

Section 8.4: Trigonometric Substitution - Worksheet Solutions

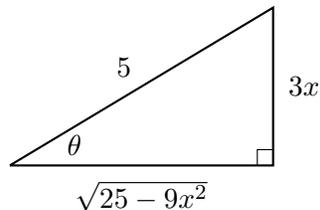
#43. Calculate the following integrals. **Note:** some of these problems use integration techniques from earlier sections.

(a) $\int \sqrt{25 - 9x^2} dx$

Solution: If you have memorized the rules for trigonometric substitution, you know that the form $a^2 - b^2x^2$ calls for the substitution $x = \frac{a}{b} \sin(\theta)$, so we will substitute $x = \frac{5}{3} \sin(\theta)$ here. In case you have not memorized this, you can recover it from simply remembering that we are trying to use the Pythagorean identity to replace a difference of squares with a perfect square. We want

$$25 - 9x^2 = 25 \left(1 - \underbrace{\left(\frac{3x}{5} \right)^2}_{\sin^2(\theta)} \right) = 25(1 - \sin^2(\theta)) = 25 \cos^2(\theta),$$

so we will want $\sin(\theta) = \frac{3x}{5}$. This gives $x = \frac{5}{3} \sin(\theta)$ and $dx = \frac{5}{3} \cos(\theta) d\theta$. The right triangle associated to this trigonometric substitution is shown below.



We get

$$\begin{aligned} \int \sqrt{25 - 9x^2} dx &= \int \sqrt{25 \cos^2(\theta)} \cdot \frac{5}{3} \cos(\theta) d\theta \\ &= \frac{25}{3} \int \cos^2(\theta) d\theta \\ &= \frac{25}{3} \int \frac{1 + \cos(2\theta)}{2} d\theta \\ &= \frac{25}{6} \left(\theta + \frac{\sin(2\theta)}{2} \right) + C \\ &= \frac{25}{6} (\theta + \sin(\theta) \cos(\theta)) + C. \end{aligned}$$

To express this result in terms of x , we use the right triangle above from which we see that $\theta = \sin^{-1} \left(\frac{3x}{5} \right)$, $\sin(\theta) = \frac{3x}{5}$ and $\cos(\theta) = \frac{\sqrt{25-9x^2}}{5}$. We obtain

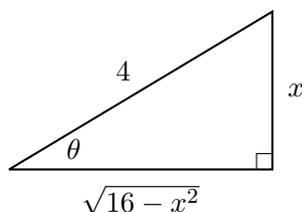
$$\int \sqrt{25 - 9x^2} dx = \boxed{\frac{25}{6} \left(\sin^{-1} \left(\frac{3x}{5} \right) + \frac{3x\sqrt{25 - 9x^2}}{25} \right) + C.}$$

$$(b) \int \frac{1}{(16 - x^2)^{3/2}} dx$$

Solution: If you have memorized the rules for trigonometric substitution, you know that the form $a^2 - b^2x^2$ calls for the substitution $x = \frac{a}{b} \sin(\theta)$, so we will substitute $x = 4 \sin(\theta)$ here. In case you have not memorized this, you can recover it from simply remembering that we are trying to use the Pythagorean identity to replace a difference of squares with a perfect square. We want

$$16 - x^2 = 16 \left(1 - \underbrace{\left(\frac{x}{4} \right)^2}_{\sin^2(\theta)} \right) = 16(1 - \sin^2(\theta)) = 16 \cos^2(\theta),$$

so we will want $\sin(\theta) = \frac{x}{4}$. This gives $x = 4 \sin(\theta)$ and $dx = 4 \cos(\theta) d\theta$. The right triangle associated to this trigonometric substitution is shown below.



We get

$$\begin{aligned} \int \frac{1}{(16 - x^2)^{3/2}} dx &= \int \frac{1}{(16 \cos^2(\theta))^{3/2}} 4 \cos(\theta) d\theta \\ &= \int \frac{4 \cos(\theta)}{64 \cos^3(\theta)} d\theta \\ &= \frac{1}{16} \int \frac{1}{\cos^2(\theta)} d\theta \\ &= \frac{1}{16} \int \sec^2(\theta) d\theta \\ &= \frac{1}{16} \tan(\theta) + C. \end{aligned}$$

To express this result in terms of x , we use the right triangle above from which we see that $\tan(\theta) = \frac{x}{\sqrt{16 - x^2}}$. We obtain

$$\int \frac{1}{(16 - x^2)^{3/2}} dx = \boxed{\frac{x}{16\sqrt{16 - x^2}} + C}.$$

$$(c) \int_0^2 \frac{1}{\sqrt{x^2 + 4}} dx$$

Solution: If you have memorized the rules for trigonometric substitution, you know that the form $a^2 + b^2x^2$ calls for the substitution $x = \frac{a}{b} \tan(\theta)$, so we will substitute $x = 2 \tan(\theta)$ here. In case you have not memorized this, you can recover it from simply remembering that we are trying to use the Pythagorean identity to replace a difference of squares with a perfect square.

We want

$$x^2 + 4 = 4 \left(\underbrace{\left(\frac{x}{2} \right)^2}_{\tan^2(\theta)} + 1 \right) = 4(\tan^2(\theta) + 1) = 4 \sec^2(\theta),$$

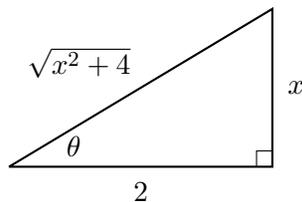
so we will want $\tan(\theta) = \frac{x}{2}$. Hence we make the substitution $x = 2 \tan(\theta)$. Then $dx = 2 \sec^2(\theta) d\theta$. The bounds become

$$\begin{aligned} x = 0 &\Rightarrow \tan(\theta) = \frac{0}{2} = 0 \Rightarrow \theta = 0, \\ x = 2 &\Rightarrow \tan(\theta) = \frac{2}{2} = 1 \Rightarrow \theta = \frac{\pi}{4}. \end{aligned}$$

So

$$\begin{aligned} \int_0^2 \frac{1}{\sqrt{x^2 + 4}} dx &= \int_0^{\pi/4} \frac{1}{\sqrt{4 + 4 \tan^2 \theta}} \cdot 2 \sec^2 \theta d\theta \\ &= \int_0^{\pi/4} \frac{1}{\sqrt{\sec^2 \theta}} \cdot \sec^2 \theta d\theta \\ &= \int_0^{\pi/4} \sec \theta d\theta \\ &= [\ln |\sec \theta + \tan \theta|]_0^{\pi/4} \\ &= \boxed{\ln(\sqrt{2} + 1)}. \end{aligned}$$

Remark: in this example, there is no need to use a right triangle to express the antiderivative in terms of x since we are computing a definite integral and we have converted the bounds in terms of θ . However, if this question had asked to compute the indefinite integral, we would have used the relation $\tan \theta = \frac{x}{2}$ and drawn the right angle triangle



to infer that $\sec \theta = \frac{\sqrt{x^2+4}}{2}$. Therefore

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 + 4}} dx &= \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{\sqrt{x^2 + 4}}{2} + \frac{x}{2} \right| + C \end{aligned}$$

Although the answer above is acceptable, observe that the last expression equals

$$\ln \left| (\sqrt{x^2 + 4} + x) \cdot \frac{1}{2} \right| + C = \ln \left| \sqrt{x^2 + 4} + x \right| + \ln \frac{1}{2} + C$$

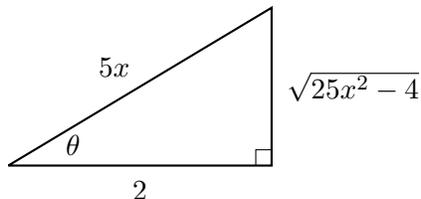
But since $\ln \frac{1}{2}$ is a constant, the expression “ $+\ln \frac{1}{2} + C$ ” represents the addition of any possible real number, just as “ $+C$ ” does. Therefore, we can simplify the expression as $\ln \left| \sqrt{x^2 + 4} + x \right| + C$.

$$(d) \int \frac{\sqrt{25x^2 - 4}}{x} dx, x > \frac{2}{5}$$

Solution: If you have memorized the rules for trigonometric substitution, you know that the form $b^2x^2 - a^2$ calls for the substitution $x = \frac{a}{b} \sec(\theta)$, so we will substitute $x = \frac{2}{5} \sec(\theta)$ here. In case you have not memorized this, you can actually recovering this from simply remembering that we are trying to use the Pythagorean identity to replace a difference of squares with a perfect square. We want

$$25x^2 - 4 = 4 \left(\left(\underbrace{\frac{5x}{2}}_{\sec(\theta)} \right)^2 - 1 \right) = 4 (\sec^2(\theta) - 1) = 4 \tan^2(\theta),$$

so we must substitute $x = \frac{2}{5} \sec(\theta)$ and $dx = \frac{2}{5} \sec(\theta) \tan(\theta) d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\sec(\theta) = \frac{5x}{2}$ as shown below.



