Section 8.8: Improper Integrals - Worksheet Solutions

#52. Calculate the following integrals or determine if they diverge.

(a) $\displaystyle\int_0^\infty e^{-5x}dx$

Solution:

$$\int_0^\infty e^{-5x}dx = \lim_{t\to\infty}\int_0^t e^{-5x}dx$$
$$= \lim_{t\to\infty}\left[-\frac{1}{5}e^{-5x}\right]_0^t$$
$$= \lim_{t\to\infty}\left(-\frac{1}{5}e^{-5t}+\frac{1}{5}e^0\right)$$
$$= \left(-\frac{1}{5}\cdot 0+\frac{1}{5}\right)$$
$$= \boxed{\frac{1}{5}}$$

(b) $\displaystyle\int_0^{\pi/4}\csc(x)dx$

Solution: This is a type II improper integral due to the vertical asymptote of $y=\csc(x)$ at $x=0$.

$$\int_0^{\pi/4}\csc(x)dx = \lim_{t\to 0^+}\int_t^{\pi/4}\csc(x)dx$$
$$= \lim_{t\to 0^+}\left[-\ln|\csc(x)+\cot(x)|\right]_t^{\pi/4}$$
$$= \lim_{t\to 0^+}\left(-\ln(\sqrt{2}+1)+\ln|\csc(t)+\cot(t)|\right)$$
$$= \infty$$

since $\cot(t),\csc(t)\to\infty$ when $t\to 0^+$, so $\ln|\csc(t)+\cot(t)|\to\infty$ when $t\to 0^+$. Therefore $\boxed{\displaystyle\int_0^{\pi/4}\csc(x)dx \text{ diverges}}$.

(c) $\displaystyle\int_0^\infty \sin(5x)dx$

Solution: We have

$$\int_0^\infty \sin(5x)dx = \lim_{t\to\infty} \int_0^t \sin(5x)dx$$

$$= \lim_{t\to\infty} \left[-\frac{1}{5}\cos(5x)\right]_0^t$$

$$= \lim_{t\to\infty} \left(\frac{1}{5} - \frac{1}{5}\cos(5t)\right).$$

This last limit does not exist due to the infinitely oscillating behavior of $\cos(5t)$ as $t \to \infty$. Hence, $\boxed{\int_0^\infty \sin(5x)dx \text{ diverges}}$.

(d) $\int_{-\infty}^0 xe^{3x}dx$

Solution: We can start by finding an antiderivative using integration by parts. We use the parts

$$u = x \implies du = dx$$

$$dv = e^{3x}dx \implies v = \frac{e^{3x}}{3}.$$

We obtain

$$\int xe^{3x}dx = \frac{xe^{3x}}{3} - \int \frac{e^{3x}}{3}dx$$

$$= \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C$$

$$= \frac{(3x-1)e^{3x}}{9} + C.$$

We can now compute the improper integral.

$$\int_{-\infty}^0 xe^{3x}dx = \lim_{t\to-\infty} \int_t^0 xe^{3x}dx$$

$$= \lim_{t\to-\infty} \left[\frac{(3x-1)e^{3x}}{9}\right]_t^0$$

$$= \lim_{t\to-\infty} \left(-\frac{1}{9} - \frac{(3t-1)e^{3t}}{9}\right)$$

$$= -\frac{1}{9} - \lim_{t\to-\infty} \frac{(3t-1)}{9e^{-3t}}$$

$$\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} -\frac{1}{9} - \lim_{t\to-\infty} \frac{3}{-27e^{-3t}}$$

$$= -\frac{1}{9} - 0$$

$$= \boxed{-\frac{1}{9}}.$$

(e) $\displaystyle\int_{-7}^{\infty} \frac{dx}{x^2 + 4x + 29}$

Solution: To find the antiderivative, we start by completing the square, which gives

$$x^2 + 4x + 29 = (x^2 + 4x + 4) - 4 + 29 = (x + 2)^2 + 25.$$

So

$$\int \frac{dx}{x^2 + 4x + 29} = \int \frac{dx}{(x + 2)^2 + 25} = \frac{1}{5} \tan^{-1}\left(\frac{x + 2}{5}\right) + C.$$

Using this for the improper integral, we obtain

$$\int_{-7}^{\infty} \frac{dx}{x^2 + 4x + 29} = \lim_{t \to \infty} \int_{-7}^{t} \frac{dx}{x^2 + 4x + 29}$$

$$= \lim_{t \to \infty} \left[\frac{1}{5} \tan^{-1}\left(\frac{x + 2}{5}\right)\right]_{-7}^{t}$$

$$= \lim_{t \to \infty} \frac{1}{5}\left(\tan^{-1}\left(\frac{t + 2}{5}\right) - \tan^{-1}(-1)\right)$$

$$= \frac{1}{5}\left(\frac{\pi}{2} - \left(-\frac{\pi}{4}\right)\right)$$

$$= \boxed{\frac{3\pi}{20}}.$$

(f) $\displaystyle\int_{-\infty}^{0} x^4 e^{3x^5} dx$

Solution: Using the substitution $u = 3x^5$, $du = 15x^4 dx$, we find that the antiderivative is

$$\int x^4 e^{3x^5} dx = \frac{1}{15} \int e^u du = \frac{e^u}{15} + C = \frac{e^{3x^5}}{15} + C.$$

