},$$

which gives

$$\int x^2 \sin^{-1}(x)dx = \frac{x^3 \sin^{-1}(x)}{3} - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}dx.$$

For this last integral, we perform the substitution $u = 1 - x^2$, $du = -2xdx$. The numerator is $x^2 xdx$, and we will replace $x^2 = 1 - u$ and $xdx = -\frac{du}{2}$. We obtain

$$\begin{aligned}
\int \frac{x^2 xdx}{\sqrt{1-x^2}} &= \int -\frac{1-u}{2\sqrt{u}}du \\
&= \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right) du \\
&= \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) \\
&= \frac{(1-x^2)^{3/2}}{3} - (1-x^2)^{1/2}.
\end{aligned}$$

Plugging back in the above equation gives

$$\begin{aligned}
\int x^2 \sin^{-1}(x)dx &= \frac{x^3 \sin^{-1}(x)}{3} - \frac{1}{3}\left(\frac{(1-x^2)^{3/2}}{3} - (1-x^2)^{1/2}\right) \\
&= \boxed{\frac{x^3 \sin^{-1}(x)}{3} - \frac{(1-x^2)^{3/2}}{9} + \frac{(1-x^2)^{1/2}}{3} + C}.
\end{aligned}$$

(g) $\int_0^{\pi/12} x \sec(4x)\tan(4x)dx$

Solution: We use IBP with

$$u = x \;\Rightarrow\; du = dx,$$
$$dv = \sec(4x)\tan(4x)dx \;\Rightarrow\; v = \frac{\sec(4x)}{4}.$$

This gives

$$\int_0^{\pi/12} x \sec(4x)\tan(4x)dx = \left[\frac{x\sec(4x)}{4}\right]_0^{\pi/12} - \int_0^{\pi/12} \frac{\sec(4x)}{4}dx$$

$$
\begin{aligned}
&= \frac{\pi \sec\left(\frac{\pi}{3}\right)}{48} - \left[\frac{\ln|\sec(4x)+\tan(4x)|}{16}\right]_0^{\pi/12} \\
&= \frac{\pi}{24} - \frac{\ln\left(\sec\left(\frac{\pi}{3}\right)+\tan\left(\frac{\pi}{3}\right)\right) - \ln(\sec(0)+\tan(0))}{16} \\
&= \boxed{\frac{\pi}{24} - \frac{\ln(2+\sqrt{3})}{16}}.
\end{aligned}
$$

(h) $\displaystyle\int \frac{\ln(x)}{x^5}dx$

Solution: We use an IBP with

$$
\begin{aligned}
u &= \ln(x) \;\Rightarrow\; du = \frac{dx}{x}, \\
dv &= \frac{dx}{x^5}, \;\Rightarrow\; v = -\frac{1}{4x^4}.
\end{aligned}
$$

This gives

$$
\begin{aligned}
\int \frac{\ln(x)}{x^5}dx &= -\frac{\ln(x)}{4x^4} + \frac{1}{4}\int \frac{1}{x^5}dx \\
&= \boxed{-\frac{\ln(x)}{4x^4} - \frac{1}{16x^4} + C}.
\end{aligned}
$$

(i) $\displaystyle\int x^3 e^{-x^2}dx$

Solution: We can start with the substitution $t = -x^2$, $dt = -2xdx$, which gives

$$
\int x^3 e^{-x^2}dx = \int x^2 e^{-x^2} x dx = \int (-t)e^t \frac{dt}{-2} = \frac{1}{2}\int te^t dt.
$$