We get $\sqrt{25x^2 - 4} = \sqrt{4 \tan^2(\theta)} = 2 \tan(\theta)$ and the integral becomes

$$\begin{aligned} \int \frac{\sqrt{25x^2 - 4}}{x} dx &= \int \frac{2 \tan(\theta)}{\frac{2}{5} \sec(\theta)} \cdot \frac{2}{5} \sec(\theta) \tan(\theta) d\theta \\ &= 2 \int \tan^2(\theta) d\theta \\ &= 2 \int (\sec^2(\theta) - 1) d\theta \\ &= 2 (\tan(\theta) - \theta) + C. \end{aligned}$$

We need to express this result in terms of x . Using the right triangle above, we see that $\tan(\theta) = \frac{\sqrt{25x^2 - 4}}{2}$ and $\theta = \sec^{-1} \left(\frac{5x}{2} \right)$. Thus

$$\int \frac{\sqrt{25x^2 - 4}}{x} dx = \sqrt{25x^2 - 4} - 2 \sec^{-1} \left(\frac{5x}{2} \right) + C.$$

$$(e) \int x\sqrt{81-x^2}dx$$

Solution: While it would give the correct answer, there is no need for trigonometric substitution here – a regular u -substitution will do. This is because we see the derivative of the inside function $81 - x^2$ appearing on the outside as a factor, up to a multiplicative constant. So we substitute $u = 81 - x^2$, $du = -2xdx$. We get

$$\begin{aligned}\int x\sqrt{81-x^2}dx &= \int -\frac{1}{2}\sqrt{u}du \\ &= -\frac{1}{3}u^{3/2} + C \\ &= \boxed{-\frac{1}{3}(81-x^2)^{3/2} + C}.\end{aligned}$$

$$(f) \int x^5 (\ln(x))^2 dx.$$

Solution: We use integration by parts twice. The first one will use

$$\begin{aligned}u &= (\ln(x))^2 \Rightarrow du = \frac{2\ln(x)dx}{x}, \\ dv &= x^5 dx, \Rightarrow v = \frac{x^6}{6}.\end{aligned}$$

This gives

$$\begin{aligned}\int x^5 (\ln(x))^2 dx &= \frac{x^6 (\ln(x))^2}{6} - \int \frac{2\ln(x)x^6}{6x} dx \\ &= \frac{x^6 (\ln(x))^2}{6} - \frac{1}{3} \int \ln(x)x^5 dx.\end{aligned}$$

For the second IBP, we take

$$\begin{aligned}u &= \ln(x) \Rightarrow du = \frac{dx}{x}, \\ dv &= x^5 dx, \Rightarrow v = \frac{x^6}{6}.\end{aligned}$$

This gives

$$\begin{aligned}\int x^5 (\ln(x))^2 dx &= \frac{x^6 (\ln(x))^2}{6} - \frac{1}{3} \left(\frac{\ln(x)x^6}{6} - \int \frac{x^6}{6x} dx \right) \\ &= \frac{x^6 (\ln(x))^2}{6} - \frac{\ln(x)x^6}{18} + \frac{1}{18} \int x^5 dx \\ &= \boxed{\frac{x^6 (\ln(x))^2}{6} - \frac{\ln(x)x^6}{18} + \frac{x^6}{108} + C}.\end{aligned}$$

$$(g) \int \frac{dt}{t\sqrt{9 + (\ln(t))^2}}$$

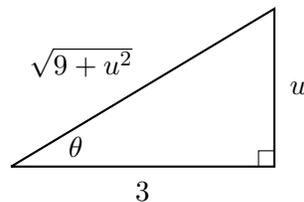
Solution: We start by using the substitution $u = \ln(t)$, which gives $du = \frac{dt}{t}$ and

$$\int \frac{dt}{t\sqrt{9 + (\ln(t))^2}} = \int \frac{du}{\sqrt{9 + u^2}}.$$

We compute this last integral using a trigonometric substitution. If you have memorized the rules for trigonometric substitution, you know that the form $a^2 + b^2x^2$ calls for the substitution $x = \frac{a}{b} \tan(\theta)$, so we substitute $x = 3 \tan(\theta)$ here. In case you do not have this memorized, observe that we want

$$9 + u^2 = 9 \left(1 + \underbrace{\left(\frac{u}{3} \right)^2}_{\tan^2(\theta)} \right) = 9 (1 + \tan^2(\theta)) = 9 \sec^2(\theta),$$

so we substitute $u = 3 \tan(\theta)$ and $du = 3 \sec^2(\theta) d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\tan(\theta) = \frac{u}{3}$ as shown below.



We get $\sqrt{9 + u^2} = \sqrt{9 \sec^2(\theta)} = 3 \sec(\theta)$ and the integral becomes

$$\begin{aligned} \int \frac{dt}{t\sqrt{9 + (\ln(t))^2}} &= \int \frac{du}{\sqrt{9 + u^2}} \\ &= \int \frac{3 \sec^2(\theta) d\theta}{3 \sec(\theta)} \\ &= \int \sec(\theta) d\theta \\ &= \ln |\tan(\theta) + \sec(\theta)| + C. \end{aligned}$$

We express this result in terms of u using the right triangle above, from which we see that

$$\tan(\theta) = \frac{u}{3}, \quad \sec(\theta) = \frac{\sqrt{9 + u^2}}{3}.$$

We get

$$\int \frac{dt}{t\sqrt{9 + (\ln(t))^2}} = \ln \left| \frac{u}{3} + \frac{\sqrt{9 + u^2}}{3} \right| + C$$

$$= \ln \left| u + \sqrt{9 + u^2} \right| + C.$$

We now finish by replacing u by $\ln(t)$ and we obtain

$$\int \frac{dt}{t\sqrt{9 + (\ln(t))^2}} = \ln \left| \ln(t) + \sqrt{9 + (\ln(t))^2} \right| + C.$$

$$(h) \int \frac{dx}{(6x - x^2 - 5)^{5/2}}$$

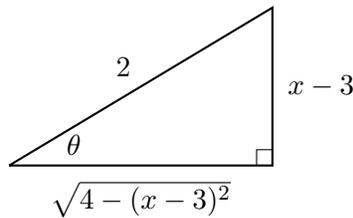
Solution: We start by completing the square in the denominator:

$$6x - x^2 - 5 = -(x^2 - 6x) - 5 = -(x^2 - 6x + 9) + 9 - 5 = 4 - (x - 3)^2.$$

We can now use a trigonometric substitution. We want

$$4 - (x - 3)^2 = 4 \left(1 - \left(\frac{x - 3}{2} \right)^2 \right) = 4(1 - \sin^2(\theta)) = 4 \cos^2(\theta),$$

so we substitute $x - 3 = 2 \sin(\theta)$ or $x = 3 + 2 \sin(\theta)$. This gives $dx = 2 \cos(\theta)d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\sin(\theta) = \frac{x-3}{2}$ as shown below.



We get $(4 - (x - 3)^2)^{5/2} = (4 \cos^2(\theta))^{5/2} = 32 \cos^5(\theta)$. The integral becomes

$$\begin{aligned} \int \frac{dx}{(6x - x^2 - 5)^{5/2}} &= \int \frac{dx}{(4 - (x - 3)^2)^{5/2}} \\ &= \int \frac{2 \cos(\theta) d\theta}{32 \cos^5(\theta)} \\ &= \frac{1}{16} \int \frac{d\theta}{\cos^4(\theta)} \\ &= \frac{1}{16} \int \sec^4(\theta) d\theta. \end{aligned}$$

Since the exponent of \sec is even, we can split off a factor $\sec^2(\theta)$, rewrite the remaining factors using the Pythagorean identity $\sec^2(\theta) = \tan^2(\theta) + 1$ and then use the substitution $u = \tan(\theta)$,

$du = \sec^2(\theta)d\theta$. This gives

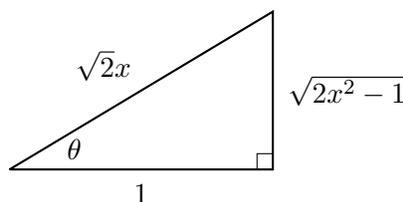
$$\begin{aligned} \int \frac{dx}{(6x - x^2 - 5)^{5/2}} &= \frac{1}{16} \int \sec^2(\theta) \sec^2(\theta) d\theta \\ &= \frac{1}{16} \int (\tan^2(\theta) + 1) \sec^2(\theta) d\theta \\ &= \frac{1}{16} \int (u^2 + 1) du \\ &= \frac{1}{16} \left(\frac{u^3}{3} + u \right) + C \\ &= \frac{1}{16} \left(\frac{\tan^3(\theta)}{3} + \tan(\theta) \right) + C \\ &= \frac{\tan(\theta)}{16} \left(\frac{\tan^2(\theta)}{3} + 1 \right) + C. \end{aligned}$$

To express this antiderivative in terms of x , we use the right triangle above, from which we see that $\tan(\theta) = \frac{x-3}{\sqrt{4-(x-3)^2}}$. So we get

$$\int \frac{dx}{(6x - x^2 - 5)^{3/2}} = \frac{x-3}{16\sqrt{4-(x-3)^2}} \left(\frac{(x-3)^2}{3(4-(x-3)^2)} + 1 \right) + C.$$

(i) $\int_1^{\sqrt{2}} \frac{dx}{x(2x^2 - 1)^{3/2}}$

Solution: We want $2x^2 - 1 = \sec^2(\theta) - 1 = \tan^2(\theta)$, so we substitute $x = \frac{\sec(\theta)}{\sqrt{2}}$ and $dx = \frac{\sec(\theta)\tan(\theta)}{\sqrt{2}}d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\sec(\theta) = \sqrt{2}x$ as shown below.