We can now use this to compute the improper integral. We have

$$\int_{-\infty}^{0} x^4 e^{3x^5} dx = \lim_{t \to -\infty} \int_{t}^{0} x^4 e^{3x^5} dx$$

$$= \lim_{t \to -\infty} \left[\frac{e^{3x^5}}{15}\right]_{t}^{0}$$

$$= \lim_{t \to -\infty} \left(\frac{1}{15} - \frac{e^{3t^5}}{15} \right)$$

$$= \boxed{\frac{1}{15}}.$$

(g) $\displaystyle \int_1^5 \frac{dx}{\sqrt[3]{10-2x}}$

Solution: This is a type II improper integral due to the vertical asymptote at $x = 5$. Therefore

$$\int_1^5 \frac{dx}{\sqrt[3]{10-2x}} = \lim_{t \to 5^-} \int_1^t \frac{dx}{\sqrt[3]{10-2x}}$$

$$= \lim_{t \to 5^-} \left[-\frac{1}{2} \cdot \frac{3}{2} (10-2x)^{2/3} \right]_1^t$$

$$= \lim_{t \to 5^-} \left(\frac{3}{4} 8^{2/3} - \frac{3}{4} (10-2t)^{2/3} \right)$$

$$= \frac{3}{4} 4 - \frac{3}{4} \cdot 0$$

$$= \boxed{3}.$$

(h) $\displaystyle \int_{-\pi/12}^{\pi/12} \csc(3\theta)d\theta$

Solution: This is a type II improper integral due to the vertical asymptote at $x = 0$. Since the discontinuity is in the interior of the interval of integration, we break up the integral as

$$\int_{-\pi/12}^{\pi/12} \csc(3\theta)d\theta = \int_{-\pi/12}^0 \csc(3\theta)d\theta + \int_0^{\pi/12} \csc(3\theta)d\theta,$$

and set up each of these two improper integral as a limit. For the first one, we have

$$\int_{-\pi/12}^0 \csc(3\theta)d\theta = \lim_{t \to 0^-} \int_{-\pi/12}^t \csc(3\theta)d\theta$$

$$= \lim_{t \to 0^-} \left[-\frac{1}{3} \ln|\csc(3\theta) + \cot(3\theta)| \right]_{-\pi/12}^t$$

$$= \lim_{t \to 0^-} \left(-\frac{1}{3} \ln|\csc(3t) + \cot(3t)| + \frac{1}{3} \ln\left| -\sqrt{2} - 1 \right| \right)$$

$$= -\infty$$

since $\csc(3t), \cot(3t) \to -\infty$ when $t \to 0^-$. Therefore, $\displaystyle\int_{-\pi/12}^{0} \csc(3\theta)d\theta$ diverges. There is no need to continue further: because one of the two integrals diverges, we know that $\boxed{\displaystyle\int_{-\pi/12}^{\pi/12} \csc(3\theta)d\theta \text{ diverges}}$.

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(16 + x^2)^{3/2}}$

Solution: We can start by finding an antiderivative of the integrand. For this, we can use the trigonometric substitution $x = 4\tan(\theta)$, $dx = 4\sec^2(\theta)d\theta$. The right triangle for this trigonometric substitution has base angle θ so that $\tan(\theta) = \frac{x}{4}$ as shown below.
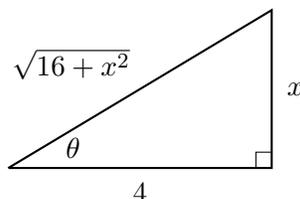
We get $(16 + u^2)^{3/2} = (16 + 16\tan^2(\theta))^{3/2} = (16\sec^2(\theta))^{3/2} = 64\sec^3(\theta)$, and the integral becomes

$$\int \frac{dx}{(16 + x^2)^{3/2}} = \int \frac{4\sec^2(\theta)d\theta}{64\sec^3(\theta)}$$
$$= \frac{1}{16}\int \frac{d\theta}{\sec(\theta)}$$
$$= \frac{1}{16}\int \cos(\theta)d\theta$$
$$= \frac{1}{16}\sin(\theta) + C.$$

In this antiderivative, we can express $\sin(\theta)$ in terms of x using the right triangle above, in which we see that $\sin(\theta) = \frac{x}{\sqrt{16+x^2}}$. Thus

$$\int \frac{dx}{(16 + x^2)^{3/2}} = \frac{x}{16\sqrt{16 + x^2}} + C.$$

We can now compute the improper integral. Observe that the integrand is even, so the integral on $(-\infty, \infty)$ is equal to two times the integral on $[0, \infty)$.

$$\int_{-\infty}^{\infty} \frac{dx}{(16 + x^2)^{3/2}} = 2\int_{0}^{\infty} \frac{dx}{(16 + x^2)^{3/2}}$$
$$= 2\lim_{t\to\infty} \int_{0}^{t} \frac{dx}{(16 + x^2)^{3/2}}$$

$$= 2 \lim_{t \to \infty} \left[\frac{x}{16\sqrt{16 + x^2}} \right]_0^t$$

$$= 2 \lim_{t \to \infty} \frac{t}{16\sqrt{16 + t^2}} \cdot \frac{\frac{1}{t}}{\frac{1}{t}}$$

$$= 2 \lim_{b \to \infty} \frac{1}{16\sqrt{\frac{16}{t^2} + 1}}$$

$$= 2 \frac{1}{16}$$

$$= \boxed{\frac{1}{8}}.$$

(j) $\displaystyle \int_0^1 \ln(x)dx$

Solution: This is a type II improper integral due to the vertical asymptote of $y = \ln(x)$ at $x = 0$. First we compute an antiderivative using integration by parts with parts

$$u = \ln(x) \implies du = \frac{dx}{x},$$

$$dv = dx \implies v = x.$$

We obtain

$$\int \ln(x)dx = x\ln(x) - \int x\frac{1}{x}dx$$

$$= x\ln(x) - \int dx$$

$$= x\ln(x) - x + C$$

$$= x(\ln(x) - 1) + C$$

Next we compute the improper integral.

