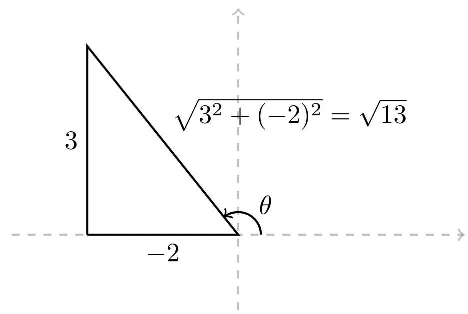
Final Exam Practice Session Solutions

1. Suppose $\tan(\theta) = -\frac{3}{2}$ and $\frac{\pi}{2} < \theta < \pi$. Evaluate the following.

   Solution. We use a right triangle in the second quadrant.



   (a) $\cos(\theta) = \boxed{-\frac{2}{\sqrt{13}}}$

   (b) $\sin(\theta) = \boxed{\frac{3}{\sqrt{13}}}$

   (c) $\cos(2\theta) = 2\cos^2(\theta) - 1 = \boxed{-\frac{5}{13}}$

   (d) $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = \boxed{-\frac{12}{13}}$

2. Calculate the following limits. You may use any valid method.

   (a) $\lim_{x\to 0} \frac{x^3 - 5x^2}{3^{x^2} - 1}$

   Solution. This limit is a $\frac{0}{0}$ indeterminate form that requires using L'Hôpital's Rule twice.

$$\begin{aligned}
\lim_{x\to 0} \frac{x^3 - 5x^2}{3^{x^2} - 1} &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to 0} \frac{3x^2 - 10x}{2\ln(3)x3^{x^2}} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to 0} \frac{6x - 10}{2\ln(3)3^{x^2} + 4(\ln(3))^2x^2e^{x^2}} \\
&= \frac{0 - 10}{2\ln(3) + 0} \\
&= \boxed{-\frac{5}{\ln(3)}}
\end{aligned}$$

   (b) $\lim_{x\to\infty} xe^{-\sqrt{x}}$

   Solution. This limit is a $\infty \cdot 0$ indeterminate form. We first write the expression as a fraction, then use L'Hôpital's Rule twice.

$$\begin{aligned}
\lim_{x\to\infty} xe^{-\sqrt{x}} &= \lim_{x\to\infty} \frac{x}{e^{\sqrt{x}}} \\
&\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{1}{\frac{1}{2\sqrt{x}}e^{\sqrt{x}}}
\end{aligned}$$

$$= \lim_{x\to\infty} \frac{2\sqrt{x}}{e^{\sqrt{x}}}$$

$$\overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}}{\frac{1}{2\sqrt{x}}e^{\sqrt{x}}}$$

$$= \lim_{x\to\infty} \frac{2}{e^{\sqrt{x}}}$$

$$= \boxed{0}$$

(c) $\displaystyle\lim_{x\to 4} \frac{\frac{2}{\sqrt{x}} - 1}{x-4}$

Solution 1. Using algebra.

$$\begin{aligned}
\lim_{x\to 4} \frac{\frac{2}{\sqrt{x}} - 1}{x-4} &= \lim_{x\to 4} \frac{\frac{2-\sqrt{x}}{\sqrt{x}}}{x-4} \\
&= \lim_{x\to 4} \frac{2-\sqrt{x}}{\sqrt{x}(x-4)} \cdot \frac{2+\sqrt{x}}{2+\sqrt{x}} \\
&= \lim_{x\to 4} \frac{4-x}{\sqrt{x}(x-4)(2+\sqrt{x})} \\
&= \lim_{x\to 4} -\frac{1}{\sqrt{x}(2+\sqrt{x})} \\
&= -\frac{1}{\sqrt{4}(2+\sqrt{4})} \\
&= \boxed{-\frac{1}{8}}
\end{aligned}$$

Solution 2. Using L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 4} \frac{\frac{2}{\sqrt{x}} - 1}{x-4} &\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to 4} \frac{-\frac{1}{x^{3/2}}}{1} \\
&= -\frac{1}{4^{3/2}} \\
&= \boxed{-\frac{1}{8}}
\end{aligned}$$