We compute this new integral with an IBP taking

$$
\begin{aligned}
u &= t \;\Rightarrow\; du = dt, \\
dv &= e^t dt \;\Rightarrow\; v = e^t,
\end{aligned}
$$

which gives

$$
\int te^t dt = te^t - \int e^t dt = te^t - e^t = e^t(t-1).
$$

We plug back and replace t by $-x^2$:

$$
\int x^3 e^{-x^2}dx = \frac{e^t}{2}(t-1)
$$

$$= \boxed{\frac{e^{-x^2}}{2}\left(-x^2-1\right)+C}.$$

(j) $\displaystyle\int_3^6 \frac{dx}{\sqrt{12x-x^2}}$

Solution: This integral can be calculated using the reference integral

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right)+C$$

after completing the square and using a substitution. We have

$$12x - x^2 = 36 - (36 - 12x + x^2) = 36 - (x-6)^2,$$

so

$$\begin{aligned}
\int_3^6 \frac{dx}{\sqrt{12x-x^2}} &= \int_3^6 \frac{dx}{\sqrt{36-(x-6)^2}} \\
&= \int_{-3}^0 \frac{du}{\sqrt{36-u^2}} \quad \text{(substitute } u = x-6\text{)} \\
&= \left[\sin^{-1}\left(\frac{u}{6}\right)\right]_{-3}^0 \\
&= \sin^{-1}(0) - \sin^{-1}\left(-\frac{1}{2}\right) \\
&= \boxed{\frac{\pi}{6}}.
\end{aligned}$$

(k) $\displaystyle\int e^{-2x}\sin(3x)dx$

Solution: We perform an IBP twice and then solve for the unknown integral. For the first IBP, the parts are

$$\begin{aligned}
u &= \sin(3x) \;\Rightarrow\; du = 3\cos(3x)dx, \\
dv &= e^{-2x}dx \;\Rightarrow\; v = -\frac{e^{-2x}}{2}.
\end{aligned}$$

This gives

$$\int e^{-2x}\sin(3x)dx = -\frac{e^{-2x}\sin(3x)}{2} - \int -\frac{e^{-2x}}{2}3\cos(3x)dx$$

$$= -\frac{e^{-2x}\sin(3x)}{2} + \frac{3}{2}\int e^{-2x}\cos(3x)dx.$$

For the second IBP, the parts are

$$u = \cos(3x) \;\Rightarrow\; du = -3\sin(3x)dx,$$
$$dv = e^{-2x}dx \;\Rightarrow\; v = -\frac{e^{-2x}}{2}.$$

This gives

$$\int e^{-2x}\sin(3x)dx = -\frac{e^{-2x}\sin(3x)}{2} + \frac{3}{2}\left(-\frac{e^{-2x}\cos(3x)}{2} - \int -\frac{e^{-2x}}{2}(-3\sin(3x))dx\right)$$
$$= -\frac{e^{-2x}\sin(3x)}{2} - \frac{3e^{-2x}\cos(3x)}{4} - \frac{9}{4}\int e^{-2x}\sin(3x)dx.$$

We can now solve the relation for the unknown integral. This gives

$$\left(1+\frac{9}{4}\right)\int e^{-2x}\sin(3x)dx = -\frac{e^{-2x}\sin(3x)}{2} - \frac{3e^{-2x}\cos(3x)}{4}$$
$$\frac{13}{4}\int e^{-2x}\sin(3x)dx = -\frac{e^{-2x}\sin(3x)}{2} - \frac{3e^{-2x}\cos(3x)}{4}$$
$$\int e^{-2x}\sin(3x)dx = \boxed{-\frac{2e^{-2x}\sin(3x)}{13} - \frac{3e^{-2x}\cos(3x)}{13} + C}$$

(l) $\int_{-2}^{-1} x\sec^{-1}(x)dx$

Solution: We use an IBP with parts

$$u = \sec^{-1}(x) \;\Rightarrow\; du = \frac{dx}{|x|\sqrt{x^2-1}},$$
$$dv = x \;\Rightarrow\; v = \frac{x^2}{2}.$$

We get

$$\int_{-2}^{-1} x\sec^{-1}(x)dx = \left[\frac{x^2\sec^{-1}(x)}{2}\right]_{-2}^{-1} - \int_{-2}^{-1}\frac{x^2}{2|x|\sqrt{x^2-1}}dx$$
$$= \frac{\sec^{-1}(-1) - 4\sec^{-1}(-2)}{2} - \int_{-2}^{-1}\frac{x^2}{2(-x)\sqrt{x^2-1}}dx \quad (x<0 \text{ so } |x| = -x)$$
$$= \frac{\pi - 4\frac{2\pi}{3}}{2} + \int_{-2}^{-1}\frac{x}{2\sqrt{x^2-1}}dx$$

$$= -\frac{5\pi}{6} + \int_3^0 \frac{dt}{4\sqrt{t}} \quad \text{(substitute } t = x^2-1, dt = 2xdx\text{)}$$
$$= -\frac{5\pi}{6} + \left[\frac{\sqrt{t}}{2}\right]_3^0$$
$$= \boxed{-\frac{5\pi}{6} - \frac{\sqrt{3}}{2}}.$$