$$\int_0^1 \ln(x)dx = \lim_{t \to 0^+} \int_t^1 \ln(x)dx$$

$$= \lim_{t \to 0^+} [x(\ln(x) - 1)]_t^1$$

$$= \lim_{t \to 0^+} (-1 - t(\ln(t) - 1))$$

$$= -1 - \lim_{t \to 0^+} t(\ln(t) - 1)$$

$$= -1 - \lim_{t \to 0^+} \frac{\ln(t) - 1}{\frac{1}{t}}$$

$$\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} -1 - \lim_{t \to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}}$$

$$= -1 - \lim_{t \to 0^+} (-t)$$
$$= -1 - 0$$
$$= \boxed{-1}.$$

(k) $\displaystyle\int_{-2}^{1} \frac{dx}{\sqrt[3]{3x-2}}$

Solution: This is a type II improper integral due to the vertical asymptote of $y = \frac{1}{\sqrt[3]{3x-2}}$ at $x = \frac{2}{3}$. Because the vertical asymptote is in the interior of the interval of integration, we need to break-up the integral into a sum of two integrals and compute each of them as a limit. We get

$$\int_{-2}^{1} \frac{dx}{\sqrt[3]{3x-2}} = \int_{-2}^{2/3} \frac{dx}{\sqrt[3]{3x-2}} + \int_{2/3}^{1} \frac{dx}{\sqrt[3]{3x-2}}$$
$$= \lim_{b \to \frac{2}{3}^-} \int_{-2}^{b} \frac{dx}{\sqrt[3]{3x-2}} + \lim_{a \to \frac{2}{3}^+} \int_{a}^{1} \frac{dx}{\sqrt[3]{3x-2}}$$
$$= \lim_{b \to \frac{2}{3}^-} \left[\frac{(3x-2)^{2/3}}{2}\right]_{-2}^{b} + \lim_{a \to \frac{2}{3}^+} \left[\frac{(3x-2)^{2/3}}{2}\right]_{a}^{1}$$
$$= \lim_{b \to \frac{2}{3}^-} \left(\frac{(3b-2)^{2/3}}{2} - 2\right) + \lim_{a \to \frac{2}{3}^+} \left(1 - \frac{(3a-2)^{2/3}}{2}\right)$$
$$= (0 - 2) + (1 - 0)$$
$$= \boxed{-1}.$$

(l) $\displaystyle\int_{0}^{3/2} \frac{dx}{\sqrt{9-4x^2}}$

Solution: This is a type II improper integral due to the vertical asymptote of $y = \frac{1}{\sqrt{9-4x^2}}$ at $x = \frac{3}{2}$.

$$\int_{0}^{3/2} \frac{dx}{\sqrt{9-4x^2}} = \lim_{t \to \frac{3}{2}^-} \int_{0}^{t} \frac{dx}{\sqrt{9-4x^2}}$$
$$= \lim_{t \to \frac{3}{2}^-} \left[\frac{1}{2} \sin^{-1}\left(\frac{2x}{3}\right)\right]_{0}^{t}$$
$$= \lim_{t \to \frac{3}{2}^-} \left(\frac{1}{2} \sin^{-1}\left(\frac{2t}{3}\right) - 0\right)$$

$$= \frac{1}{2}\sin^{-1}(1)$$

$$= \boxed{\frac{\pi}{4}}.$$

(m) $\displaystyle\int_e^\infty \frac{dx}{x\ln(x)}$

Solution: We use the substitution $u = \ln(x)$, $du = \frac{dx}{x}$ to compute the antiderivative.

$$\int \frac{dx}{x\ln(x)} = \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|\ln(x)| + C.$$

We can now use this antiderivative to compute the improper integral.

$$\int_e^\infty \frac{dx}{x\ln(x)} = \lim_{t\to\infty} \int_e^t \frac{dx}{x\ln(x)}$$

$$= \lim_{t\to\infty} \left[\ln|\ln(x)|\right]_e^t$$

$$= \lim_{t\to\infty} \left(\ln|\ln(t)| - \ln|\ln(e)|\right)$$

$$= \infty$$

since $\ln(t) \to \infty$ when $t \to \infty$. Therefore $\boxed{\displaystyle\int_e^\infty \frac{dx}{x\ln(x)} \text{ diverges}}$.