(d) $\displaystyle\lim_{x\to 0} \left(\cos(3x) + \tan(5x)\right)^{1/x}$

Solution. This limit is an indeterminate power 1^∞. **Warning:** limits of the form 1^∞ need not be equal to 1! This is because the base is not equal to 1, it is *approaching* 1. We can resolve the indeterminate form by taking the ln of the limit L and applying L'Hôpital's Rule. This gives:

$$\begin{aligned}
\ln(L) &= \lim_{x\to 0} \frac{\ln\left(\cos(3x) + \tan(5x)\right)}{x} \\
&\overset{\text{L'H}}{\underset{\frac{0}{0}}{=}} \lim_{x\to 0} \frac{\frac{-3\sin(3x)+5\sec^2(5x)}{\cos(3x)+\tan(5x)}}{1}
\end{aligned}$$

$$= \frac{-3\sin(0) + 5\sec^2(0)}{\cos(0) + \tan(0)}$$
$$= 5.$$

This is the ln of the original limit, so we now solve for L and we get $\boxed{L = e^5}$.

3. Calculate $\frac{dy}{dx}$ for the following curves. You do not have to simplify your answers.

(a) $y = \sqrt{9x^2 + 16x + 4}$

Solution. $\boxed{\frac{dy}{dx} = \frac{1}{2\sqrt{9x^2+16x+4}}(18x+16)}$

(b) $y = \frac{\sin(13x)}{(2x+5)^{10}}$

Solution. $\boxed{\frac{dy}{dx} = \frac{13\cos(13x)(2x+5)^{10} - \sin(13x)\cdot 10(2x+5)^9 \cdot 2}{(2x+5)^{20}}}$

(c) $y = \sin^{-1}\left(\sqrt[4]{x}\right)$

Solution. $\frac{dy}{dx} = \frac{1}{\sqrt{1 - (\sqrt[4]{x})^2}} \cdot \frac{x^{-3/4}}{4}$

(d) $y = x^{\arctan(2x)}$

Solution. Since both the base and the exponent depend on x, we will need to use logarithmic differentiation.

$$\ln(y) = \arctan(2x)\ln(x)$$
$$\Rightarrow \frac{y'}{y} = \frac{2\ln(x)}{1+4x^2} + \frac{\arctan(2x)}{x}$$
$$\Rightarrow y' = y\left(\frac{2\ln(x)}{1+4x^2} + \frac{\arctan(2x)}{x}\right)$$
$$\boxed{\frac{dy}{dx} = x^{\arctan(2x)}\left(\frac{2\ln(x)}{1+4x^2} + \frac{\arctan(2x)}{x}\right)}$$

(e) $y = \int_{\ln(x)}^{\pi} \cos(t^3)dt$

Solution. $\boxed{\frac{dy}{dx} = -\cos\left((\ln(x))^3\right)\cdot\frac{1}{x}}$

(f) $y = \tan(xe^{7x})$

Solution. $\boxed{\frac{dy}{dx} = \sec^2(xe^{7x})\left(e^{7x} + 7xe^{7x}\right)}$

4. Calculate the following integrals.

(a) $\int \left(\frac{9}{x^3} - \frac{1}{\sqrt{1-25x^2}} \right) dx$

Solution.

$$\begin{aligned}\int \left(\frac{9}{x^3} - \frac{1}{\sqrt{1-25x^2}} \right) dx &= \int \left(9x^{-3} - \frac{1}{\sqrt{1-(5x)^2}} \right) dx \\ &= \boxed{\frac{9x^{-2}}{-2} - \frac{1}{5}\sin^{-1}(5x) + C}\end{aligned}$$

(b) $\int \tan^7(3\theta)\sec^2(3\theta)d\theta$

Solution. This integral can be evaluated with the substitution $u = \tan(3\theta)$, so $du = 3\sec^2(3\theta)d\theta$. We obtain

$$\begin{aligned}\int \tan^7(3\theta)\sec^2(3\theta)d\theta &= \int \frac{1}{3}u^7 du \\ &= \frac{u^8}{24} + C \\ &= \boxed{\frac{\tan^8(3\theta)}{24} + C}.\end{aligned}$$

(c) $\int_0^1 xe^{x^2}\cos\left(e^{x^2}\right) dx$

Solution. We use the substitution $u = e^{x^2}$, which gives $du = 2xe^{x^2}dx$. The bounds change as follows.

$$\begin{aligned}x = 0 &\Rightarrow u = e^{0^2} = 1, \\ x = 1 &\Rightarrow u = e^{1^2} = e.\end{aligned}$$

We get

$$\begin{aligned}\int_0^1 xe^{x^2}\cos\left(e^{x^2}\right) dx &= \int_1^e \frac{1}{2}\cos(u)\,du \\ &= \left[\frac{1}{2}\sin(u)\right]_1^e \\ &= \boxed{\frac{\sin(e) - \sin(1)}{2}}.\end{aligned}$$