We get $(2x^2 - 1)^{3/2} = (\tan^2(\theta))^{3/2} = \tan^3(\theta)$. The bounds change as follows:

$$\begin{aligned} x = 1 &\Rightarrow \sec(\theta) = \sqrt{2} \cdot 1 = \sqrt{2} \Rightarrow \theta = \sec^{-1}(\sqrt{2}) = \frac{\pi}{4}, \\ x = \sqrt{2} &\Rightarrow \sec(\theta) = \sqrt{2} \cdot \sqrt{2} = 2 \Rightarrow \theta = \sec^{-1}(2) = \frac{\pi}{3}. \end{aligned}$$

The integral becomes

$$\begin{aligned}
 \int_1^{\sqrt{2}} \frac{dx}{x(2x^2 - 1)^{3/2}} &= \int_{\pi/4}^{\pi/3} \frac{\frac{\sec(\theta)\tan(\theta)}{\sqrt{2}} d\theta}{\frac{\sec(\theta)}{\sqrt{2}} \tan^3(\theta)} \\
 &= \int_{\pi/4}^{\pi/3} \frac{d\theta}{\tan^2(\theta)} \\
 &= \int_{\pi/4}^{\pi/3} \cot^2(\theta) d\theta \\
 &= \int_{\pi/4}^{\pi/3} (\csc^2(\theta) - 1) d\theta \\
 &= [-\cot(\theta) - \theta]_{\pi/4}^{\pi/3} \\
 &= -\cot\left(\frac{\pi}{3}\right) - \frac{\pi}{3} + \cot\left(\frac{\pi}{4}\right) + \frac{\pi}{4} \\
 &= \boxed{1 - \frac{1}{\sqrt{3}} - \frac{\pi}{12}}.
 \end{aligned}$$

(j) $\int_0^{5/2} \sqrt{25 - 4x^2} dx$

Solution: We use the trigonometric substitution $x = \frac{5}{2} \sin(\theta)$. This gives $dx = \frac{5}{2} \cos(\theta) d\theta$ and $\sqrt{25 - 4x^2} = \sqrt{25 - 25 \sin^2(\theta)} = \sqrt{25 \cos^2(\theta)} = 5 \cos(\theta)$. The bounds change as follows:

$$\begin{aligned}
 x = 0 &\Rightarrow \theta = \sin^{-1}(0) = 0, \\
 x = \frac{5}{2} &\Rightarrow \theta = \sin^{-1}(1) = \frac{\pi}{2}.
 \end{aligned}$$

The integral becomes

$$\begin{aligned}
 \int_0^{5/2} \sqrt{25 - 4x^2} dx &= \int_0^{\pi/2} (5 \cos(\theta)) \frac{5}{2} \cos(\theta) d\theta \\
 &= \frac{25}{2} \int_0^{\pi/2} \cos^2(\theta) d\theta \\
 &= \frac{25}{2} \left[\frac{1 + \cos(2\theta)}{2} \right]_0^{\pi/2} \\
 &= \frac{25}{4} \left[\theta + \frac{\sin(2\theta)}{2} \right]_0^{\pi/2} \\
 &= \boxed{\frac{25\pi}{8}}.
 \end{aligned}$$

$$(k) \int \frac{e^{6x}}{\sqrt{16 - e^{4x}}} dx$$

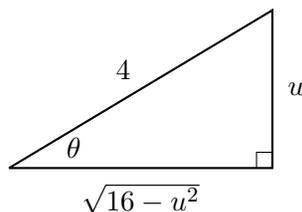
Solution: We start with the substitution $u = e^{2x}$, so that $du = 2e^{2x} dx$. The extraneous factor e^{4x} in the numerator can be expressed as $e^{4x} = (e^{2x})^2 = u^2$. So the integral becomes

$$\int \frac{e^{6x}}{\sqrt{16 - e^{4x}}} dx = \int \frac{e^{4x}}{\sqrt{16 - e^{4x}}} e^{2x} dx = \int \frac{u^2}{2\sqrt{16 - u^2}} du.$$

We can now use a trigonometric substitution. We want

$$16 - u^2 = 16 \left(1 - \left(\frac{u}{4} \right)^2 \right) = 16(1 - \sin^2(\theta)) = 16 \cos^2(\theta),$$

so we substitute $u = 4 \sin(\theta)$ and $du = 4 \cos(\theta) d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\sin(\theta) = \frac{u}{4}$ as shown below.



We get $\sqrt{16 - u^2} = \sqrt{16 - 16 \sin^2(\theta)} = \sqrt{16 \cos^2(\theta)} = 4 \cos(\theta)$. The integral becomes

$$\begin{aligned} \int \frac{e^{6x}}{\sqrt{16 - e^{4x}}} dx &= \int \frac{u^2}{2\sqrt{16 - u^2}} du \\ &= \int \frac{(4 \sin(\theta))^2}{2(4 \cos(\theta))} 4 \cos(\theta) d\theta \\ &= 8 \int \sin^2(\theta) d\theta. \end{aligned}$$

We can compute this integral using the double angle formulas $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$. We get

$$\begin{aligned} \int \frac{e^{6x}}{\sqrt{16 - e^{4x}}} dx &= 8 \int \frac{1 - \cos(2\theta)}{2} 2 d\theta \\ &= 4 \left(\theta - \frac{\sin(2\theta)}{2} \right) + C \\ &= 4(\theta - \cos(\theta) \sin(\theta)) + C \end{aligned}$$

where we have used the trigonometric identity $\sin(2\theta) = 2 \cos(\theta) \sin(\theta)$ in the last step. We can use the right triangle above to express this result in terms of u , observing that

$$\theta = \sin^{-1} \left(\frac{u}{4} \right), \quad \cos(\theta) = \frac{\sqrt{16 - u^2}}{4}, \quad \sin(\theta) = \frac{u}{4}.$$

We can then replace $u = e^{2x}$ and we get

$$\begin{aligned} \int \frac{e^{6x}}{\sqrt{16 - e^{4x}}} dx &= 4 \left(\sin^{-1} \left(\frac{u}{4} \right) - \frac{\sqrt{16 - u^2} u}{4} \right) + C \\ &= \boxed{\frac{1}{4} \sin^{-1} \left(\frac{e^{2x}}{4} \right) - \frac{e^{2x} \sqrt{16 - e^{4x}}}{4} + C}. \end{aligned}$$

$$(1) \int_0^{\pi/18} \cos^5(3x) dx$$

Solution: Because the exponent of \cos is odd, we can split off a factor $\cos(3x)$, rewrite the remaining powers in terms of $\sin(3x)$ and use the substitution $u = \sin(3x)$, which gives $du = 3 \cos(3x) dx$. The bounds will become

$$\begin{aligned} x = 0 &\Rightarrow u = \sin(0) = 0, \\ x = \frac{\pi}{18} &\Rightarrow u = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}. \end{aligned}$$

We obtain

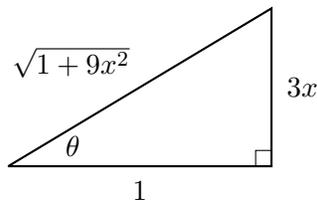
$$\begin{aligned} \int_0^{\pi/18} \cos^5(3x) dx &= \int_0^{\pi/18} \cos^4(3x) \cos(3x) dx \\ &= \int_0^{\pi/18} (1 - \sin^2(3x))^2 \cos(3x) dx \\ &= \int_0^{1/2} (1 - u^2)^2 \frac{1}{3} du \\ &= \frac{1}{3} \int_0^{1/2} (1 - 2u^2 + u^4) du \\ &= \frac{1}{3} \left[u - \frac{2u^3}{3} + \frac{u^5}{5} \right]_0^{1/2} \\ &= \boxed{\frac{203}{1440}}. \end{aligned}$$

$$(m) \int \frac{x^2}{(1 + 9x^2)^{5/2}} dx$$

Solution: We use a trigonometric substitution. We will want

$$1 + 9x^2 = 1 + (3x)^2 = 1 + \tan^2(\theta) = \sec^2(\theta),$$

so we substitute $x = \frac{1}{3} \tan(\theta)$ and $dx = \frac{1}{3} \sec(\theta)^2 d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\tan(\theta) = 3x$ as shown below.



We get $(1 + 9x^2)^{5/2} = (\sec^2(\theta))^{5/2} = \sec^5(\theta)$ and the integral becomes

$$\begin{aligned}
 \int \frac{x^2}{(1 + 9x^2)^{5/2}} dx &= \int \frac{\left(\frac{1}{3} \tan(\theta)\right)^2 \frac{1}{3} \sec^2(\theta) d\theta}{\sec^5(\theta)} \\
 &= \frac{1}{27} \int \frac{\tan^2(\theta)}{\sec^3(\theta)} d\theta \\
 &= \frac{1}{27} \int \frac{\frac{\cos^2(\theta)}{\sin^2(\theta)}}{\frac{1}{\cos^3(\theta)}} d\theta \\
 &= \frac{1}{27} \int \sin^2(\theta) \cos(\theta) d\theta \\
 &= \frac{1}{81} \sin^3(\theta) + C \\
 &= \frac{1}{81} \left(\frac{3x}{\sqrt{1 + 9x^2}} \right)^3 + C \\
 &= \boxed{\frac{x^3}{3(1 + 9x^2)^{3/2}} + C}.
 \end{aligned}$$

(n) $\int_5^{11} \frac{dx}{(x^2 - 10x + 61)^{5/2}}$

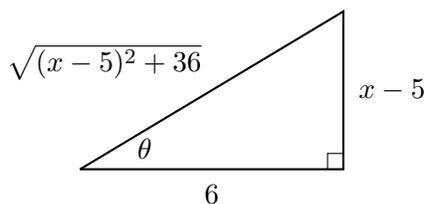
Solution: We start by completing the square in the denominator:

$$x^2 - 10x + 61 = (x^2 - 10x + 25) - 25 + 61 = (x - 5)^2 + 36.$$

We can now use a trigonometric substitution. We want

$$(x - 5)^2 + 36 = 36 \left(\left(\frac{x - 5}{6} \right)^2 + 1 \right) = 36(\tan^2(\theta) + 1) = 36 \sec^2(\theta),$$

so we substitute $x - 5 = 6 \tan(\theta)$, or $x = 5 + 6 \tan(\theta)$. This gives $dx = 6 \sec^2(\theta) d\theta$ and the following right triangle with base angle θ such that $\tan(\theta) = \frac{x-5}{6}$.