(n) $\displaystyle\int_7^{14} \frac{dx}{\sqrt{14x - x^2}}$

Solution: This integral is a type II improper integral because the upper bound of integration $x = 14$ is a vertical asymptote of the integrand. We start by finding an antiderivative by completing the square, which gives

$$14x - x^2 = 49 - (49 - 14x + x^2) = 49 - (x - 7)^2.$$

So

$$\int \frac{dx}{\sqrt{14x - x^2}} = \frac{dx}{\sqrt{49 - (x - 7)^2}} = \sin^{-1}\left(\frac{x - 7}{7}\right) + C.$$

Hence

$$\int_7^{14} \frac{dx}{\sqrt{14x - x^2}} = \lim_{t\to 14^-} \int_7^t \frac{dx}{\sqrt{14x - x^2}}$$

$$= \lim_{t \to 14^-} \left[\sin^{-1}\left(\frac{x-7}{7} \right) \right]_7^t$$

$$= \lim_{t \to 14^-} \left(\sin^{-1}\left(\frac{t-7}{7} \right) - 0 \right)$$

$$= \sin^{-1}(1)$$

$$= \boxed{\frac{\pi}{2}}.$$

(o) $\displaystyle\int_0^\infty e^{-x} \sin(x)dx$

Solution: We start by computing an antiderivative, using two successive integration by parts. For the first IBP, the parts are

$$u = \sin(x) \;\Rightarrow\; du = \cos(x)dx,$$
$$dv = e^{-x}dx \;\Rightarrow\; v = -e^{-x}.$$

This gives

$$\int e^{-x} \sin(x)dx = -e^{-x}\sin(x) - \int -e^{-x}\cos(x)dx$$

$$= -e^{-x}\sin(x) + \int e^{-x}\cos(x)dx.$$

The second IBP uses the parts

$$u = \cos(x) \;\Rightarrow\; du = -\sin(x)dx,$$
$$dv = e^{-x}dx \;\Rightarrow\; v = -e^{-x}.$$

We get

$$\int e^{-x} \sin(x)dx = -e^{-x}\sin(x) - e^{-x}\cos(x) - \int (-e^{-x})(-\sin(x))dx$$

$$= -e^{-x}\sin(x) - e^{-x}\cos(x) - \int e^{-x}\sin(x)dx.$$

We can solve this relation for the unknown integral by moving all terms $\displaystyle\int e^{-x}\sin(x)dx$ to the left-hand side and we get

$$2\int e^{-x}\sin(x)dx = -e^{-x}\sin(x) - e^{-x}\cos(x)$$

$$\Rightarrow \int e^{-x}\sin(x)dx = -\frac{e^{-x}(\sin(x)+\cos(x))}{2} + C$$

We can now compute the improper integral.

$$\int_0^\infty e^{-x}\sin(x)dx = \lim_{t\to\infty}\int_0^t e^{-x}\sin(x)dx$$

$$= \lim_{t\to\infty}\left[-\frac{e^{-x}(\sin(x)+\cos(x))}{2}\right]_0^t$$

$$= \lim_{t\to\infty}\left(-\frac{e^{-b}(\sin(t)+\cos(t))}{2}+\frac{1}{2}\right)$$

$$= \frac{1}{2}-\lim_{t\to\infty}\frac{\sin(t)+\cos(t)}{e^t}.$$

This last limit can be computed using the Sandwich Theorem. Because $-1\leqslant\sin(t)\leqslant 1$ and $-1\leqslant\cos(t)\leqslant 1$, we have the inequalities

$$-2\leqslant\sin(t)+\cos(t)\leqslant 2$$

$$\Rightarrow\quad -\frac{2}{e^t}\leqslant\frac{\sin(t)+\cos(t)}{e^t}\leqslant\frac{2}{e^t}$$

Since $\displaystyle\lim_{t\to\infty}\frac{2}{e^t}=0=\lim_{t\to\infty}-\frac{2}{e^t}$, it follows by the Squeeze Theorem that

$$\lim_{t\to\infty}\frac{\sin(t)+\cos(t)}{e^t}=0.$$

Hence

$$\boxed{\int_0^\infty e^{-x}\sin(x)dx = \frac{1}{2}.}$$

(p) $\displaystyle\int_0^\infty\frac{dx}{(x^2+16)^2}$

Solution: We can set-up the improper integral as a limit as follows:

$$\int_0^\infty\frac{dx}{(x^2+16)^2}=\lim_{t\to\infty}\int_0^t\frac{dx}{(x^2+16)^2}.$$

We can compute the proper integral using the trigonometric substitution $x=4\tan(\theta)$, which gives $dx=4\sec^2(\theta)d\theta$. The bounds of the integral become

$$x=0\ \Rightarrow\ \theta=\tan^{-1}\left(\frac{0}{4}\right)=0,$$

$$x=t\ \Rightarrow\ \theta=\tan^{-1}\left(\frac{t}{4}\right).$$

So

$$\int_0^\infty \frac{dx}{(x^2+16)^2} = \lim_{t\to\infty} \int_0^{\tan^{-1}(t/4)} \frac{4\sec^2(\theta)d\theta}{(16\tan^2(\theta)+16)^2}$$

$$= \lim_{t\to\infty} \int_0^{\tan^{-1}(t/4)} \frac{4\sec^2(\theta)d\theta}{(16\sec^2(\theta))^2}$$

$$= \lim_{t\to\infty} \frac{1}{64} \int_0^{\tan^{-1}(t/4)} \cos^2(\theta)d\theta$$

$$= \frac{1}{64} \int_0^{\pi/2} \cos^2(\theta)d\theta$$

since $\tan^{-1}\left(\frac{t}{4}\right) \to \frac{\pi}{2}$ as $t \to \infty$. We can compute this last integral with a double-angle formula to obtain

$$\int_0^\infty \frac{dx}{(x^2+16)^2} = \frac{1}{64} \int_0^{\pi/2} \frac{1+\cos(2\theta)}{2} d\theta$$