(d) $\int \frac{6t+21}{t^2+7t+3}dt$

Solution. We use the substitution $u = t^2 + 7t + 3$, so $du = (2t+7)dt$. Then we have

$$\int \frac{6t+21}{t^2+7t+3}dt = \int \frac{3(2t+7)}{t^2+7t+3}dt$$

$$
\begin{aligned}
&= \int \frac{3}{u} du \\
&= 3 \ln |u| + C \\
&= \boxed{3 \ln \left|t^2 + 7t + 3\right| + C}.
\end{aligned}
$$

(e) $\int_0^6 x\sqrt{36 - x^2}dx$

Solution. We can use the substitution $u = 36 - x^2$, so that $du = -2xdx$. The bounds change as follows:

$$
\begin{aligned}
x = 0 \;&\Rightarrow\; u = 36 - 0 = 36, \\
x = 6 \;&\Rightarrow\; u = 36 - 36 = 0.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\int_0^6 x\sqrt{36 - x^2} &= \int_{36}^0 -\frac{1}{2}\sqrt{u}du \\
&= \left[\frac{1}{3}u^{3/2}\right]_{36}^0 \\
&= \frac{36^{3/2}}{3} \\
&= \boxed{72}.
\end{aligned}
$$

(f) $\int_0^6 \sqrt{36 - x^2}dx$

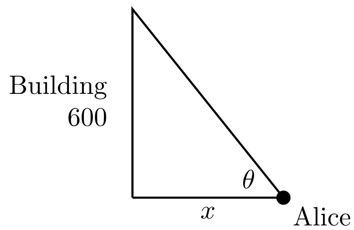
Solution. Substitution will be no use here because the derivative of the inside function $36 - x^2$ does not appear anywhere in the integrand. Instead, we can use geometry. The graph of the equation $y = \sqrt{36 - x^2}$ is a semi-circle of radius 6 centered at $(0, 0)$ and located above the x-axis. (This can be seen by squaring both sides and moving the x^2 to the left-hand side to obtain $x^2 + y^2 = 36$.) Therefore, this integral calculates the area of the region between the semi-circle and the x-axis on $[0, 6]$, which is a quarter of disk of radius 6. Hence,

$$
\int_0^6 \sqrt{36 - x^2}dx = \boxed{\frac{36\pi}{4}}.
$$

5. The two parts of this problem are independent.

(a) Alice walks at 4 ft/sec towards a building of height 600 ft. At what rate is the viewing angle between Alice and the top of the building changing when Alice is 200 ft away from the building?

Solution. We start by drawing a picture and naming the two relevant variables, which are the distance x between Alice and the building and the angle θ between the ground and the line of sight from Alice to the top of the building.



We can use right triangle trigonometry to get a relation between the variables. One possible option would be to use the relation $\cot(\theta) = \frac{x}{600}$. (Other correct relations will lead to the same final answer.) Differentiating this relation with respect to the time t gives

$$-\csc^2(\theta)\frac{d\theta}{dt} = \frac{1}{600}\frac{dx}{dt}.$$

We can now plug-in the known values $x = 200$ and $\dfrac{dx}{dt} = -4$, which gives

$$\begin{cases} \cot(\theta) = \frac{200}{600} = \frac{1}{3}, \\ -\csc^2(\theta)\frac{d\theta}{dt} = -\frac{4}{600} = -\frac{1}{150}. \end{cases}$$

To solve this for the unknown rate $\frac{d\theta}{dt}$, we'll need to first find $\csc^2(\theta)$. Using the Pythagorean identity, we have

$$\csc^2(\theta) = \cot^2(\theta) + 1 = \frac{1}{9} + 1 = \frac{10}{9}.$$

So we get

$$-\frac{10}{9}\frac{d\theta}{dt} = -\frac{1}{150} \quad \Rightarrow \quad \boxed{\frac{d\theta}{dt} = \frac{3}{500} \text{ rad/sec}}.$$

(b) A rectangular ice block with square base melts at a rate of 60 in^3/min. When the side length of the base is 4 in, the height is 6 in and decreases at a rate of 2 in/min. At what rate is the side length of the base decreasing at that time?