Then $((x-5)^2 + 36)^{5/2} = (36 \sec^2(\theta))^{5/2} = 6^5 \sec^5(\theta)$. The bounds change as follows:

$$x = 5 \Rightarrow \tan(\theta) = \frac{5-5}{6} = 0 \Rightarrow \theta = \tan^{-1}(0) = 0,$$

$$x = 11 \Rightarrow \tan(\theta) = \frac{11-5}{6} = 1 \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}.$$

The integral becomes

$$\begin{aligned} \int_5^{11} \frac{dx}{(x^2 - 10x + 61)^{5/2}} &= \int_5^{11} \frac{dx}{((x-5)^2 + 36)^{5/2}} \\ &= \int_0^{\pi/4} \frac{6 \sec^2(\theta) d\theta}{6^5 \sec^5(\theta)} \\ &= \frac{1}{1296} \int_0^{\pi/4} \frac{d\theta}{\sec^3(\theta)} \\ &= \frac{1}{1296} \int_0^{\pi/4} \cos^3(\theta) d\theta. \end{aligned}$$

Since the exponent of \cos is odd, we can compute this integral by splitting off a factor $\cos(\theta)$, rewriting the remaining factors with the trigonometric identity $\cos^2(\theta) = 1 - \sin^2(\theta)$ and using the substitution $u = \sin(\theta)$, $du = \cos(\theta)d\theta$. The bounds will change as follows

$$\theta = 0 \Rightarrow u = \sin(0) = 0,$$

$$\theta = \frac{\pi}{4} \Rightarrow u = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

The integral becomes

$$\begin{aligned} \int_5^{11} \frac{dx}{(x^2 - 10x + 61)^{5/2}} &= \frac{1}{1296} \int_0^{\pi/4} \cos^2(\theta) \cos(\theta) d\theta \\ &= \frac{1}{1296} \int_0^{\pi/4} (1 - \sin^2(\theta)) \cos(\theta) d\theta \\ &= \frac{1}{1296} \int_0^{\sqrt{2}/2} (1 - u^2) du \\ &= \frac{1}{1296} \left[u - \frac{u^3}{3} \right]_0^{\sqrt{2}/2} \\ &= \frac{1}{1296} \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}^3}{2^3 \cdot 3} \right) \end{aligned}$$

$$= \frac{5\sqrt{2}}{15552}.$$

(o) $\int \sin^4(5x) dx$

Solution: We use double angle formulas twice. We have

$$\begin{aligned} \sin^4(5x) &= \left(\frac{1 - \cos(10x)}{2} \right)^2 \\ &= \frac{1}{4} (1 - 2\cos(10x) + \cos^2(10x)) \\ &= \frac{1}{4} \left(1 - 2\cos(10x) + \frac{1 + \cos(20x)}{2} \right) \\ &= \frac{1}{8} (3 - 4\cos(10x) + \cos(20x)). \end{aligned}$$

So

$$\begin{aligned} \int \sin^4(5x) dx &= \frac{1}{8} \int (3 - 4\cos(10x) + \cos(20x)) dx \\ &= \frac{1}{8} \left(3x - \frac{2\sin(10x)}{5} + \frac{\sin(20x)}{20} \right) + C. \end{aligned}$$

(p) $\int \frac{dx}{\sqrt{x^2 - 6x}}, x > 6$

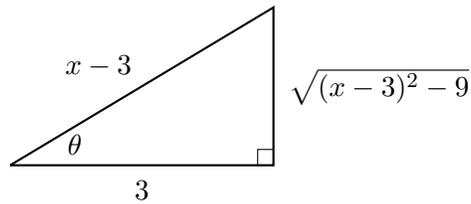
Solution: We start by completing the square to see a difference of squares appear in the square root:

$$x^2 - 6x = (x^2 - 6x + 9) - 9 = (x - 3)^2 - 9.$$

We will therefore use a trigonometric substitution. We want

$$(x - 3)^2 - 9 = 9 \left(\left(\frac{x - 3}{3} \right)^2 - 1 \right) = 9(\sec^2(\theta) - 1) = 9 \tan^2(\theta),$$

so we substitute $x - 3 = 3\sec(\theta)$ and $dx = 3\sec(\theta)\tan(\theta)d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\sec(\theta) = \frac{x-3}{3}$ as shown below.



We get $\sqrt{(x-3)^2 - 9} = \sqrt{9 \tan^2(\theta)} = 3 \tan(\theta)$, and the integral becomes

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x^2 - 6x}} &= \int \frac{dx}{\sqrt{(x-3)^2 - 9}} \\
 &= \int \frac{3 \sec(\theta) \tan(\theta) d\theta}{3 \tan(\theta)} \\
 &= \int \sec(\theta) d\theta \\
 &= \ln |\sec(\theta) + \tan(\theta)| + C \\
 &= \ln \left| \frac{x-3}{3} + \frac{\sqrt{(x-3)^2 - 9}}{3} \right| + C.
 \end{aligned}$$

(q) $\int_{\sqrt{2}}^2 \sec^{-1}(t) dt$

Solution: *Method 1:* first an IBP and then a trigonometric substitution. For the IBP, we use the parts

$$\begin{aligned}
 u &= \sec^{-1}(t) \Rightarrow du = \frac{dt}{|t|\sqrt{t^2 - 1}}, \\
 dv &= dt, \Rightarrow v = t.
 \end{aligned}$$

This gives

$$\begin{aligned}
 \int_{\sqrt{2}}^2 \sec^{-1}(t) dt &= [t \sec^{-1}(t)]_{\sqrt{2}}^2 - \int_{\sqrt{2}}^2 \frac{t dt}{|t|\sqrt{t^2 - 1}} \\
 &= 2 \sec^{-1}(2) - \sqrt{2} \sec^{-1}(\sqrt{2}) - \int_{\sqrt{2}}^2 \frac{t dt}{\sqrt{2} t \sqrt{t^2 - 1}} \quad (\text{since } t > 0) \\
 &= \frac{2\pi}{3} - \frac{\sqrt{2}\pi}{4} - \int_{\sqrt{2}}^2 \frac{dt}{\sqrt{2} \sqrt{t^2 - 1}}.
 \end{aligned}$$

In this last integral, we can use the trigonometric substitution $t = \sec(\theta)$, $dt = \tan(\theta) \sec(\theta) d\theta$. The bounds will become

$$t = \sqrt{2} \Rightarrow \theta = \sec^{-1}(\sqrt{2}) = \frac{\pi}{4},$$

$$t = 2 \Rightarrow \theta = \sec^{-1}(2) = \frac{\pi}{3}.$$

We therefore obtain

$$\begin{aligned} \int_{\sqrt{2}}^2 \sec^{-1}(t) dt &= \frac{2\pi}{3} - \frac{\sqrt{2}\pi}{4} - \int_{\pi/4}^{\pi/3} \frac{\sec(\theta) \tan(\theta) d\theta}{\tan(\theta)} \\ &= \frac{2\pi}{3} - \frac{\sqrt{2}\pi}{4} - \int_{\pi/4}^{\pi/3} \sec(\theta) d\theta \\ &= \frac{2\pi}{3} - \frac{\sqrt{2}\pi}{4} - [\ln |\sec(\theta) + \tan(\theta)|]_{\pi/4}^{\pi/3} \\ &= \boxed{\frac{2\pi}{3} - \frac{\sqrt{2}\pi}{4} - \ln(2 + \sqrt{3}) + \ln(\sqrt{2} + 1)}. \end{aligned}$$

Method 2: first a trigonometric substitution and then an IBP. As before, the trigonometric substitution uses $t = \sec(\theta)$ and gives

$$\int_{\sqrt{2}}^2 \sec^{-1}(t) dt = \int_{\pi/4}^{\pi/3} \sec^{-1}(\sec(\theta)) \sec(\theta) \tan(\theta) d\theta = \int_{\pi/4}^{\pi/3} \theta \sec(\theta) \tan(\theta) d\theta.$$

In the new integral, with use an IBP with parts

$$\begin{aligned} u &= \theta \Rightarrow du = d\theta, \\ dv &= \sec(\theta) \tan(\theta) d\theta, \Rightarrow v = \sec(\theta). \end{aligned}$$