(m) $\int \theta \sec^2(7\theta) d\theta$

Solution: We use an IBP with the parts

$$u = \theta \;\Rightarrow\; du = d\theta,$$
$$dv = \sec^2(7\theta)dx \;\Rightarrow\; v = \frac{\tan(7\theta)}{7},$$

$$\int \theta \sec^2(7\theta)d\theta = \frac{1}{7}\theta\tan(7\theta) - \frac{1}{7}\int \tan(7\theta)d\theta$$
$$= \boxed{\frac{1}{7}\theta\tan(7\theta) - \frac{1}{49}\ln|\sec(7\theta)| + C}.$$

(n) $\int \sin(\sqrt{3x+5})dx$

Solution: We start with the substitution $t = \sqrt{3x+5}$, which gives $dt = \frac{3dx}{2\sqrt{3x+5}}$. This last relation can be written $dx = \frac{2}{3}\sqrt{3x+5}dt = \frac{2}{3}tdt$. Therefore,

$$\int \sin(\sqrt{3x+5})dx = \int \frac{2}{3}t\sin(t)dt = \frac{2}{3}\int t\sin(t)dt.$$

This new integral can be evaluated with IBP using the parts

$$u = t \;\Rightarrow\; du = dt,$$
$$dv = \sin(t)dt \;\Rightarrow\; v = -\cos(t).$$

We obtain

$$\int t\sin(t)dt = -t\cos(t) + \int \cos(t)dt = -t\cos(t) + \sin(t) + C.$$

Going back to the original integral, we have

$$\int \sin(\sqrt{3x+5})dx = \frac{2}{3}\int t\sin(t)dt$$

$$= \frac{2}{3}\left(-t\cos(t) + \sin(t)\right) + C$$

$$= \boxed{\frac{2}{3}\left(-\sqrt{3x+5}\cos(\sqrt{3x+5}) + \sin\left(\sqrt{3x+5}\right)\right) + C}.$$

(o) $\displaystyle\int \frac{x}{\sqrt{5x+1}}dx$

Solution: This integral can be calculated using three methods: substituting $u = 5x + 1$, substituting $u = \sqrt{5x+1}$ or integrating by parts.

Method 1: substitute $u = 5x + 1$, so $du = 5dx$. The extraneous factor x in the denominator can be expressed in terms of u as $x = \frac{u-1}{5}$. We get

$$\begin{aligned}
\int \frac{x}{\sqrt{5x+1}}dx &= \int \frac{u-1}{25\sqrt{u}}du \\
&= \frac{1}{25}\int \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right) du \\
&= \frac{1}{25}\int \left(u^{1/2} - u^{-1/2}\right) du \\
&= \frac{1}{25}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) + C \\
&= \boxed{\frac{2}{25}\left(\frac{1}{3}(5x+1)^{3/2} - (5x+1)^{1/2}\right) + C}.
\end{aligned}$$

Method 2: substitute $u = \sqrt{5x+1}$, so $du = \frac{5}{2\sqrt{5x+1}}dx$. The extraneous factor x in the denominator can be expressed in terms of u as $x = \dfrac{u^2-1}{5}$. We get

$$\begin{aligned}
\int \frac{x}{\sqrt{5x+1}}dx &= \int \frac{2(u^2-1)}{25}du \\
&= \frac{2}{25}\int (u^2-1)du \\
&= \frac{2}{25}\left(\frac{1}{3}u^3 - u\right) + C \\
&= \boxed{\frac{2}{25}\left(\frac{1}{3}(5x+1)^{3/2} - (5x+1)^{1/2}\right) + C}.
\end{aligned}$$