Solution. Call x the side length of the base of the block and h the height of the block. We have $V = x^2 h$. Differentiating this relation with respect to the time t gives

$$\frac{dV}{dt} = 2xh\frac{dx}{dt} + x^2\frac{dh}{dt}.$$

We can now plug-in the known values $\frac{dV}{dt} = -60$, $x = 4$, $h = 6$ and $\frac{dh}{dt} = -2$. This gives

$$-60 = 2 \cdot 4 \cdot 6\frac{dx}{dt} + 4^2 \cdot (-2) \quad \Rightarrow \quad \boxed{\frac{dx}{dt} = -\frac{7}{12} \text{ in/min}}.$$

6. A closed cylindrical box has volume 250π ft^3. Find the dimensions of the box (height and radius) that give the minimal possible surface area.

Solution. Call r the radius of the cylinder and h its height. The objective function is the surface area of the box $S = 2\pi r^2 + 2\pi rh$. The volume being 250π gives the constraint $\pi r^2 h = 250\pi$, so $h = \frac{250}{r^2}$. Therefore, we can write the surface area in terms of the variable r only as

$$S(r) = 2\pi r^2 + 2\pi r\frac{250}{r^2} = 2\pi\left(r^2 + \frac{250}{r}\right).$$

The interval of interest is $(0, \infty)$. We now find the absolute minimum of $S(r)$ on that interval. We have

$$S'(r) = 2\pi \left(2r - \frac{250}{r^2}\right).$$

The equation $S'(r) = 0$ gives $r^3 = 125$, so $r = 5$ is the only critical point of $S(r)$. We can test whether $S(r)$ has a local maximum or minimum at $r = 5$ using the second derivative test. We have

$$S''(r) = 2\pi \left(2 + \frac{500}{r^3}\right) > 0 \text{ on } (0, \infty).$$

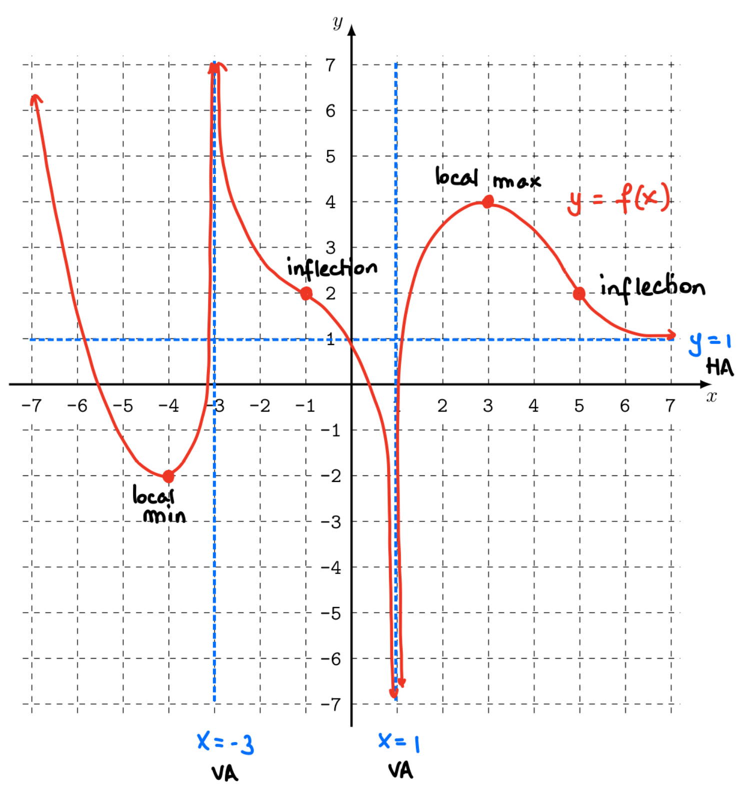
Hence, $S(r)$ is concave up on $(0, \infty)$, so there is a local minimum at $r = 5$. Since $r = 5$ is the only critical point of $S(r)$, it must be the location of the absolute minimum of $S(r)$. In conclusion, the box with the minimal possible surface area has radius $\boxed{r = 5 \text{ ft}}$ and height $\boxed{h = 10 \text{ ft}}$.

7. Sketch the graph of a function f with the following features. Label all asymptotes, local extrema and inflection points.

- The domain of f is $(-\infty, -3) \cup (-3, 1) \cup (1, \infty)$ and the lines $x = -3$ and $x = 1$ are asymptotes of f.
- $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = 1$.
- $f(-3) = -2$, $f(-1) = 2$, $f(3) = 4$ and $f(5) = 2$.
- The signs of f' and f'' are given by the following charts.

x	$(-\infty, -4)$	$(-4, -3)$	$(-3, -1)$	$(-1, 1)$	$(1, 3)$	$(3, 5)$	$(5, \infty)$
$f'(x)$	$-$	$+$	$-$	$-$	$+$	$-$	$-$
$f''(x)$	$+$	$+$	$+$	$-$	$-$	$-$	$+$

Solution.