$$= \frac{1}{64} \left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4} \right]_0^{\pi/2}$$

$$= \boxed{\frac{\pi}{256}}.$$

(q) $\displaystyle\int_0^1 \frac{\ln(x)}{\sqrt[3]{x}} dx$

Solution: Note that this ia type II improper integral due to the vertical asymptote of the integrand at $x = 0$. We can start by finding an antiderivative using an IBP with parts

$$u = \ln(x) \;\Rightarrow\; du = \frac{dx}{x},$$

$$dv = \frac{dx}{\sqrt[3]{x}} \;\Rightarrow\; v = \frac{3x^{2/3}}{2}.$$

We get

$$\int \frac{\ln(x)}{\sqrt[3]{x}} dx = \frac{3x^{2/3}\ln(x)}{2} - \int \frac{3x^{2/3}}{2} \cdot \frac{dx}{x}$$

$$= \frac{3x^{2/3}\ln(x)}{2} - \frac{3}{2} \int \frac{dx}{x^{1/3}}$$

$$= \frac{3x^{2/3}\ln(x)}{2} - \frac{3}{2} \cdot \frac{3x^{2/3}}{2} + C$$

$$= \frac{3x^{2/3}}{4} (2\ln(x) - 3) + C.$$

We can now use this to compute the improper integral, as follows:

$$\int_0^1 \frac{\ln(x)}{\sqrt[3]{x}}dx = \lim_{t\to 0}\int_t^1 \frac{\ln(x)}{\sqrt[3]{x}}dx$$

$$= \lim_{t\to 0^+}\left[\frac{3x^{2/3}}{4}\left(2\ln(x)-3\right)\right]_t^1$$

$$= \lim_{t\to 0^+}\left(-\frac{9}{4}-\frac{3t^{2/3}}{4}\left(2\ln(t)-3\right)\right)$$

$$= -\frac{9}{4}-\frac{3}{4}\lim_{t\to 0^+}\frac{2\ln(t)-3}{t^{-2/3}}$$

$$\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} -\frac{9}{4}-\frac{3}{4}\lim_{t\to 0^+}\frac{\frac{2}{t}}{-\frac{2}{3}t^{-5/3}}$$

$$= -\frac{9}{4}-\frac{3}{4}\lim_{t\to 0^+}3t^{2/3}$$

$$= \boxed{-\frac{9}{4}}.$$

(r) $\displaystyle\int_0^\infty e^{-\sqrt{z}}dz$

Solution: We can start by computing an antiderivative using a substitution followed by an IBP. For the substitution, we will let $w = \sqrt{z}$, so that $dw = \frac{dz}{2\sqrt{z}}$, or $dz = 2w\,dw$. This gives

$$\int e^{-\sqrt{z}}dz = 2\int we^{-w}dw.$$

This last integral can be computed using an IBP with $u = w$, $du = dw$ and $dv = e^{-w}dw$, $v = -e^{-w}$. We obtain

$$\int e^{-\sqrt{z}}dz = 2\left(-we^{-w}-\int(-e^{-w})dw\right)$$

$$= 2\left(-we^{-w}-e^{-w}\right)+C$$

$$= -2e^{-w}(w+1)+C$$

$$= -2e^{-\sqrt{z}}(\sqrt{z}+1)+C.$$

We now use this for the improper integral. We obtain

$$\int_0^\infty e^{-\sqrt{z}}dz = \lim_{t\to\infty}\int_0^t e^{-\sqrt{z}}dz$$

$$= \lim_{t\to\infty}\left[-2e^{-\sqrt{z}}(\sqrt{z}+1)\right]_0^t$$

$$= \lim_{t \to \infty} \left(2 - 2e^{-\sqrt{t}}(\sqrt{t} + 1) \right)$$

$$= 2 - 2 \lim_{t \to \infty} \frac{\sqrt{t} + 1}{e^{\sqrt{t}}}$$

$$\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} 2 - 2 \lim_{t \to \infty} \frac{\frac{1}{2\sqrt{t}}}{\frac{1}{2\sqrt{t}} e^{\sqrt{t}}}$$

$$= 2 - 2 \lim_{t \to \infty} \frac{1}{e^{\sqrt{t}}}$$

$$= \boxed{2}.$$

#53. Consider the unbounded region $\mathcal{R}$ between the graph of $y = \dfrac{\ln(x)}{x}$ and the x-axis for $x \geqslant 1$.

    (a) Find the area of the region $\mathcal{R}$ or determine if $\mathcal{R}$ has infinite area.

Solution: The area of $\mathcal{R}$ is given by

$$A = \int_1^\infty \frac{\ln(x)}{x} \, dx.$$

The antiderivative of the integrand can be found with the substitution $u = \ln(x)$, $du = \frac{dx}{x}$, which gives

$$\int \frac{\ln(x)}{x} \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{(\ln(x))^2}{2} + C.$$

We can use this to compute the area, as follows

$$A = \int_1^\infty \frac{\ln(x)}{x} \, dx$$

$$= \lim_{t \to \infty} \int_1^t \frac{\ln(x)}{x} \, dx$$

$$= \lim_{t \to \infty} \left[\frac{(\ln(x))^2}{2} \right]_1^t$$

$$= \lim_{t \to \infty} \frac{(\ln(t))^2}{2}$$

$$= \infty.$$

So $\boxed{\mathcal{R} \text{ has infinite area}}$.