Method 3: use an IBP with parts

$$u = x \;\Rightarrow\; du = dx,$$

$$dv = \frac{dx}{\sqrt{5x+1}} \;\Rightarrow\; v = \frac{2}{5}\sqrt{5x+1}.$$

We obtain

$$\begin{aligned}\int \frac{x}{\sqrt{5x+1}}dx &= \frac{2}{5}x\sqrt{5x+1} - \frac{2}{5}\int \sqrt{5x+1}dx \\ &= \boxed{\frac{2}{5}x\sqrt{5x+1} - \frac{4}{75}(5x+1)^{3/2} + C}.\end{aligned}$$

Remark: This answer may appear to be different from the other two, but a bit of algebra shows that they are the same.

#34. Calculate the volume of the solid obtained by revolving the given region about the given axis using (i) the method of disks/washers and (ii) the method of cylindrical shells.

(a) The region between the graph of $y = \sqrt{\tan^{-1}(x)}$ and the x-axis for $0 \leqslant x \leqslant 1$ revolved about the x-axis.

Solution: (i) Revolving the vertical strip at x in the region about the x-axis forms a disk of radius $r(x) = \sqrt{\tan^{-1}(x)}$. So the volume is

$$\begin{aligned}V &= \int_0^1 \pi r(x)^2 dx \\ &= \int_0^1 \pi \sqrt{\tan^{-1}(x)}^2 dx \\ &= \pi \int_0^1 \tan^{-1}(x) dx.\end{aligned}$$

We calculate this integral with an IBP using the parts

$$\begin{aligned}u &= \tan^{-1}(x) \;\Rightarrow\; du = \frac{dx}{x^2+1}, \\ dv &= dx \;\Rightarrow\; v = x.\end{aligned}$$

We obtain

$$\begin{aligned}V &= \pi\left(\left[x\tan^{-1}(x)\right]_0^1 - \int_0^1 \frac{x}{x^2+1}dx\right) \\ &= \pi\left(\frac{\pi}{4} - \int_0^1 \frac{x}{x^2+1}dx\right).\end{aligned}$$

This remaining integral can be evaluated with the substitution $u = x^2+1$, so that $du = 2xdx$. The bounds are $u = 1$ when $x = 0$ and $u = 2$ when $x = 1$. We get

$$V = \pi\left(\frac{\pi}{4} - \int_1^2 \frac{du}{2u}\right)$$

$$
\begin{aligned}
&= \pi\left(\frac{\pi}{4} - \frac{1}{2}\left[\ln|u|\right]_1^2\right) \\
&= \boxed{\pi\left(\frac{\pi}{4} - \frac{\ln(2)}{2}\right) \text{ cubic units}}.
\end{aligned}
$$

(ii) We will need to express the curve as a function of y:

$$
y = \sqrt{\tan^{-1}(x)} \ \Rightarrow\ \tan^{-1}(x) = y^2 \ \Rightarrow\ x = \tan\left(y^2\right).
$$

Then the region is bounded between the curves $x = \tan\left(y^2\right)$ and $x = 1$ for $0 \leqslant y \leqslant \sqrt{\frac{\pi}{4}}$. Revolving the horizontal strip at y in the region about the x-axis forms a cylindrical shell of radius $r(y) = y$ and height $h(y) = 1 - \tan(y^2)$. So the volume is

$$
\begin{aligned}
V &= \int_0^{\sqrt{pi/4}} 2\pi r(y)h(y)dy \\
&= \int_0^{\sqrt{pi/4}} 2\pi y\left(1 - \tan\left(y^2\right)\right) dy.
\end{aligned}
$$

We can calculate this integral with the substituion $u = y^2$, which gives $du = 2ydy$. The new bounds are $u = 0$ to $u = \frac{\pi}{4}$. We get

$$
\begin{aligned}
V &= \int_0^{\pi/4} \pi\left(1 - \tan(u)\right) du \\
&= \pi\left[u - \ln|\sec(u)|\right]_0^{\pi/4} \\
&= \pi\left(\frac{\pi}{4} - \ln\left(\sec\left(\frac{\pi}{4}\right)\right) + \ln\left(\sec(0)\right)\right) \\
&= \pi\left(\frac{\pi}{4} - \ln\left(\sqrt{2}\right) + \ln(1)\right) \\
&= \boxed{\pi\left(\frac{\pi}{4} - \frac{\ln(2)}{2}\right) \text{ cubic units}}.
\end{aligned}
$$

(b) The region bounded by the y-axis, the graph of $y = \sin(x)$ and the line $y = 1$ revolved about the y-axis.