8. Find all horizontal and vertical asymptotes of the function $f(x) = \dfrac{\sqrt{4x^2+9}+5x}{x+3}$.

Solution. To find potential vertical asymptotes, we set the denominator equal to 0. This gives $x+3=0$, or $x=-3$. Substituting this value in $f(x)$ gives the form $\frac{\text{non-zero number}}{0}$. It follows that $\boxed{x=-3 \text{ is the vertical asymptote of } f}$.

To find the horizontal asymptotes of f, we calculate the limits at ∞ and $-\infty$. We have

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{4x^2+9}+5x}{x+3} &= \lim_{x\to-\infty} \frac{\sqrt{x^2\left(4+\frac{9}{x^2}\right)}+5x}{x+3} \\
&= \lim_{x\to-\infty} \frac{|x|\sqrt{4+\frac{9}{x^2}}+5x}{x+3} \\
&= \lim_{x\to-\infty} \frac{-x\sqrt{4+\frac{9}{x^2}}+5x}{x+3} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \quad (x<0) \\
&= \lim_{x\to-\infty} \frac{-\sqrt{4+\frac{9}{x^2}}+5}{1+\frac{3}{x}} \\
&= \frac{-\sqrt{4+0}+5}{1+0} \\
&= 3,
\end{aligned}$$

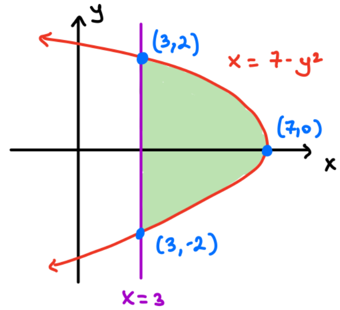
$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{4x^2+9}+5x}{x+3} &= \lim_{x\to-\infty} \frac{\sqrt{x^2\left(4+\frac{9}{x^2}\right)}+5x}{x+3} \\
&= \lim_{x\to-\infty} \frac{|x|\sqrt{4+\frac{9}{x^2}}+5x}{x+3} \\
&= \lim_{x\to-\infty} \frac{x\sqrt{4+\frac{9}{x^2}}+5x}{x+3} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \quad (x>0) \\
&= \lim_{x\to-\infty} \frac{\sqrt{4+\frac{9}{x^2}}+5}{1+\frac{3}{x}} \\
&= \frac{\sqrt{4+0}+5}{1+0} \\
&= 7,
\end{aligned}$$

So $\boxed{y=3 \text{ and } y=7 \text{ are the two horizontal asymptotes of } f}$.

9. For each region described below, (i) sketch the region, then use (ii) integration with respect to x and (iii) integration with respect to y to set-up expression with integrals calculating the area of the region.

   (a) The region bounded by the parabola $x = 7-y^2$ and the line $x=3$.

   Solution. (i)



(ii) The vertical strip at x in the region is bounded by both branches of the parabola. To find the length of the strip $\ell(x)$, we need to solve the equation of the parabola for x.

$$x = 7 - y^2 \ \Rightarrow \ y^2 = 7 - x \ \Rightarrow \ y = \pm\sqrt{7-x}.$$

The equation $y = \sqrt{7-x}$ corresponds to the top branch of the parabola, and the equation $y = -\sqrt{7-x}$ corresponds to the bottom branch. Therefore, the length of the vertical strip at x is $\ell(x) = \sqrt{7-x} - (-\sqrt{7-x}) = 2\sqrt{7-x}$. The region is located between $x = 3$ and $x = 7$, therefore

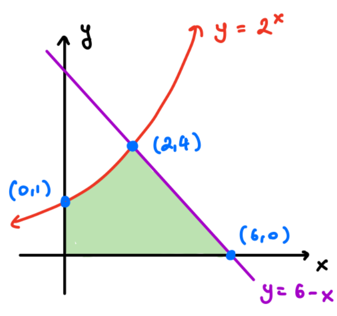
$$\boxed{A = \int_3^7 2\sqrt{7-x}dx}.$$

(iii) The horizontal strip at y is bounded by the line $x = 3$ on the left and $x = 7 - x^2$ on the right, so it has length $\ell(y) = (7 - y^2) - 3 = 4 - y^2$. The region is located between $y = -2$ and $y = 2$, so

$$\boxed{A = \int_{-2}^2 (4 - y^2)dy}.$$

(b) The region bounded by the x-axis, the y-axis, the curve $y = 2^x$ and the line $y = 6 - x$. *(You can use fact that the curve and the line intersect at the point* $(2, 4)$ *only.)*