We obtain

$$\begin{aligned} \int_{\sqrt{2}}^2 \sec^{-1}(t) dt &= [\theta \sec(\theta)]_{\pi/4}^{\pi/3} - \int_{\pi/4}^{\pi/3} \sec(\theta) d\theta \\ &= \frac{2\pi}{3} - \frac{\sqrt{2}\pi}{4} - [\ln |\sec(\theta) + \tan(\theta)|]_{\pi/4}^{\pi/3} \\ &= \boxed{\frac{2\pi}{3} - \frac{\sqrt{2}\pi}{4} - \ln(2 + \sqrt{3}) + \ln(\sqrt{2} + 1)}. \end{aligned}$$

$$(r) \int x^2 \tan^{-1}(x) dx$$

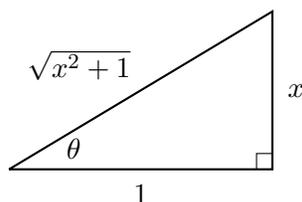
Solution: *Method 1:* first an IBP and then a trigonometric substitution. For the IBP, we use the parts

$$\begin{aligned} u &= \tan^{-1}(x) \Rightarrow du = \frac{dx}{x^2 + 1}, \\ dv &= x^2 dx \Rightarrow v = \frac{x^3}{3}. \end{aligned}$$

This gives

$$\begin{aligned}\int x^2 \tan^{-1}(x) dx &= \frac{x^3 \tan^{-1}(x)}{3} - \int \frac{x^3}{3(x^2 + 1)} dx \\ &= \frac{x^3 \tan^{-1}(x)}{3} - \frac{1}{3} \int \frac{x^3}{x^2 + 1} dx\end{aligned}$$

In this last integral, we can use the trigonometric substitution $x = \tan(\theta)$, so that $dx = \sec^2(\theta)d\theta$ and $x^2 + 1 = \tan^2(\theta) + 1 = \sec^2(\theta)$. The right triangle for this trigonometric substitution has base angle θ so that $\tan(\theta) = x$ as shown below.



We obtain

$$\begin{aligned}\int \frac{x^3}{x^2 + 1} dx &= \int \frac{\tan^3(\theta)}{\sec^2(\theta)} \sec^2(\theta) d\theta \\ &= \int \tan^3(\theta) d\theta \\ &= \int \tan(\theta) \tan^2(\theta) d\theta \\ &= \int \tan(\theta) (\sec^2(\theta) - 1) d\theta \\ &= \int (\tan(\theta) \sec(\theta)^2 - \tan(\theta)) d\theta \\ &= \frac{\tan^2(\theta)}{2} - \ln |\sec(\theta)| + C \\ &= \frac{x^2}{2} - \ln \left| \sqrt{x^2 + 1} \right| + C \\ &= \frac{x^2}{2} - \frac{1}{2} \ln(x^2 + 1) + C.\end{aligned}$$

Putting everything together, we get

$$\begin{aligned}\int x^2 \tan^{-1}(x) dx &= \frac{x^3 \tan^{-1}(x)}{3} - \frac{1}{3} \int \frac{x^3}{x^2 + 1} dx \\ &= \frac{x^3 \tan^{-1}(x)}{3} - \frac{1}{3} \left(\frac{x^2}{2} - \frac{1}{2} \ln(x^2 + 1) \right) + C \\ &= \boxed{\frac{x^3 \tan^{-1}(x)}{3} - \frac{x^2}{6} + \frac{1}{6} \ln(x^2 + 1) + C}.\end{aligned}$$

Method 2: first a trigonometric substitution and then an IBP. As before, the trigonometric substitution uses $x = \tan(\theta)$, $dx = \sec^2(\theta)d\theta$ so that

$$\begin{aligned}\int x^2 \tan^{-1}(x)dx &= \int \tan^2(\theta) \tan^{-1}(\tan(\theta)) \sec^2(\theta)d\theta \\ &= \int \theta \tan^2(\theta) \sec^2(\theta)d\theta.\end{aligned}$$

We can compute this integral using an IBP with parts

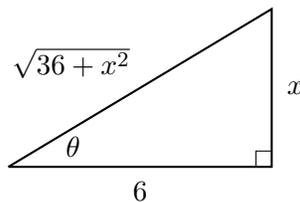
$$\begin{aligned}u = \theta &\Rightarrow du = d\theta, \\ dv = \tan^2(\theta) \sec^2(\theta)d\theta &\Rightarrow v = \frac{\tan^3(\theta)}{3}.\end{aligned}$$

We obtain

$$\begin{aligned}\int x^2 \tan^{-1}(x)dx &= \int \theta \tan^2(\theta) \sec^2(\theta)d\theta \\ &= \frac{\theta \tan^3(\theta)}{3} - \frac{1}{3} \int \tan^3(\theta)d\theta \\ &= \frac{\theta \tan^3(\theta)}{3} - \frac{1}{3} \int (\tan(\theta) \sec^2(\theta) - \tan(\theta)) d\theta \\ &= \frac{\theta \tan^3(\theta)}{3} - \frac{1}{3} \left(\frac{\tan^2(\theta)}{2} - \ln |\sec(\theta)| \right) + C \\ &= \boxed{\frac{x^3 \tan^{-1}(x)}{3} - \frac{x^2}{6} + \frac{1}{6} \ln(x^2 + 1) + C}.\end{aligned}$$

(s) $\int \frac{dx}{(36 + x^2)^2}$

Solution: We use the trigonometric substitution $x = 6 \tan(\theta)$, which gives $dx = 6 \sec^2(\theta)$ and $36 + x^2 = 36 + 36 \tan^2(\theta) = 36 \sec^2(\theta)$. The right triangle for this trigonometric substitution has base angle θ so that $\tan(\theta) = \frac{x}{6}$ as shown below.



The integral becomes

$$\int \frac{dx}{(36 + x^2)^2} = \int \frac{6 \sec^2(\theta)d\theta}{(36 \sec^2(\theta))^2}$$

$$\begin{aligned}
&= \frac{1}{216} \int \frac{d\theta}{\sec^2(\theta)} \\
&= \frac{1}{216} \int \cos^2(\theta) d\theta \\
&= \frac{1}{216} \int \frac{1 + \cos(2\theta)}{2} d\theta \\
&= \frac{1}{432} \left(\theta + \frac{\sin(2\theta)}{2} \right) + C \\
&= \frac{1}{432} (\theta + \sin(\theta) \cos(\theta)) + C \\
&= \boxed{\frac{1}{432} \left(\tan^{-1} \left(\frac{x}{6} \right) + \frac{6x}{36 + x^2} \right) + C}
\end{aligned}$$

(t) $\int \sin^{-1}(x)^2 dx$

Solution: We use an IBP with parts

$$\begin{aligned}
u &= \sin^{-1}(x)^2 \Rightarrow du = \frac{2 \sin^{-1}(x)}{\sqrt{1-x^2}} dx, \\
dv &= dx, \Rightarrow v = x.
\end{aligned}$$

This gives

$$\begin{aligned}
\int \sin^{-1}(x)^2 dx &= x \sin^{-1}(x)^2 - \int \frac{2x \sin^{-1}(x)}{\sqrt{1-x^2}} dx \\
&= x \sin^{-1}(x)^2 - 2 \int \frac{x \sin^{-1}(x)}{\sqrt{1-x^2}} dx.
\end{aligned}$$

In this new integral, we use another IBP with parts

$$\begin{aligned}
u &= \sin^{-1}(x) \Rightarrow du = \frac{dx}{\sqrt{1-x^2}}, \\
dv &= \frac{x}{\sqrt{1-x^2}}, \Rightarrow v = -\sqrt{1-x^2}.
\end{aligned}$$

We obtain

$$\begin{aligned}
\int \sin^{-1}(x)^2 dx &= x \sin^{-1}(x)^2 - 2 \int \frac{x \sin^{-1}(x)}{\sqrt{1-x^2}} dx \\
&= x \sin^{-1}(x)^2 - 2 \left(-\sin^{-1}(x) \sqrt{1-x^2} - \int (-\sqrt{1-x^2}) \frac{dx}{\sqrt{1-x^2}} \right) \\
&= x \sin^{-1}(x)^2 + 2 \sin^{-1}(x) \sqrt{1-x^2} - 2 \int dx
\end{aligned}$$

$$= \boxed{x \sin^{-1}(x)^2 + 2 \sin^{-1}(x) \sqrt{1-x^2} - 2x + C}.$$

(u) $\int_0^8 \sqrt{64+x^2} dx$

Solution: We substitute $x = 8 \tan(\theta)$, so that $dx = 8 \sec^2(\theta) d\theta$ and $\sqrt{64+x^2} = \sqrt{64+64 \tan^2(\theta)} = \sqrt{64 \sec^2(\theta)} = 8 \sec(\theta)$. The bounds change as follows:

$$x = 0 \Rightarrow \theta = \tan^{-1}\left(\frac{0}{8}\right) = 0,$$

$$x = 8 \Rightarrow \theta = \tan^{-1}\left(\frac{8}{8}\right) = \frac{\pi}{4}.$$

The integral becomes

$$\begin{aligned} \int_0^8 \sqrt{64+x^2} dx &= \int_0^{\pi/4} 8 \sec(\theta) \cdot 8 \sec^2(\theta) d\theta \\ &= 64 \int_0^{\pi/4} \sec^3(\theta) d\theta. \end{aligned}$$

Using integration by parts, we have previously found that

$$\int \sec^3(\theta) d\theta = \frac{1}{2} (\sec(\theta) \tan(\theta) - \ln |\sec(\theta) + \tan(\theta)|) + C.$$

So

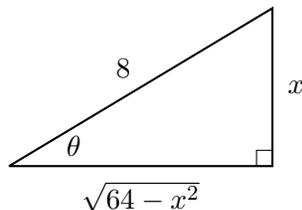
$$\begin{aligned} \int_0^8 \sqrt{64+x^2} dx &= 64 \left[\frac{1}{2} (\sec(\theta) \tan(\theta) - \ln |\sec(\theta) + \tan(\theta)|) \right]_0^{\pi/4} \\ &= \boxed{32 (\sqrt{2} - \ln(\sqrt{2} + 1))}. \end{aligned}$$

#44. Calculate the average value of the function $f(x) = \frac{1}{x\sqrt{64-x^2}}$ on the interval $[4, 4\sqrt{2}]$

Solution: The average value on the interval $[4, 4\sqrt{2}]$ is given by

$$\text{av}(f) = \frac{1}{4\sqrt{2} - 4} \int_4^{4\sqrt{2}} \frac{dx}{x\sqrt{64-x^2}}.$$

We compute this integral using the substitution $x = 8 \sin(\theta)$ and $dx = 8 \cos(\theta)d\theta$. The right triangle for this trigonometric substitution has base angle θ such that $\sin(\theta) = \frac{x}{8}$ as shown below.