    (b) We now revolve the region $\mathcal{R}$ about the x-axis to form a solid of revolution. Calculate the volume of the solid or determine if the solid has infinite volume.

Solution: Revolving the vertical strip at x in the region about the x-axis forms a disk of radius $r(x) = \frac{\ln(x)}{x}$. So the volume is given by

$$V = \int_1^\infty \pi r(x)^2 dx = \pi \int_1^\infty \frac{(\ln(x))^2}{x^2} dx.$$

To compute the antiderivative of the integrand, we use two successive IPBs. The first one will use the parts

$$u = (\ln(x))^2 \implies du = \frac{2\ln(x)dx}{x},$$

$$dv = \frac{dx}{x^2} \implies v = -\frac{1}{x}.$$

This gives

$$\int \frac{(\ln(x))^2}{x^2} dx = -\frac{(\ln(x))^2}{x} - \int \frac{2\ln(x)}{x}\left(-\frac{1}{x}\right) dx$$

$$= -\frac{(\ln(x))^2}{x} + 2\int \frac{\ln(x)}{x^2} dx.$$

For the second IBP, we take

$$u = \ln(x) \implies du = \frac{dx}{x},$$

$$dv = \frac{dx}{x^2} \implies v = -\frac{1}{x}.$$

We obtain

$$\int \frac{(\ln(x))^2}{x^2} dx = -\frac{(\ln(x))^2}{x} + 2\left(-\frac{\ln(x)}{x} - \int \frac{1}{x}\left(-\frac{1}{x}\right) dx\right)$$

$$= -\frac{(\ln(x))^2}{x} - 2\frac{\ln(x)}{x} + 2\int \frac{1}{x^2} dx$$

$$= -\frac{(\ln(x))^2}{x} - 2\frac{\ln(x)}{x} - \frac{2}{x} + C$$

$$= -\frac{(\ln(x))^2 + 2\ln(x) + 2}{x} + C.$$

We can now use this to compute the volume.

$$V = \pi \int_1^\infty \frac{(\ln(x))^2}{x^2} dx$$

$$= \pi \lim_{t \to \infty} \int_1^t \frac{(\ln(x))^2}{x^2} dx$$

$$= \pi \lim_{t \to \infty} \left[-\frac{(\ln(x))^2 + 2\ln(x) + 2}{x} \right]_1^t$$

$$= \pi \lim_{t \to \infty} \left(2 - \frac{(\ln(t))^2 + 2\ln(t) + 2}{t} \right)$$

$$= \pi \left(2 - \lim_{t \to \infty} \frac{(\ln(t))^2 + 2\ln(t) + 2}{t} \right).$$

To compute the remaining limit, we use L'Hôpital's Rule twice for the indeterminate form $\frac{\infty}{\infty}$.

$$V \overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \pi \left(2 - \lim_{t \to \infty} \frac{2\frac{\ln(t)}{t} + \frac{2}{t}}{1} \right)$$

$$= \pi \left(2 - \lim_{t \to \infty} \frac{2\ln(t) + 2}{t} \right)$$

$$\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \pi \left(2 - \lim_{t \to \infty} \frac{\frac{2}{t}}{1} \right)$$

$$= \pi (2 - 0)$$

$$= \boxed{2\pi \text{ cubic units}}.$$

(c) We now revolve the region $\mathcal{R}$ about the y-axis to form a solid of revolution. Calculate the volume of the solid or determine if the solid has infinite volume.

Solution: Revolving the vertical strip at x about the y-axis forms a shell with radius $r(x) = x$ and height $h(x) = \frac{\ln(x)}{x}$. So the volume is

$$V = \int_1^\infty 2\pi r(x) h(x) dx = \int_1^\infty 2\pi x \frac{\ln(x)}{x} dx = 2\pi \int_1^\infty \ln(x) dx.$$

We have previously computed the antiderivative of the integrand using integration by parts and found that
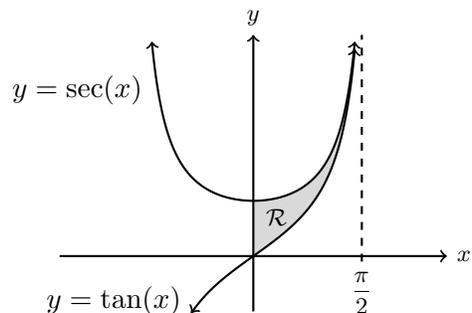
$$\int \ln(x) dx = x(\ln(x) - 1) + C.$$

So the volume is

$$V = 2\pi \lim_{t \to \infty} \int_1^t \ln(x) dx$$

$$= 2\pi \lim_{t \to \infty} [x(\ln(x) - 1)]_1^t$$

$$= 2\pi \lim_{t \to \infty} t(\ln(t) - 1)$$

$$= \infty,$$

so the solid has infinite volume.

#54. Find the area of the unbounded region $\mathcal{R}$ trapped between the graphs of $y = \sec(x)$ and $y = \tan(x)$ for $0 \leqslant x < \frac{\pi}{2}$ (see graph to the right) or show that $\mathcal{R}$ has infinite area.