Solution.(i)



(ii) The vertical strip at x in the region is bounded by $y = 2^x$ between $x = 0$ and $x = 2$, and $y = 6-x$ between $x = 2$ and $x = 6$. Therefore, the area is

$$\boxed{A = \int_0^2 2^x dx + \int_2^6 (6-x)dx}.$$

(iii) The horizontal strip at y is bounded on the right by $y = 6 - x$, so $x = 6 - y$. On the left, the strip is bounded by the y-axis when $0 \leqslant y \leqslant 1$. When $1 \leqslant y \leqslant 4$, the strip is bounded on the right by $y = 2^x$, that is $x = \log_2(y)$. Hence, the area is

$$\boxed{A = \int_0^1 (6-y)dy + \int_1^4 (6 - y - \log_2(y))dy}.$$

10. Let $f(x) = \sin(x)\sqrt[3]{\cos(x)}$. Find the absolute maximum and minimum values of f on the interval $[0, \pi]$ and where they occur.

Solution. First, we find the critical points of f. The derivative is given by

$$f'(x) = \cos(x)(\cos(x))^{1/3} - \frac{\sin^2(x)}{3(\cos(x))^{2/3}} = \frac{3\cos^2(x) - \sin^2(x)}{2(\cos(x))^{2/3}}.$$

- $f'(x) = 0$ gives $3\cos^2(x) = \sin^2(x)$, or $\tan^2(x) = 3$, which gives $\tan(x) = \pm\sqrt{3}$. This equation has two solution in $[0, \pi]$, namely $x = \frac{\pi}{3}, \frac{2\pi}{3}$.
- $f'(x)$ is undefined when $\cos(x) = 0$, which occurs when $x = \frac{\pi}{2}$ in the interval $[0, \pi]$.

We now evaluate $f(x)$ at the critical points in $[0, \pi]$ and the endpoints of $[0, \pi]$.

- $f(0) = 0$,
- $f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}\sqrt[3]{\frac{1}{2}} = \frac{\sqrt{3}}{2^{4/3}}$,
- $f\left(\frac{\pi}{2}\right) = 0$,
- $f\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}\sqrt[3]{-\frac{1}{2}} = -\frac{\sqrt{3}}{2^{4/3}}$,
- $f(\pi) = 0$.

Hence the absolute maximum of $f(x)$ on $[0, \pi]$ is $\boxed{\frac{\sqrt{3}}{2^{4/3}}}$ and it occurs at $\boxed{x = \frac{\pi}{3}}$ and the absolute minimum of $f(x)$ on $[0, \pi]$ is $\boxed{-\frac{\sqrt{3}}{2^{4/3}}}$ and it occurs at $\boxed{x = \frac{2\pi}{3}}$.

11. Suppose that f is a one-to-one differentiable function. The following table of values is given for f and f'.

x	-1	0	1	2
$f(x)$	2	3	6	11
$f'(x)$	7	2	8	5

(a) Find an equation of the tangent line to the graph of $y = f(x)$ at the point $x = 1$.

Solution. $\boxed{y - 6 = 8(x - 1)}$

(b) Find an equation of the tangent line to the graph of $y = f^{-1}(x)$ at the point $x = 2$.

Solution. $\boxed{y - (-1) = \frac{1}{7}(x - 2)}$

(c) Let $G(x) = \arccos(2x) f(3x)$. Calculate $G'(0)$.

Solution. We have

$$G'(x) = \frac{-2}{\sqrt{1-4x^2}} f(3x) + 3\arccos(2x) f'(3x).$$

So

$$G'(0) = -2f(0) + 3\arccos(0) f'(0) = \boxed{-6 + 3\pi}.$$

12. Let $f(x) = \frac{1}{2}x^{2/3}(x+5)$. Find the open intervals where f is increasing, decreasing, concave up, concave down, the x-coordinates of the local maxima, local minima and inflection points of f. Then sketch the graph of f.

Solution. First, we calculate the derivatives of f and chart their sign.