Then $\sqrt{64 - x^2} = \sqrt{64 - 64 \sin^2(\theta)} = \sqrt{64 \cos^2(\theta)} = 8 \cos(\theta)$. The bounds change as follows:

$$x = 4 \Rightarrow \sin(\theta) = \frac{4}{8} = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6},$$

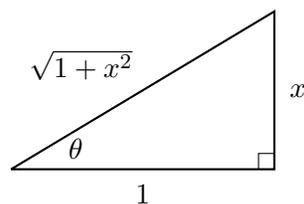
$$x = 4\sqrt{2} \Rightarrow \sin(\theta) = \frac{4\sqrt{2}}{8} = \frac{\sqrt{2}}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

Therefore

$$\begin{aligned} \text{av}(f) &= \frac{1}{4(\sqrt{2} - 1)} \int_{\pi/6}^{\pi/4} \frac{8 \cos(\theta)d\theta}{8 \sin(\theta)8 \cos(\theta)} \\ &= \frac{1}{32(\sqrt{2} - 1)} \int_{\pi/6}^{\pi/4} \frac{d\theta}{\sin(\theta)} \\ &= \frac{1}{32(\sqrt{2} - 1)} \int_{\pi/6}^{\pi/4} \csc(\theta)d\theta \\ &= \frac{1}{32(\sqrt{2} - 1)} [\ln |\csc(\theta) - \cot(\theta)|]_{\pi/6}^{\pi/4} \\ &= \frac{1}{32(\sqrt{2} - 1)} \left(\ln \left| \csc\left(\frac{\pi}{4}\right) - \cot\left(\frac{\pi}{4}\right) \right| - \ln \left| \csc\left(\frac{\pi}{6}\right) - \cot\left(\frac{\pi}{6}\right) \right| \right) \\ &= \boxed{\frac{1}{32(\sqrt{2} - 1)} \left(\ln(\sqrt{2} - 1) - \ln(2 - \sqrt{3}) \right)}. \end{aligned}$$

#45. (a) Evaluate $\int \sqrt{1 + x^2} dx$.

Solution: We want $1 + x^2 = 1 + \tan^2(\theta)$, so we substitute $x = \tan(\theta)$, $dx = \sec^2(\theta)d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\tan(\theta) = x$, as shown below.



Then $\sqrt{1+x^2} = \sqrt{1+\tan^2(\theta)} = \sqrt{\sec^2(\theta)} = \sec(\theta)$. The integral becomes

$$\begin{aligned}\int \sqrt{1+x^2} dx &= \int \sec(\theta) \sec^2(\theta) d\theta \\ &= \int \sec^3(\theta) d\theta.\end{aligned}$$

We can evaluate $\int \sec^3(\theta) d\theta$ with an IBP and solving for the unknown integral when it reappears on the right-hand side. For the IBP we use the parts

$$\begin{aligned}u &= \sec(\theta) \Rightarrow du = \sec(\theta) \tan(\theta) d\theta, \\ dv &= \sec^2(\theta) d\theta \Rightarrow v = \tan(\theta).\end{aligned}$$

We get

$$\begin{aligned}\int \sec^3(\theta) d\theta &= \int \sec^2(\theta) \sec(\theta) d\theta \\ \int \sec^3(\theta) d\theta &= \tan(\theta) \sec(\theta) - \int \tan(\theta) \sec(\theta) \tan(\theta) d\theta \\ \int \sec^3(\theta) d\theta &= \tan(\theta) \sec(\theta) - \int \tan^2(\theta) \sec(\theta) d\theta\end{aligned}$$

We will use the Pythagorean identity $\tan^2(\theta) = \sec^2(\theta) - 1$ to see the original integral reappear on the right-hand side.

$$\begin{aligned}\int \sec^3(\theta) d\theta &= \tan(\theta) \sec(\theta) - \int (\sec^2(\theta) - 1) \sec(\theta) d\theta \\ \int \sec^3(\theta) d\theta &= \tan(\theta) \sec(\theta) - \int \sec^3(\theta) d\theta + \int \sec(\theta) d\theta \\ \int \sec^3(\theta) d\theta &= \tan(\theta) \sec(\theta) - \int \sec^3(\theta) d\theta + \ln |\sec(\theta) + \tan(\theta)|\end{aligned}$$

We can now move the term $-\int \sec^3(\theta) d\theta$ to the left hand side and finish solving

$$2 \int \sec^3(\theta) d\theta = \tan(\theta) \sec(\theta) + \ln |\sec(\theta) + \tan(\theta)|$$

$$\Rightarrow \int \sec^3(\theta) d\theta = \frac{1}{2} (\tan(\theta) \sec(\theta) + \ln |\sec(\theta) + \tan(\theta)|) + C.$$

Using the right triangle above, we can express this result in terms of x , observing that $\tan(\theta) = x$ and $\sec(\theta) = \sqrt{x^2 + 1}$. We get

$$\int \sqrt{1 + x^2} dx = \frac{1}{2} \left(x\sqrt{x^2 + 1} + \ln \left| x + \sqrt{x^2 + 1} \right| \right) + C.$$

(b) Use your result from part (a) for the following applications.

i. Calculate the length of the curve $y = x^2$, $0 \leq x \leq 1$.

Solution: The arc length is given by

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_0^1 \sqrt{1 + (2x)^2} dx \\ &= \frac{1}{2} \int_0^2 \sqrt{1 + u^2} du \quad (u = 2x) \\ &= \frac{1}{2} \left[\frac{1}{2} \left(u\sqrt{u^2 + 1} + \ln \left| u + \sqrt{u^2 + 1} \right| \right) \right]_0^2 \\ &= \boxed{\frac{1}{4} \left(2\sqrt{5} + \ln(2 + \sqrt{5}) \right) \text{ units}}. \end{aligned}$$

ii. Calculate the area of the surface obtained by revolving the curve $y = e^x$, $0 \leq x \leq \ln(2)$, about the x -axis.

Solution: The area of a surface of revolution about the x -axis is given by

$$\begin{aligned} A &= \int_0^{\ln(2)} 2\pi y(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_0^{\ln(2)} 2\pi e^x \sqrt{1 + e^{2x}} dx \\ &= 2\pi \int_1^2 \sqrt{1 + u^2} du \quad (u = e^x, du = e^x dx) \\ &= 2\pi \left[\frac{1}{2} \left(u\sqrt{u^2 + 1} + \ln \left| u + \sqrt{u^2 + 1} \right| \right) \right]_1^2 \end{aligned}$$

$$= \pi \left(2\sqrt{5} + \ln(2 + \sqrt{5}) - \sqrt{2} - \ln(1 + \sqrt{2}) \right) \text{ square units}.$$

- iii. Calculate the area of the surface obtained by revolving the curve $y = \sin^{-1}(x)$, $0 \leq x \leq 1$ about the y -axis.

Solution: Note that the curve can be expressed as a function of y as $x = \sin(y)$, $0 \leq y \leq \frac{\pi}{2}$. The area of a surface of revolution about the y -axis is given by

$$\begin{aligned} A &= \int_0^{\pi/2} 2\pi x(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \\ &= \int_0^{\pi/2} 2\pi \sin(y) \sqrt{1 + \cos^2(y)} dy \\ &= 2\pi \int_1^0 -\sqrt{1 + u^2} du && (u = \cos(y), du = -\sin(y)dy) \\ &= 2\pi \int_0^1 \sqrt{1 + u^2} du \\ &= 2\pi \left[\frac{1}{2} \left(u\sqrt{u^2 + 1} + \ln \left| u + \sqrt{u^2 + 1} \right| \right) \right]_0^1 \\ &= \pi \left(\sqrt{2} + \ln(1 + \sqrt{2}) \right) \text{ square units}. \end{aligned}$$

- #46. Calculate the area of the region between the semi-circle $y = \sqrt{2 - x^2}$ and the x -axis for $\frac{\sqrt{2}}{2} \leq x \leq 1$.