Solution: (i) The region can be described as the region between the y-axis and $x = \sin^{-1}(y)$ for $0 \leqslant y \leqslant 1$. Revolving the horizontal strip at y in the region will form a disk of radius $r(y) = \sin^{-1}(y)$. So the volume is

$$
V = \int_0^1 \pi r(y)^2 dy
$$

$$= \int_0^1 \pi \left[\sin^{-1}(y)\right]^2 dy.$$

We can calculate this integral with two successive IBPs. The first one uses the parts

$$u = \left[\sin^{-1}(y)\right]^2 \;\Rightarrow\; du = \frac{2\sin^{-1} dy}{\sqrt{1-y^2}},$$
$$dv = dy \;\Rightarrow\; v = y.$$

This gives

$$V = \pi\left(\left[y\left[\sin^{-1}(y)\right]^2\right]_0^1 - \int_0^1 \frac{2y\sin^{-1}(y)}{\sqrt{1-y^2}}dy\right)$$
$$= \pi\left(\frac{\pi^2}{4} - 2\int_0^1 \frac{y\sin^{-1}(y)}{\sqrt{1-y^2}}dy\right).$$

In this last integral, we use an IBP with parts

$$u = \sin^{-1}(y) \;\Rightarrow\; du = \frac{dy}{\sqrt{1-y^2}},$$
$$dv = \frac{ydy}{\sqrt{1-y^2}} \;\Rightarrow\; v = -\sqrt{1-y^2}.$$

We get

$$V = \pi\left(\frac{\pi^2}{4} - 2\left(\left[-\sin^{-1}(y)\sqrt{1-y^2}\right]_0^1 - \int_0^1 \frac{-\sqrt{1-y^2}}{\sqrt{1-y^2}}dy\right)\right)$$
$$= \pi\left(\frac{\pi^2}{4} - 2\left(0 + \int_0^1 dy\right)\right)$$
$$= \boxed{\pi\left(\frac{\pi^2}{4} - 2\right) \text{ cubic units}}.$$

(ii) Revolving the vertical strip at x in the region about the y-axis creates a cylindrical shell of radius $r(x) = x$ and height $h(x) = 1 - \sin(x)$. Therefore the volume is

$$V = \int_0^{\pi/2} 2\pi r(x)h(x)dx$$
$$= 2\pi\int_0^{\pi/2} x\left(1-\sin(x)\right)dx.$$

We can compute this integral with an IBP, taking the parts

$$u = x \;\Rightarrow\; du = dx,$$

$$dv = 1 - \sin(x) \;\Rightarrow\; v = x + \cos(x).$$

We get

$$\begin{aligned}
V &= 2\pi\left([x(x+\cos(x))]_0^{\pi/2} - \int_0^{\pi/2}(x+\cos(x))dx\right)\\
&= 2\pi\left(\frac{\pi}{2}\left(\frac{\pi}{2}+0\right) - 0 - \left[\frac{x^2}{2}+\sin(x)\right]_0^{\pi/2}\right)\\
&= 2\pi\left(\frac{\pi^2}{4} - \frac{\pi^2}{8} - 1\right)\\
&= \boxed{\pi\left(\frac{\pi^2}{4} - 2\right) \text{ cubic units}}.
\end{aligned}$$

(c) The region between the graph of $y = \ln(x)$ and the x-axis for $1 \leqslant x \leqslant e$ revolved about the line $x = -2$.