Solution: The area of the region is computed by the integral

$$A = \int_0^{\pi/2} (\sec(x) - \tan(x))dx.$$

This is an improper integral since the integrand is discontinuous at $x = \frac{\pi}{2}$. So

$$
\begin{aligned}
A &= \lim_{t \to \pi/2^-} \int_0^t (\sec(x) - \tan(x))dx \\
&= \lim_{t \to \pi/2^-} [\ln|\sec(x) + \tan(x)| - \ln|\sec(x)|]_0^t \\
&= \lim_{t \to \pi/2^-} \left[\ln\left|\frac{\sec(x) + \tan(x)}{\sec(x)}\right|\right]_0^t \\
&= \lim_{t \to \pi/2^-} [\ln|1 + \sin(x)|]_0^t \\
&= \lim_{t \to \pi/2^-} \ln(1 + \sin(t)) \\
&= \boxed{\ln(2) \text{ square units}}.
\end{aligned}
$$

Remark: we cannot compute the area as

$$\int_0^{\pi/2} \sec(x)dx - \int_0^{\pi/2} \tan(x)dx$$

because both of these integrals diverge. In other words, the regions under the graphs of $y = \sec(x)$ and $y = \tan(x)$ on $[0, \frac{\pi}{2})$ have infinite area, but the region *between* the two curves has finite area.

#55. For the region $\mathcal{R}$ defined in the previous problem, we consider the solid obtained by revolving $\mathcal{R}$ about the x-axis. Find the volume of the solid or determine if the solid has infinite volume.

Solution: We use the washer method. Revolving the vertical strip at x about the x-axis creates a washer with inner radius $r_{\text{in}}(x) = \tan(x)$ and outer radius $r_{\text{out}}(x) = \sec(x)$. So the volume is

$$V = \int_0^{\pi/2} \pi \left(r_{\text{out}}(x)^2 - r_{\text{in}}(x)^2 \right) dx = \int_0^{\pi/2} \pi \left(\sec^2(x) - \tan^2(x) \right) dx.$$

This integral is a type II improper integral because the integrand is discontinuous at $x = \frac{\pi}{2}$. However, we can compute using limits by observing that the Pythagorean identity $\sec^2(x) = \tan^2(x) + 1$ allows us to simplify the integrand as

$$\sec^2(x) - \tan^2(x) = (\tan^2(x) + 1) - \tan^2(x) = 1,$$

which is continuous on $\left[0, \frac{\pi}{2} \right]$. Hence

$$V = \int_0^{\pi/2} \pi \, dx$$
$$= \pi \, [x]_0^{\pi/2}$$
$$= \boxed{\frac{\pi^2}{2} \text{ cubic units}}.$$

#56. Use a convergence test to determine if the following improper integrals converge or diverge.

(a) $\displaystyle \int_3^\infty \frac{dx}{xe^x}$

Solution: We use the DCT. Observe that for x in $[3, \infty)$, we have

$$0 \leqslant e^x \leqslant xe^x$$
$$\Rightarrow \quad 0 \leqslant \frac{1}{xe^x} \leqslant \frac{1}{e^x}.$$

Furthermore, $\displaystyle \int_3^\infty \frac{dx}{e^x}$ converges since

$$\int_3^\infty \frac{dx}{e^x} = \lim_{b \to \infty} \int_3^b e^{-x} dx$$
$$= \lim_{b \to \infty} \left(-e^{-b} + e^3 \right)$$
$$= e^3.$$

Thus, $\boxed{\displaystyle\int_3^\infty \frac{dx}{xe^x} \text{ converges}}$ as well.

Remark: the inequality

$$0 \leqslant \frac{1}{xe^x} \leqslant \frac{1}{x}$$

is also true, but it does not help establish the convergence of $\displaystyle\int_3^\infty \frac{dx}{xe^x}$ since the integral of the "bigger function" $\displaystyle\int_3^\infty \frac{dx}{x}$ diverges (type I p-integral with $p = 1$).

(b) $\displaystyle\int_1^\infty \frac{dx}{x^2 + 3x + 1}$

Solution: We use the DCT. Observe that for x in $[1, \infty)$ we have the inequalities

$$0 \leqslant x^2 \leqslant x^2 + 3x + 1$$
$$\Rightarrow \quad 0 \leqslant \frac{1}{x^2 + 3x + 1} \leqslant \frac{1}{x^2}.$$

Furthermore, the integral $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges since it is a type I p-integral with $p = 2 > 1$. Therefore, $\boxed{\displaystyle\int_1^\infty \frac{dx}{x^2 + 3x + 1} \text{ converges}}$ as well.

Remark. We can also use the LCT, observing that

$$\lim_{x \to \infty} \frac{\frac{1}{x^2 + 3x + 1}}{\frac{1}{x^2}} = \lim_{x \to \infty} \frac{1}{1 + 3/x + 1/x^2} = 1 > 0$$

and the integral $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges since it is a type I p-integral with $p = 2 > 1$.

(c) $\displaystyle\int_4^\infty \frac{\cos(x) + 5}{x^{3/5}}\,dx$

Solution: We use the DCT, observing that for x in $[4, \infty)$ we have

$$-1 \leqslant \cos(x)$$
$$\Rightarrow \quad 0 \leqslant 4 \leqslant \cos(x) + 5$$
$$\Rightarrow \quad 0 \leqslant \frac{4}{x^{3/5}} \leqslant \frac{\cos(x) + 5}{x^{3/5}}$$

Furthermore, the integral $\displaystyle\int_4^\infty \frac{4}{x^{3/5}}\,dx$ diverges since it is a type I p-integral with $p = \frac{3}{5} \leqslant 1$. It

follows that $\boxed{\displaystyle\int_4^\infty \frac{\cos(x)+5}{x^{3/5}}\,dx \text{ diverges}}$ as well.