Solution: The area is given by

$$A = \int_{\sqrt{2}/2}^1 \sqrt{2 - x^2} dx.$$

This integral can be calculated using the substitution $x = \sqrt{2} \sin(\theta)$, which gives $dx = \sqrt{2} \cos(\theta) d\theta$ and $2 - x^2 = 2 - 2 \sin^2(\theta) = 2 \cos^2(\theta)$. The bounds become

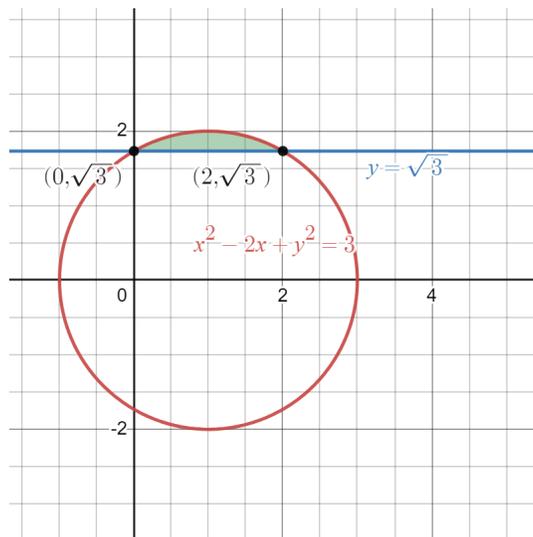
$$\begin{aligned} x = \frac{\sqrt{2}}{2} &\Rightarrow \sin(\theta) = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \\ x = 1 &\Rightarrow \sin(\theta) = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}. \end{aligned}$$

Hence we get

$$\begin{aligned}
 A &= \int_{\pi/6}^{\pi/4} \sqrt{2 \cos^2(\theta)} \sqrt{2} \cos(\theta) d\theta \\
 &= 2 \int_{\pi/6}^{\pi/4} \cos^2(\theta) d\theta \\
 &= 2 \int_{\pi/6}^{\pi/4} \frac{1 + \cos(2\theta)}{2} d\theta \\
 &= \left[\theta + \frac{\sin(2\theta)}{2} \right]_{\pi/6}^{\pi/4} \\
 &= \frac{\pi}{4} + \sin\left(\frac{\pi}{2}\right) - \frac{\pi}{6} - \sin\left(\frac{\pi}{3}\right) \\
 &= \boxed{\frac{\pi}{12} + 1 - \frac{\sqrt{3}}{2} \text{ units}}.
 \end{aligned}$$

#47. Calculate the area of the region inside the circle of equation $x^2 - 2x + y^2 = 3$ and above the line $y = \sqrt{3}$

Solution: The region is sketched below.



Note that the upper half semi-circle can be expressed as a function of x as $y = \sqrt{3 + 2x - x^2}$. We will compute the area using vertical strips. The vertical strip at x in the region has length

$\ell(x) = \sqrt{3 + 2x - x^2} - \sqrt{3}$. Therefore, the area is given by

$$A = \int_0^2 \ell(x) dx = \int_0^2 \left(\sqrt{3 + 2x - x^2} - \sqrt{3} \right) dx = \int_0^2 \sqrt{3 + 2x - x^2} dx - 2\sqrt{3}.$$

To compute the remaining integral, we start by completing the square in the square root:

$$3 + 2x - x^2 = 3 - (x^2 - 2x) = 3 - (x^2 - 2x + 1) + 1 = 4 - (x - 1)^2.$$

We can then use a trigonometric substitution. We want $4 - (x - 1)^2 = 4(1 - \sin^2(\theta))$, so we substitute $x - 1 = 2 \sin(\theta)$ or $x = 1 + 2 \sin(\theta)$. This gives $dx = 2 \cos(\theta) d\theta$ and $\sqrt{4 - (x - 1)^2} = \sqrt{4 - 4 \sin^2(\theta)} = \sqrt{4 \cos^2(\theta)} = 2 \cos(\theta)$. The bounds of the integral become

$$\begin{aligned} x = 0 &\Rightarrow \sin(\theta) = \frac{0 - 1}{2} = -\frac{1}{2} \Rightarrow \theta = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}, \\ x = 2 &\Rightarrow \sin(\theta) = \frac{2 - 1}{2} = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^2 \sqrt{3 + 2x - x^2} dx &= \int_0^2 \sqrt{4 - (x - 1)^2} dx \\ &= \int_{-\pi/6}^{\pi/6} 2 \cos(\theta) 2 \cos(\theta) d\theta \\ &= 4 \int_{-\pi/6}^{\pi/6} \cos^2(\theta) d\theta \\ &= 8 \int_0^{\pi/6} \cos^2(\theta) d\theta \end{aligned}$$

where we have used the fact that the integrand is even in the last step. We can now compute this integral with the double angle formula as follows:

$$\begin{aligned} \int_0^2 \sqrt{3 + 2x - x^2} dx &= 8 \int_0^{\pi/6} \frac{1 + \cos(2\theta)}{2} 2 d\theta \\ &= 4 \left[\theta + \frac{\sin(2\theta)}{2} \right]_0^{\pi/6} \\ &= 4 \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4} \right) \\ &= \frac{2\pi}{3} + \sqrt{3}. \end{aligned}$$

In conclusion, the area is

$$A = \int_0^2 \sqrt{3 + 2x - x^2} dx - 2\sqrt{3}$$

$$\begin{aligned}
 &= \frac{2\pi}{3} + \sqrt{3} - 2\sqrt{3} \\
 &= \boxed{\frac{2\pi}{3} - \sqrt{3} \text{ square units}}.
 \end{aligned}$$

- #48. Consider the region \mathcal{R} bounded between the graph of $y = \frac{1}{16-x^2}$ and the x -axis for $0 \leq x \leq 2$. Find the volume of the solid obtained by revolving \mathcal{R} about the line $x = -3$.

Solution: We use the shell method. Revolving the vertical strip at x about the line $x = -3$ forms a cylindrical shell of radius $r(x) = x + 3$ and height $h(x) = \frac{1}{16-x^2}$. Therefore

$$\begin{aligned}
 V &= \int_0^2 2\pi r(x)h(x)dx \\
 &= 2\pi \int_0^2 \frac{x+3}{16-x^2} dx.
 \end{aligned}$$

We can evaluate this integral with a trigonometric substitution. We substitute $x = 4\sin(\theta)$, so $dx = 4\cos(\theta)d\theta$ and $16-x^2 = 16-16\sin^2(\theta) = 16\cos^2(\theta)$. The bounds become

$$\begin{aligned}
 x = 0 &\Rightarrow \sin(\theta) = \frac{0}{4} = 0 \Rightarrow \theta = \sin^{-1}(0) = 0, \\
 x = 2 &\Rightarrow \sin(\theta) = \frac{2}{4} = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.
 \end{aligned}$$

So

$$\begin{aligned}
 V &= 2\pi \int_0^{\pi/6} \frac{4\sin(\theta) + 3}{16\cos^2(\theta)} 4\cos(\theta)d\theta \\
 &= \frac{\pi}{2} \int_0^{\pi/6} \frac{4\sin(\theta) + 3}{\cos(\theta)} d\theta \\
 &= \frac{\pi}{2} \int_0^{\pi/6} (4\tan(\theta) + 3\sec(\theta)) d\theta \\
 &= \frac{\pi}{2} [4\ln|\sec(\theta)| + 3\ln|\sec(\theta) + \tan(\theta)|]_0^{\pi/6} \\
 &= \frac{\pi}{2} \left(4\ln\left(\frac{2}{\sqrt{3}}\right) + 3\ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) \right) \\
 &= \boxed{\frac{\pi}{2} \left(4\ln(2) - \frac{1}{2}\ln(3) \right) \text{ cubic units}}.
 \end{aligned}$$

#49. (a) Evaluate $\int \arcsin(x)dx$.

Solution: We use an integration by parts with parts

$$u = \arcsin(x) \Rightarrow du = \frac{dx}{\sqrt{1-x^2}},$$

$$dv = dx \Rightarrow v = x.$$

We get

$$\begin{aligned} \int \arcsin(x)dx &= x \arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}dx \\ &= \boxed{x \arcsin(x) + \sqrt{1-x^2} + C}. \end{aligned}$$

(b) Using your answer to part (a), evaluate $\int \ln(x) \arcsin(x)dx$. (*Hint: LIATE.*)

Solution: We use an integration by parts with

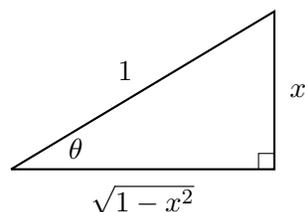
$$u = \ln(x) \Rightarrow du = \frac{dx}{x},$$

$$dv = \arcsin(x)dx \Rightarrow v = x \arcsin(x) + \sqrt{1-x^2}.$$

This gives

$$\begin{aligned} \int \ln(x) \arcsin(x)dx &= \ln(x) \left(x \arcsin(x) + \sqrt{1-x^2} \right) - \int \left(x \arcsin(x) + \sqrt{1-x^2} \right) \frac{1}{x} dx \\ &= \ln(x) \left(x \arcsin(x) + \sqrt{1-x^2} \right) - \int \arcsin(x)dx - \int \frac{\sqrt{1-x^2}}{x} dx \\ &= \ln(x) \left(x \arcsin(x) + \sqrt{1-x^2} \right) - \left(x \arcsin(x) + \sqrt{1-x^2} \right) - \int \frac{\sqrt{1-x^2}}{x} dx \\ &= (\ln(x) - 1) \left(x \arcsin(x) + \sqrt{1-x^2} \right) - \int \frac{\sqrt{1-x^2}}{x} dx. \end{aligned}$$

The remaining integral can be evaluated using the trigonometric substitution $x = \sin(\theta)$, which gives $dx = \cos(\theta)d\theta$. The right triangle for this substitution has base angle θ so that $\sin(\theta) = x$, as shown below.