Solution: (i) The region can be described as the region between $x = e^y$ and $x = e$ for $0 \leqslant y \leqslant 1$. Revolving the horizontal strip at y in the region about the line $x = -2$ forms a washer with inner radius $r_{\text{in}}(y) = e^y - (-2) = e^y + 2$ and outer radius $r_{\text{out}}(y) = e - (-2) = e + 2$. So the volume is

$$\begin{aligned}
V &= \int_0^1 \pi\left(r_{\text{out}}(y)^2 - r_{\text{in}}(y)^2\right)dy\\
&= \int_0^1 \pi\left((e+2)^2 - (e^y+2)^2\right)dy\\
&= \pi\int_0^1 \left((e+2)^2 - 4 - e^{2y} - 4e^y\right)dy\\
&= \pi\left[((e+2)^2-4)y - \frac{e^{2y}}{2} - 4e^y\right]_0^1\\
&= \pi\left((e+2)^2 - 4 - \frac{e^2}{2} - 4e + \frac{1}{2} + 4\right)\\
&= \boxed{\frac{\pi(e^2+9)}{9} \text{ cubic units}}.
\end{aligned}$$

(ii) Revolving the vertical strip at x about the line $x = -2$ forms a cylindrical shell with radius $r(x) = x - (-2) = x + 2$ and height $h(x) = \ln(x)$. So the volume is

$$V = \int_1^e 2\pi r(x)h(x)dx$$

$$= 2\pi \int_1^e (x+2)\ln(x)dx.$$

We compute this integral with an IBP, taking the parts

$$u = \ln(x) \;\Rightarrow\; du = \frac{dx}{x},$$
$$dv = (x+2)dx \;\Rightarrow\; v = \frac{x^2}{2} + 2x.$$

This gives

$$\begin{aligned} V &= 2\pi \left(\left[\left(\frac{x^2}{2} + 2x \right) \ln(x) \right]_1^e - \int_1^e \left(\frac{x^2}{2} + 2x \right) \frac{1}{x} dx \right) \\ &= 2\pi \left(\frac{e^2}{2} + 2e - \int_1^e \left(\frac{x}{2} + 2 \right) dx \right) \\ &= 2\pi \left(\frac{e^2}{2} + 2e - \left[\frac{x^2}{4} + 2x \right]_1^e \right) \\ &= 2\pi \left(\frac{e^2}{2} + 2e - \frac{e^2}{4} - 2e + \frac{1}{4} + 2 \right) \\ &= \boxed{\frac{\pi(e^2+9)}{9} \text{ cubic units}}. \end{aligned}$$

#35. Express $\int \sin^7(2x)dx$ in terms of $\int \sin^5(2x)dx$.

Solution: We split off a factor $\sin(2x)$ and use IBP with parts

$$u = \sin^6(2x) \;\Rightarrow\; du = 12\sin^5(2x)\cos(2x),$$
$$dv = \sin(2x)dx \;\Rightarrow\; v = -\frac{1}{2}\cos(2x).$$

This gives

$$\begin{aligned} \int \sin^7(2x)dx &= \int \sin^6(2x)\sin(2x)dx \\ &= -\frac{\sin^6(2x)\cos(2x)}{2} - \int 12\sin^5(2x)\cos(2x)\left(-\frac{\cos(2x)}{2}\right)dx \\ &= -\frac{\sin^6(2x)\cos(2x)}{2} + 6\int \sin^5(2x)\cos^2(2x)dx. \end{aligned}$$

In this last integral, we use the Pythagorean identity $\cos^2(2x) = 1 - \sin^2(2x)$ to obtain

$$\begin{aligned}\int \sin^7(2x)dx &= -\frac{\sin^6(2x)\cos(2x)}{2} + 6\int \sin^5(2x)\left(1 - \sin^2(2x)\right)dx \\ &= -\frac{\sin^6(2x)\cos(2x)}{2} + 6\int \sin^5(2x)dx - 6\int \sin^7(2x)dx.\end{aligned}$$