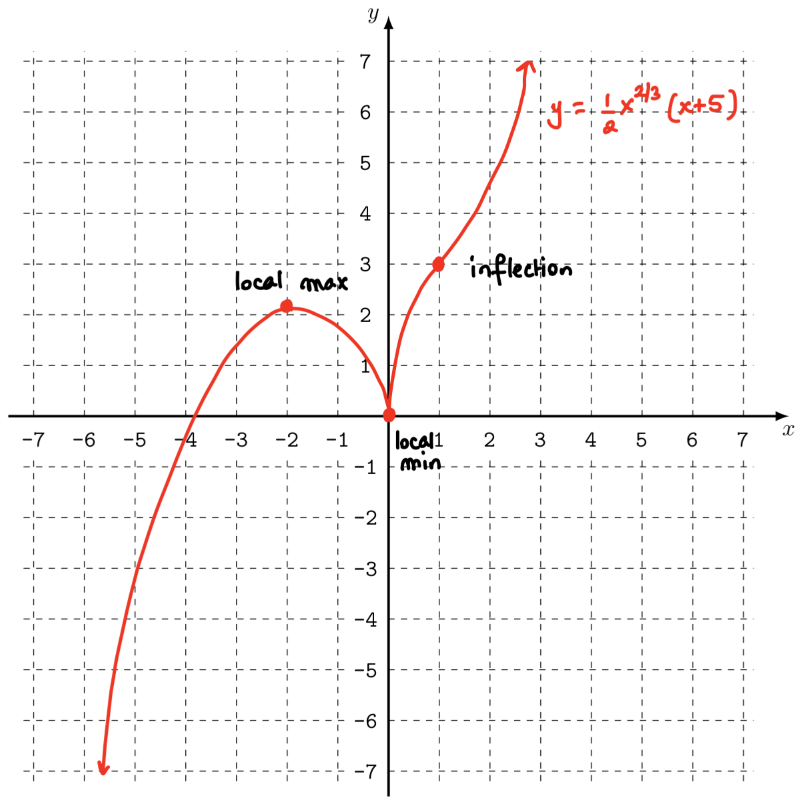
$$\begin{aligned} f'(x) &= \frac{x+5}{3x^{1/3}} + \frac{x^{2/3}}{2} \\ &= \frac{2x + 10 + 3x}{6x^{1/3}} \\ &= \frac{5(x+2)}{6x^{1/3}} \end{aligned}$$

x	$(-\infty, -2)$	$(-2, 0)$	$(0, \infty)$
$f'(x)$	$+$	$-$	$+$

$$\begin{aligned} f''(x) &= \frac{5}{6x^{1/3}} - \frac{5(x+2)}{18x^{4/3}} \\ &= \frac{15x - 5x - 10}{18x^{4/3}} \\ &= \frac{5(x-1)}{9x^{4/3}} \end{aligned}$$

x	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$f''(x)$	$-$	$-$	$+$

- Increasing: $\boxed{(-\infty, -2), (0, \infty)}$
- Decreasing: $\boxed{(-2, 0)}$
- Concave up: $\boxed{(1, \infty)}$
- Concave down: $\boxed{(-\infty, 0), (0, 1)}$
- Local maximum at: $\boxed{x = -2}$
- Local minimum at: $\boxed{x = 0}$
- Inflection point at: $\boxed{x = 1}$



13. Let $F(x) = \displaystyle\int_0^{x^2} e^{-t^2/4} dt$. Find the open intervals where F is concave up, concave down and the x-coordinates of the inflection points of F.

Solution. First, we must find $F''(x)$. For the first derivative, we use the Fundamental Theorem combined with the Chain Rule to get

$$F'(x) = e^{-(x^2)^2/4} \cdot (2x) = 2xe^{-x^4/4}.$$

We differentiate one more time and we obtain

$$\begin{aligned} F''(x) &= 2e^{-x^4/4} + 2xe^{-x^4/4} \cdot (-x^3) \\ &= 2e^{-x^4/4}\left(1 - x^4\right) \\ &= 2e^{-x^4/4}(1-x)(1+x)(1+x^2). \end{aligned}$$

The sign chart of $F''(x)$ is given below.

x	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$F''(x)$	$-$	$+$	$-$

We can now conclude that

- F is concave up on $\boxed{(-1, 1)}$.
- F is concave down on $\boxed{(-\infty, -1), (1, \infty)}$.

- F has inflection points at $\boxed{x = -1, 1}$.

14. Find the points on the ellipse of equation $x^2 + xy + y^2 = 12$ where the tangent line is (a) horizontal and (b) vertical.