We get

$$\begin{aligned}
 \int \frac{\sqrt{1-x^2}}{x} dx &= \int \frac{\cos(\theta)}{\sin(\theta)} \cos(\theta) d\theta \\
 &= \int \frac{\cos^2(\theta)}{\sin(\theta)} d\theta \\
 &= \int \frac{1 - \sin^2(\theta)}{\sin(\theta)} d\theta \\
 &= \int (\csc(\theta) - \sin(\theta)) d\theta \\
 &= \cos(\theta) - \ln |\csc(\theta) + \cot(\theta)| + C \\
 &= \sqrt{1-x^2} - \ln \left| \frac{1}{x} + \frac{\sqrt{1-x^2}}{x} \right| + C \\
 &= \sqrt{1-x^2} - \ln \left| \frac{1 + \sqrt{1-x^2}}{x} \right| + C.
 \end{aligned}$$

Putting everything together, we obtain the (rather lengthy) antiderivative

$$\begin{aligned}
 \int \ln(x) \arcsin(x) dx &= (\ln(x) - 1) \left(x \arcsin(x) + \sqrt{1-x^2} \right) - \int \frac{\sqrt{1-x^2}}{x} dx \\
 &= (\ln(x) - 1) \left(x \arcsin(x) + \sqrt{1-x^2} \right) - \sqrt{1-x^2} + \ln \left| \frac{1 + \sqrt{1-x^2}}{x} \right| + C.
 \end{aligned}$$

#50. Calculate the surface area obtained by revolving the curve $y = \frac{1}{2}x^2$, $0 \leq x \leq 1$, about the x -axis.

Solution: The surface area for revolution about the x -axis is given by

$$A = \int_0^1 2\pi y(x) \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx$$

$$\begin{aligned}
&= \int_0^1 2\pi \cdot \frac{1}{2} x^2 \sqrt{1+x^2} dx \\
&= \pi \int_0^1 x^2 \sqrt{1+x^2} dx.
\end{aligned}$$

Using the trigonometric substitution $x = \tan(\theta)$, $dx = \sec^2(\theta)d\theta$, we get

$$\begin{aligned}
A &= \pi \int_0^{\pi/4} \tan^2(\theta) \sec^3(\theta) d\theta \\
&= \pi \int_0^{\pi/4} (\sec^2(\theta) - 1) \sec^3(\theta) d\theta \\
&= \pi \int_0^{\pi/4} (\sec^5(\theta) - \sec^3(\theta)) d\theta.
\end{aligned}$$

We have seen previously that

$$\int \sec^3(\theta) d\theta = \frac{1}{2} (\sec(\theta) \tan(\theta) - \ln |\sec(\theta) + \tan(\theta)|) + C,$$

and we can express $\int \sec^5(\theta) d\theta$ in terms of $\int \sec^3(\theta) d\theta$ using an IBP with $u = \sec^3(\theta)$, $du = 3 \sec^3(\theta) \tan(\theta)$ and $dv = \sec^2(\theta) d\theta$, $v = \tan(\theta)$. We get

$$\begin{aligned}
\int \sec^5(\theta) d\theta &= \int \sec^3(\theta) \sec^2(\theta) d\theta \\
&= \sec^3(\theta) \tan(\theta) - 3 \int \sec^3(\theta) \tan^2(\theta) d\theta \\
&= \sec^3(\theta) \tan(\theta) - 3 \int \sec^3(\theta) (\sec^2(\theta) - 1) d\theta \\
&= \sec^3(\theta) \tan(\theta) - 3 \int \sec^5(\theta) + 3 \int \sec^3(\theta) d\theta \\
&= \sec^3(\theta) \tan(\theta) - 3 \int \sec^5(\theta) + \frac{3}{2} (\sec(\theta) \tan(\theta) - \ln |\sec(\theta) + \tan(\theta)|).
\end{aligned}$$

Collecting the terms involving $\int \sec^5(\theta) d\theta$ to the left-hand side gives

$$\begin{aligned}
4 \int \sec^5(\theta) d\theta &= \sec^3(\theta) \tan(\theta) + \frac{3}{2} (\sec(\theta) \tan(\theta) - \ln |\sec(\theta) + \tan(\theta)|) \\
\Rightarrow \int \sec^5(\theta) d\theta &= \frac{1}{4} \sec^3(\theta) \tan(\theta) + \frac{3}{8} (\sec(\theta) \tan(\theta) - \ln |\sec(\theta) + \tan(\theta)|) + C.
\end{aligned}$$

Going back to the surface area, we get

$$A = \pi \int_0^{\pi/4} (\sec^5(\theta) - \sec^3(\theta)) d\theta$$

$$\begin{aligned}
&= \left[\frac{1}{4} \sec^3(\theta) \tan(\theta) - \frac{1}{8} (\sec(\theta) \tan(\theta) - \ln |\sec(\theta) + \tan(\theta)|) \right]_0^{\pi/4} \\
&= \boxed{\frac{\sqrt{2}}{2} - \frac{1}{8} (\sqrt{2} + \ln |\sqrt{2} + 1|)} \text{ square units.}
\end{aligned}$$

#51. Let \mathcal{R} be the region between the graph $y = \frac{1}{x\sqrt{1+x^2}}$ and the x -axis for $1 \leq x \leq \sqrt{3}$.

(a) Find the area of \mathcal{R} .

Solution: The area is given by

$$A = \int_1^{\sqrt{3}} \frac{dx}{x\sqrt{1+x^2}}.$$

This integral can be calculated with the trigonometric substitution $x = \tan(\theta)$, which gives $dx = \sec^2(\theta)d\theta$. Then we have $\sqrt{1+x^2} = \sqrt{1+\tan^2(\theta)} = \sqrt{\sec^2(\theta)} = \sec(\theta)$. The bounds change as follows:

$$\begin{aligned}
x = 1 &\Rightarrow \tan(\theta) = 1 \Rightarrow \theta = \frac{\pi}{4}, \\
x = \sqrt{3} &\Rightarrow \tan(\theta) = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}.
\end{aligned}$$

We obtain

$$\begin{aligned}
A &= \int_{\pi/4}^{\pi/3} \frac{\sec^2(\theta)d\theta}{\tan(\theta)\sec(\theta)} \\
&= \int_{\pi/4}^{\pi/3} \frac{\sec(\theta)}{\tan(\theta)} d\theta \\
&= \int_{\pi/4}^{\pi/3} \csc(\theta)d\theta \\
&= [-\ln |\csc(\theta) + \cot(\theta)|]_{\pi/4}^{\pi/3} \\
&= -\ln \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) + \ln(\sqrt{2} + 1) \\
&= \boxed{\ln \left(\frac{\sqrt{2} + 1}{\sqrt{3}} \right)} \text{ square units.}
\end{aligned}$$

(b) Find the volume of the solid obtained by revolving \mathcal{R} about the x -axis.

Solution: We use the disk method. Revolving the vertical strip at x about the x -axis forms a disk of radius $r(x) = \frac{1}{x\sqrt{1+x^2}}$. So

$$\begin{aligned} V &= \int_1^{\sqrt{3}} \pi r(x)^2 dx \\ &= \pi \int_1^{\sqrt{3}} \frac{dx}{x^2(1+x^2)}. \end{aligned}$$

This integral can be calculated with the same trigonometric substitution as the previous question. We get

$$\begin{aligned} V &= \pi \int_{\pi/4}^{\pi/3} \frac{\sec^2(\theta) d\theta}{\tan^2(\theta) \sec^2(\theta)} \\ &= \pi \int_{\pi/4}^{\pi/3} \frac{d\theta}{\tan^2(\theta)} \\ &= \pi \int_{\pi/4}^{\pi/3} \cot^2(\theta) d\theta \\ &= \pi \int_{\pi/4}^{\pi/3} (\csc^2(\theta) - 1) d\theta \\ &= \pi [-\cot(\theta) - \theta]_{\pi/4}^{\pi/3} \\ &= \pi \left(-\frac{1}{\sqrt{3}} - \frac{\pi}{3} + 1 + \frac{\pi}{4} \right) \\ &= \boxed{\pi \left(1 - \frac{1}{\sqrt{3}} - \frac{\pi}{12} \right) \text{ cubic units}}. \end{aligned}$$

(c) Find the volume of the solid obtained by revolving \mathcal{R} about the y -axis.

Solution: We use the shell method. Revolving the vertical strip at x about the y -axis forms a shell with radius $r(x) = x$ and height $h(x) = \frac{1}{x\sqrt{x^2+1}}$.

$$\begin{aligned} V &= \int_1^{\sqrt{3}} 2\pi r(x)h(x) dx \\ &= 2\pi \int_1^{\sqrt{3}} x \frac{1}{x\sqrt{1+x^2}} dx \\ &= 2\pi \int_1^{\sqrt{3}} \frac{dx}{\sqrt{1+x^2}}. \end{aligned}$$

We can calculate this integral with the same trigonometric substitution as the previous two

questions. We get

$$\begin{aligned} V &= 2\pi \int_{\pi/4}^{\pi/3} \frac{\sec^2(\theta)d\theta}{\sec(\theta)} \\ &= 2\pi \int_{\pi/4}^{\pi/3} \sec(\theta)d\theta \\ &= 2\pi [\ln |\sec(\theta) + \tan(\theta)|]_{\pi/4}^{\pi/3} \\ &= 2\pi \left(\ln(2 + \sqrt{3}) - \ln(\sqrt{2} + 1) \right) \\ &= 2\pi \ln \left(\frac{2 + \sqrt{3}}{1 + \sqrt{2}} \right) \text{ cubic units} \end{aligned}$$