We can now solve for the original integral by moving the term $-6\int \sin^7(2x)dx$ to the left-hand side. We get

$$\begin{aligned}&\int \sin^7(2x)dx + 6\int \sin^7(2x)dx = -\frac{\sin^6(2x)\cos(2x)}{2} + 6\int \sin^5(2x)dx \\ \Rightarrow\ &7\int \sin^7(2x)dx = -\frac{\sin^6(2x)\cos(2x)}{2} + 6\int \sin^5(2x)dx \\ \Rightarrow\ &\boxed{\int \sin^7(2x)dx = -\frac{\sin^6(2x)\cos(2x)}{14} + \frac{6}{7}\int \sin^5(2x)dx}.\end{aligned}$$

#36. Express $\int \sec^9(4x)dx$ in terms of $\int \sec^7(4x)dx$.

Solution: We split off a factor $\sec^2(4x)$ and use IBP with parts

$$\begin{aligned}&u = \sec^7(4x) \ \Rightarrow\ du = 7\sec^6(4x)\sec(4x)\tan(4x)(4)dx = 28\sec^7(4x)\tan(4x)dx, \\ &dv = \sec^2(4x)dx \ \Rightarrow\ v = \frac{\tan(4x)}{4}.\end{aligned}$$

We get

$$\begin{aligned}\int \sec^9(4x)dx &= \int \sec^7(4x)\sec^2(4x)dx \\ &= \frac{\sec^7(4x)\tan(4x)}{4} - \int 28\sec^7(4x)\tan(4x)\frac{\tan(4x)}{4}dx \\ &= \frac{\sec^7(4x)\tan(4x)}{4} - 7\int \sec^7(4x)\tan^2(4x)dx.\end{aligned}$$

In this last integral, we use the Pyhtagorean identity $\tan^2(4x) = \sec^2(4x) - 1$. This gives

$$\begin{aligned}\int \sec^9(4x)dx &= \frac{\sec^7(4x)\tan(4x)}{4} - 7\int \sec^7(4x)\left(\sec^2(4x) - 1\right)dx \\ &= \frac{\sec^7(4x)\tan(4x)}{4} - 7\int \sec^9(4x)dx + 7\int \sec^7(4x)dx.\end{aligned}$$

We can now solve for the original integral by moving the term $-7\int \sec^9(4x)dx$ to the left-hand side. We get

$$\begin{aligned}
\int \sec^9(4x)dx + 7\int \sec^9(4x)dx &= \frac{\sec^7(4x)\tan(4x)}{4} + 7\int \sec^7(4x)dx \\
\Rightarrow 8\int \sec^9(4x)dx &= \frac{\sec^7(4x)\tan(4x)}{4} + 7\int \sec^7(4x)dx \\
\Rightarrow &\boxed{\int \sec^9(4x)dx = \frac{\sec^7(4x)\tan(4x)}{32} + \frac{7}{8}\int \sec^7(4x)dx}.
\end{aligned}$$

#37. Find reduction formulas for the following integrals.

(a) $\int \cos^n(3x)dx$

Solution: We split off a factor $\cos(3x)$ and use IBP with parts

$$\begin{aligned}
u &= \cos^{n-1}(3x) \ \Rightarrow \ du = -3(n-1)\cos^{n-2}(3x)\sin(3x)dx, \\
dv &= \cos(3x)dx \ \Rightarrow \ v = \frac{\sin(3x)}{3}.
\end{aligned}$$

This gives

$$\begin{aligned}
\int \cos^n(3x)dx &= \int \cos^{n-1}(3x)\cos(3x)dx \\
\int \cos^n(3x)dx &= \frac{\cos^{n-1}(3x)\sin(3x)}{3} - \int -3(n-1)\cos^{n-2}(3x)\sin(3x)\frac{\sin(3x)}{3}dx \\
\int \cos^n(3x)dx &= \frac{\cos^{n-1}(3x)\sin(3x)}{3} + (n-1)\int \cos^{n-2}(3x)\sin^2(3x)dx.
\end{aligned}$$