    Solution. We use implicit differentiation to find $y' = \frac{dy}{dx}$. Differentiating the equation $x^2 + xy + y^2 = 12$ with respect to x gives

    $$\begin{aligned} &2x + y + xy' + 2yy' = 0 \\ &\Rightarrow\ y'(x + 2y) = -2x - y \\ &\Rightarrow\ y' = \frac{-2x - y}{x + 2y}. \end{aligned}$$

    (a) The tangent line is horizontal when $y' = 0$. This gives $-2x - y = 0$, so $y = -2x$. Substituting this back in the equation of the ellipse gives

    $$\begin{aligned} &x^2 + x(-2x) + (-2x)^2 = 12 \\ &3x^2 = 12 \\ &x^2 = 4 \\ &x = \pm 2 \end{aligned}$$

    Using $y = -2x$, we get the points $\boxed{(2, -4), (-2, 4)}$.

    (b) The tangent line is vertical when y' is undefined, which happens when the denominator of y' is equal to 0. This gives $x + 2y = 0$, so $x = -2y$. Substituting this back in the equation of the ellipse gives

    $$\begin{aligned} &(-2y)^2 + (-2y)y + y^2 = 12 \\ &3y^2 = 12 \\ &y^2 = 4 \\ &y = \pm 2 \end{aligned}$$

    Using $x = -2y$, we get the points $\boxed{(4, -2), (-4, 2)}$.

15. A particle is moving along an axis with acceleration $a(t) = \frac{6t}{(9+t^2)^2}$, initial velocity $v(0) = 1$ and initial position $s(0) = -2$. Find the position $s(t)$ of the particle.

    Solution. First, we find the velocity $v(t)$ of the particle. The velocity is an antiderivative of the acceleration, so

    $$\begin{aligned} v(t) &= \int a(t)dt \\ &= \int \frac{6t}{(9+t^2)^2} dt \\ &= \int \frac{3}{u^2} du \quad \left(u = 9 + t^2, du = 2tdt\right) \qquad = -\frac{3}{u} + C \\ &= -\frac{3}{9+t^2} + C. \end{aligned}$$

To find the constant C, we use the initial velocity $v(0) = 1$, which gives $-\frac{3}{9+0} + C = 1$, so $C = \frac{4}{3}$. Hence, the velocity is $v(t) = -\frac{3}{9+t^2} + \frac{4}{3}$.

We can now find the position $s(t)$ by taking an antiderivative of the velocity. This gives

$$\begin{aligned} s(t) &= \int \left(-\frac{3}{9+t^2} + \frac{4}{3} \right) dt \\ &= \int \left(-\frac{3}{3^2+t^2} + \frac{4}{3} \right) dt \\ &= -3\frac{1}{3} \tan^{-1}\left(\frac{t}{3}\right) + \frac{4}{3}t + D \\ &= -\tan^{-1}\left(\frac{t}{3}\right) + \frac{4}{3}t + D. \end{aligned}$$

To find the constant D, we use the initial position $s(0) = -2$, which gives $-\tan^{-1}(0) + 0 + D = -2$, so $D = -2$. Hence we have obtained $\boxed{s(t) = -\tan^{-1}\left(\frac{t}{3}\right) + \frac{4}{3}t - 2}$.

16. Let $f(x) = \begin{cases} Ax + B & \text{if } x \leq 0, \\ \arcsin\left(\frac{1}{x+2}\right) & \text{if } 0 < x. \end{cases}$

(a) Find the value of the constant B for which f is continuous for all real numbers.

Solution. Each piece of f is continuous on its given domain, so it suffices to ensure continuity at the transition point $x = 0$. For this, we need $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$. This gives

$$A \cdot 0 + B = \arcsin\left(\frac{1}{2}\right) \quad \Rightarrow \quad \boxed{B = \frac{\pi}{6}}.$$

(b) Find the values of the constants A, B for which f satisfies the conditions of the Mean Value Theorem on the interval $[-1, 1]$.

Solution. To satisfy the assumptions of the MVT, we will need $f(x)$ to be continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. We already know from (a) that continuity requires $\boxed{B = \frac{\pi}{6}}$.

For differentiability, observe that each piece is differentiable on its given domain. So we just need to ensure differentiability at $x = 0$. We have

- $\frac{d}{dx}(Ax + B)_{|x=0} = A$,
- $\frac{d}{dx}\left(\arcsin\left(\frac{1}{x+2}\right)\right)_{|x=0} = \left(\frac{1}{\sqrt{1 - \frac{1}{(x+2)^2}}} \cdot \frac{-1}{(x+2)^2} \right)_{|x=0} = \frac{1}{\sqrt{1 - \frac{1}{4}}} \cdot \frac{-1}{2} = -\frac{1}{\sqrt{3}}$.

So we need $\boxed{A = -\frac{1}{\sqrt{3}}}$.