Remark: we would not be able to use the LCT to compare with the divergent p-integral integral $\displaystyle\int_4^\infty \frac{dx}{x^{3/5}}$ since

$$\lim_{x\to\infty} \frac{\frac{\cos(x)+5}{x^{3/5}}}{\frac{1}{x^{3/5}}} = \lim_{x\to\infty} \left(\cos(x)+5\right) \text{ does not exist.}$$

(d) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}+x^2}$

Solution: We use the DCT, observing that for x in $(0,1]$ we have

$$0 \leqslant \sqrt{x} \leqslant \sqrt{x}+x^2$$
$$\Rightarrow \quad 0 \leqslant \frac{1}{x^2+\sqrt{x}} \leqslant \frac{1}{\sqrt{x}}$$

Furthermore, the integral $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ converges since it is a type II p-integral with $p=\frac{1}{2}<1$. It

follows that $\boxed{\displaystyle\int_0^1 \frac{dx}{x^2+\sqrt{x}} \text{ converges}}$ as well.

Remark 1. The inequality

$$0 \leqslant \frac{1}{x^2+\sqrt{x}} \leqslant \frac{1}{x^2}$$

is also true, but it does not help establish the convergence of $\displaystyle\int_0^1 \frac{dx}{x^2+\sqrt{x}}$ since the integral of

the "bigger function" $\displaystyle\int_0^1 \frac{dx}{x^2}$ diverges (type II p-integral with $p=2 \geqslant 1$).

Remark 2. We could have also used the LCT to compare with the convergent type II p-integral $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$, remarking that

$$\lim_{x\to 0^+} \frac{\frac{1}{x^2+\sqrt{x}}}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0^+} \frac{\sqrt{x}}{x^2+\sqrt{x}}$$
$$= \lim_{x\to 0^+} \frac{\sqrt{x}}{x^2+\sqrt{x}} \cdot \frac{\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x}}}$$
$$= \lim_{x\to 0^+} \frac{1}{x^{3/2}+1}$$
$$= 1 > 0.$$

(e) $\displaystyle\int_5^\infty \frac{xdx}{x^4-1}$

Solution: We use the LCT, comparing with $\frac{1}{x^3}$. We have

$$\lim_{x\to\infty} \frac{\frac{x}{x^4-1}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{x^4}{x^4-1}$$

$$= \lim_{x\to\infty} \frac{x^4}{x^4-1} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}}$$

$$= \lim_{x\to\infty} \frac{1}{1-\frac{1}{x^4}}$$

$$= 1 > 0.$$

Furthermore, the integral $\displaystyle\int_5^\infty \frac{dx}{x^3}$ converges since it is a type I p-integral with $p = 3 > 1$. Hence, $\boxed{\displaystyle\int_5^\infty \frac{xdx}{x^4-1} \text{ converges}}$ as well.

Remark. The DCT cannot be used to compare with the convergent type I p-integral $\displaystyle\int_5^\infty \frac{dx}{x^3}$ since we have the inequalities

$$0 \leqslant x^4 - 1 \leqslant x^4$$

$$\Rightarrow \quad 0 \leqslant \frac{1}{x^4} \leqslant \frac{1}{x^4-1}$$

$$\Rightarrow \quad 0 \leqslant \frac{1}{x^3} \leqslant \frac{x}{x^4-1}$$

and knowing that the integral of the "smaller function" converges does not say anything about the integral of the "bigger function".

(f) $\displaystyle\int_1^\infty \frac{x^3+5x^2+1}{\sqrt{x^7+4x+2}}dx$

Solution: We use the LCT. To find a good function to compare to, we keep the terms of the numerator and denominator that are dominant when $x \to \infty$:

$$\frac{x^3+5x^2+1}{\sqrt{x^7+4x+2}} \sim \frac{x^3}{\sqrt{x^7}} = \frac{x^3}{x^{7/2}} = \frac{1}{x^{1/2}}.$$

Now that we have found our reference function, we properly establish the limit comparison.

$$\lim_{x\to\infty} \frac{\frac{x^3+5x^2+1}{\sqrt{x^7+4x+2}}}{\frac{1}{x^{1/2}}} = \lim_{x\to\infty} \frac{x^{7/2}+5x^{5/2}+x^{1/2}}{\sqrt{x^7+4x+2}}$$

$$= \lim_{x \to \infty} \frac{x^{7/2} + 5x^{5/2} + x^{1/2}}{\sqrt{x^7 + 4x + 2}} \cdot \frac{\frac{1}{x^{7/2}}}{\frac{1}{x^{7/2}}}$$

$$= \lim_{x \to \infty} \frac{1 + \frac{5}{x^2} + \frac{1}{x^3}}{\sqrt{1 + \frac{4}{x^6} + \frac{2}{x^7}}}$$

$$= 1 > 0.$$

We also know that the integral $\displaystyle\int_1^\infty \frac{dx}{x^{1/2}}$ diverges since it is a type I p-integral with $p = \frac{1}{2} \leqslant 1$.

Therefore, $\boxed{\displaystyle\int_1^\infty \frac{x^3 + 5x^2 + 1}{\sqrt{x^7 + 4x + 2}} dx \text{ diverges}}$ as well.