In this last integral, we use the Phytagorean identity $\sin^2(3x) = 1 - \cos^2(3x)$ to obtain

$$\begin{aligned}
\int \cos^n(3x)dx &= \frac{\cos^{n-1}(3x)\sin(3x)}{3} + (n-1)\int \cos^{n-2}(3x)\left(1 - \cos(3x)^2\right)dx \\
\int \cos^n(3x)dx &= \frac{\cos^{n-1}(3x)\sin(3x)}{3} + (n-1)\int \cos^{n-2}(3x)dx - (n-1)\int \cos^n(3x)dx
\end{aligned}$$

We can now solve for the original integral by moving the term $-(n-1)\int \cos^n(3x)dx$ to the left-hand side.

$$\int \cos^n(3x)dx + (n-1)\int \cos^n(3x)dx = \frac{\cos^{n-1}(3x)\sin(3x)}{3} + (n-1)\int \cos^{n-2}(3x)dx$$

$$n\int \cos^n(3x)dx = \frac{\cos^{n-1}(3x)\sin(3x)}{3} + (n-1)\int \cos^{n-2}(3x)dx$$

We now divide by n to obtain the reduction formula

$$\boxed{\int \cos^n(3x)dx = \frac{\cos^{n-1}(3x)\sin(3x)}{3n} + \frac{(n-1)}{n}\int \cos^{n-2}(3x)dx}.$$

(b) $\int (\ln(x))^n \, dx$

Solution: We use IBP with parts

$$u = (\ln(x))^n \ \Rightarrow\ du = \frac{n(\ln(x))^{n-1}\,dx}{x},$$
$$dv = dx \ \Rightarrow\ v = x.$$

This gives

$$\int (\ln(x))^n \, dx = x(\ln(x))^n - \int x\frac{n(\ln(x))^{n-1}}{x}dx$$

$$\boxed{\int (\ln(x))^n \, dx = x(\ln(x))^n - n\int (\ln(x))^{n-1}\, dx}$$

(c) $\int \sec^n(5x)dx$

Solution: We separate a factor $\sec^2(5x)$ and use IBP with parts

$$u = \sec^{n-2}(5x) \ \Rightarrow\ du = 5(n-2)\sec^{n-2}(5x)\tan(5x)dx,$$
$$dv = \sec^2(5x)dx \ \Rightarrow\ v = \frac{\tan(5x)}{5}.$$

We get

$$\int \sec^n(5x)dx = \int \sec^{n-2}(5x)\sec^2(5x)dx$$
$$\int \sec^n(5x)dx = \frac{\sec^{n-2}(5x)\tan(5x)}{5} - \int 5(n-2)\sec^{n-2}(5x)\tan(5x)\frac{\tan(5x)}{5}dx$$
$$\int \sec^n(5x)dx = \frac{\sec^{n-2}(5x)\tan(5x)}{5} - (n-2)\int \sec^{n-2}(5x)\tan^2(5x)dx$$

In this last integral, we use the Pythagorean identity $\tan^2(5x) = \sec^2(5x) - 1$, which gives

$$\int \sec^n(5x)dx = \frac{\sec^{n-2}(5x)\tan(5x)}{5} - (n-2)\int \sec^{n-2}(5x)\left(\sec^2(5x) - 1\right) dx$$
$$\int \sec^n(5x)dx = \frac{\sec^{n-2}(5x)\tan(5x)}{5} - (n-2)\int \sec^n(5x)dx + (n-2)\int \sec^{n-2}(5x)dx$$

We can now solve for the original integral by moving the term $-(n-2)\int \sec^n(5x)dx$ to the left-hand side.

$$\int \sec^n(5x)dx + (n-2)\int \sec^n(5x)dx = \frac{\sec^{n-2}(5x)\tan(5x)}{5} + (n-2)\int \sec^{n-2}(5x)dx$$
$$(n-1)\int \sec^n(5x)dx = \frac{\sec^{n-2}(5x)\tan(5x)}{5} + (n-2)\int \sec^{n-2}(5x)dx$$

Dividing by $n-1$ gives the reduction formula

$$\boxed{\int \sec^n(5x)dx = \frac{\sec^{n-2}(5x)\tan(5x)}{5(n-1)} + \frac{n-2}{n-1}\int \sec^{n-2}(5x)dx}.$$